Superfluid density and condensate fraction in the BCS-BEC crossover regime at finite temperatures

N. Fukushima,[1] Y. Ohashi,[1,2] E. Taylor,[3] and A. Griffin[3]
[1]*Institute of Physics, University of Tsukuba, Tsukuba, Ibaraki 305, Japan*
[2]*Faculty of Science and Technology, Keio University, Hiyoshi, Yokohama, 223, Japan*
[3]*Department of Physics, University of Toronto, Toronto, Ontario, Canada M5S 1A7*
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The superfluid density is a fundamental quantity describing the response to a rotation as well as in two-fluid collisional hydrodynamics. We present extensive calculations of the superfluid density ρ_s in the BCS-BEC crossover regime of a uniform superfluid Fermi gas at finite temperatures. We include strong-coupling or fluctuation effects on these quantities within a Gaussian approximation. We also incorporate the same fluctuation effects into the BCS single-particle excitations described by the superfluid order parameter Δ and Fermi chemical potential μ, using the Nozières–Schmitt-Rink approximation. This treatment is shown to be necessary for consistent treatment of ρ_s over the entire BCS-BEC crossover. We also calculate the condensate fraction N_c as a function of the temperature, a quantity which is quite different from the superfluid density ρ_s. We show that the mean-field expression for the condensate fraction N_c is a good approximation even in the strong-coupling BEC regime. Our numerical results show how ρ_s and N_c depend on temperature, from the weak-coupling BCS region to the BEC region of tightly bound Cooper pair molecules. In a companion paper [Phys. Rev. A **74**, 063626 (2006)], we derive an equivalent expression for ρ_s from the thermodynamic potential, which exhibits the role of the pairing fluctuations in a more explicit manner.

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I. INTRODUCTION

In the last few years, the BCS-BEC crossover in two-component Fermi superfluids has become a central topic in ultracold atom physics [1–4]. This crossover is of special interest since the superfluidity continuously changes from the weak-coupling BCS-type to the Bose-Einstein condensation (BEC) of tightly bound Cooper pairs, as one increases the strength of a pairing interaction [5]. Thus the BCS-BEC crossover enables us to study fermion superfluidity and boson superfluidity in a unified manner.

The superfluid density ρ_s is a fundamental quantity which describes the response of a superfluid which arises from a BEC [6]. The superfluid density was first introduced by Landau as part of the two-fluid theory of superfluid ^{4}He [7]. At $T=0$, the value of ρ_s always equals the total carrier density n. (In the BCS-BEC crossover, n is the total number density of fermions.) This property is satisfied in both the Fermi superfluids and Bose superfluids, irrespective of the strength of the interaction between particles. This is quite different from what is called the condensate fraction N_c, which describes the number of Bose-condensed particles [7,8]. For example, in superfluid ^{4}He, only about 10% of atoms are Bose condensed even at $T=0$, due to the strong repulsion between the ^{4}He atoms (for a review, see Ref. [9]). In contrast, all the atoms contribute to the superfluid density at $T=0$: namely, $\rho_s(T=0)=n$.

In a companion paper, we have discussed some analytical results for the superfluid density ρ_s in the BCS-BEC crossover regime of a uniform superfluid Fermi gas [10]. Going past the weak-coupling BCS theory, we derived an expression for ρ_s in the Gaussian fluctuation level in terms of the fluctuations in the Cooper channel. The resulting expression for the normal fluid density, $\rho_n \equiv n-\rho_s$, is given by the sum of the usual BCS normal fluid density ρ_n^F and a bosonic fluctuation contribution ρ_n^B. While the superfluid density from fermions dominates in the weak-coupling BCS regime, the bosonic fluctuation contribution ρ_n^B becomes dominant in the strong-coupling BEC regime. Since ρ_n^B is absent in the mean-field BCS theory, inclusion of fluctuations in the Cooper channel is clearly essential in considering the superfluid density in the BCS-BEC crossover.

In Ref. [10], our expression for ρ_s was obtained using the thermodynamic potential in the presence of a superfluid flow. In the present paper, we derive a second expression for ρ_s by calculating the effect of pairing fluctuations on the single-particle Green's function with a supercurrent. In this paper, we use this expression to numerically calculate ρ_s in the entire BCS-BEC crossover regime at finite temperatures. However, this expression for ρ_s (given in Sec. III) is equivalent to the result derived in Ref. [10]. In calculating the superfluid order parameter Δ as well as Fermi chemical potential μ, we include the effect of the pairing fluctuations following the approach given in Ref. [4]. This self-consistent treatment of Δ and μ is crucial in calculating ρ_s as a function of the temperature in the BCS-BEC crossover.

Besides the superfluid density, we also calculate the condensate fraction N_c describing the number of Bose-condensed particles [8]. N_c is of special interest in superfluid Fermi gases, since it can be observed experimentally. Indeed, a finite value of N_c is the signature of the BCS-BEC superfluid phase [5]. In this paper, we show that strong-coupling pair fluctuations have little effect on N_c in the BCS-BEC crossover. We note that N_c has been recently calculated at $T=0$ within a simple mean-field BCS approach [11] and using Monte Carlo (MC) techniques [12]. In this paper, we present detailed results for N_c at finite temperatures in the BCS-BEC crossover, based on the Nozières–Schmitt-Rink (NSR) theory of fluctuations [2].

The present paper is organized as follows. In Sec. II, the BCS Green's functions are solved numerically for the superfluid order parameter Δ and chemical potential μ self-consistently. This is done for the entire BCS-BEC crossover region and at finite temperatures. In Sec. III, we calculate the superfluid density ρ_s treating the strong-coupling pair fluctuation effects within a Gaussian approximation [2–4,13,14]. Numerical results for ρ_s are presented in Sec. IV. In Sec. V, we define and calculate the condensate fraction N_c.

Throughout this paper, we take $\hbar = k_B = 1$. We also set the volume $V=1$, so that the number of atoms, N, and the number density n are the same.

II. BCS-BEC CROSSOVER IN THE SUPERFLUID PHASE

In superfluid Fermi gases, current experiments make use of a broad Feshbach resonance to tune the magnitude and sign of the pairing interaction [5]. In this case, the superfluid properties can be studied by using the single-channel BCS model, described by the Hamiltonian

$$H = \sum_{\mathbf{p},\sigma} \xi_{\mathbf{p}} c^\dagger_{\mathbf{p}\sigma} c_{\mathbf{p}\sigma} - U \sum_{\mathbf{p},\mathbf{p}',\mathbf{q}} c^\dagger_{\mathbf{p}+\mathbf{q}\uparrow} c^\dagger_{\mathbf{p}'-\mathbf{q}\downarrow} c_{\mathbf{p}'\downarrow} c_{\mathbf{p}\uparrow}. \tag{2.1}$$

Here, $c^\dagger_{\mathbf{p}\sigma}$ is a creation operator of a Fermi atom with pseudospin $\sigma = \uparrow, \downarrow$ (which describes the two atomic hyperfine states). The Fermi atoms have kinetic energy $\xi_{\mathbf{p}} = \varepsilon_{\mathbf{p}} - \mu = p^2/2m - \mu$, measured from the Fermi chemical potential μ. $-U$ describes a pairing interaction between different Fermi atoms. The magnitude and sign of U can be tuned using the Feshbach resonance using an external magnetic field. The weak-coupling BCS limit corresponds to $U \to +0$. We only consider a uniform gas in this paper.

Nozières and Schmitt-Rink first discussed the BCS-BEC crossover behavior of the Hamiltonian in Eq. (2.1) to determine T_c [2] based on a Gaussian approximation for pair fluctuations [3,13]. The NSR theory has been extended to the superfluid phase below T_c [4,14,15]. In the extended NSR theory, the superfluid order parameter Δ and chemical potential μ are determined from the coupled equations [4,14]

$$1 = U \sum_{\mathbf{p}} \frac{1}{2E_{\mathbf{p}}} \tanh \frac{\beta E_{\mathbf{p}}}{2}, \tag{2.2}$$

$$N = N_F^0 - \frac{1}{2\beta} \frac{\partial}{\partial \mu} \sum_{\mathbf{q},\nu_n} \ln \det[1 + U \hat{\Xi}(\mathbf{q}, i\nu_n)]. \tag{2.3}$$

Here, $E_{\mathbf{p}} = \sqrt{\xi_{\mathbf{p}}^2 + \Delta^2}$ describes Bogoliubov single-particle excitations. In Eq. (2.3),

$$N_F^0 = \sum_{\mathbf{p}} \left(1 - \frac{\xi_{\mathbf{p}}}{E_{\mathbf{p}}} \tanh \frac{\beta}{2} E_{\mathbf{p}}\right) \tag{2.4}$$

is the number of Fermi atoms in the mean-field approximation. The second term in Eq. (2.3) describes contribution from bosonic collective pair fluctuations [16,17], where $\hat{\Xi}$ is a (2×2)-matrix correlation function given by [4]

$$\hat{\Xi}(\mathbf{q}, i\nu_n) = \frac{1}{4} \begin{pmatrix} \Pi^0_{11} + \Pi^0_{22} + i(\Pi^0_{12} - \Pi^0_{21}) & \Pi^0_{11} - \Pi^0_{22} \\ \Pi^0_{11} - \Pi^0_{22} & \Pi^0_{11} + \Pi^0_{22} - i(\Pi^0_{12} - \Pi^0_{21}) \end{pmatrix}. \tag{2.5}$$

The correlation functions Π^0_{ij} are given within the mean-field approximation by [18]

$$\Pi^0_{ij}(\mathbf{q}, i\nu_n) = \frac{1}{\beta} \sum_{\mathbf{p},\omega_n} \mathrm{Tr}[\tau_i \hat{G}_0(\mathbf{p} + q/2, i\omega_n + i\nu_n) \tau_j \hat{G}_0(\mathbf{p} - q/2, i\omega_n)], \tag{2.6}$$

where

$$\hat{G}_0(\mathbf{p}, i\omega_m) = \frac{i\omega_m + \xi_{\mathbf{p}}\tau_3 - \Delta\tau_1}{(i\omega_m)^2 - E_{\mathbf{p}}^2} \tag{2.7}$$

is the (2×2)-matrix single-particle Green's function [19] (where ω_m is the fermion Matsubara frequency) and τ_j are the Pauli operators. For detailed expressions of Π^0_{ij}, we refer to Refs. [4,18].

We solve Eq. (2.2) together with Eq. (2.3) to give Δ and μ self-consistently. As usual, we need to introduce a high-energy cutoff ω_c in these coupled equations. This cutoff can be formally eliminated by introducing the two-body s-wave scattering length a_s [3],

$$\frac{4\pi a_s}{m} \equiv - \frac{U}{1 - U \sum_{\mathbf{p}}^{\omega_c} \frac{1}{2\varepsilon_{\mathbf{p}}}}. \tag{2.8}$$

Using a_s in place of U, one can rewrite Eqs. (2.2) and (2.3) in the form

$$1 = - \frac{4\pi a_s}{m} \sum_{\mathbf{p}} \left(\frac{1}{2E_{\mathbf{p}}} \tanh \frac{\beta}{2} E_{\mathbf{p}} - \frac{1}{2\varepsilon_{\mathbf{p}}}\right), \tag{2.9}$$

$$N = N_{\mathrm{F}}^0 - \frac{1}{2\beta} \frac{\partial}{\partial \mu} \sum_{\mathbf{q},\nu_n} \ln \det\left[1 - \frac{4\pi a_s}{m}\left(\hat{\Xi}(\mathbf{q}, i\nu_n) + \frac{1}{2\varepsilon_{\mathbf{p}}}\right)\right], \tag{2.10}$$

where the momentum sums are now no longer divergent. The weak-coupling BCS regime and the strong-coupling BEC regime are, respectively, given by $(k_{\mathrm{F}} a_s)^{-1} \lesssim -1$ and $(k_{\mathrm{F}} a_s)^{-1} \gtrsim 1$ (where k_{F} is the Fermi momentum). The region $-1 \lesssim (k_{\mathrm{F}} a_s)^{-1} \lesssim 1$ is referred to as the "crossover regime."

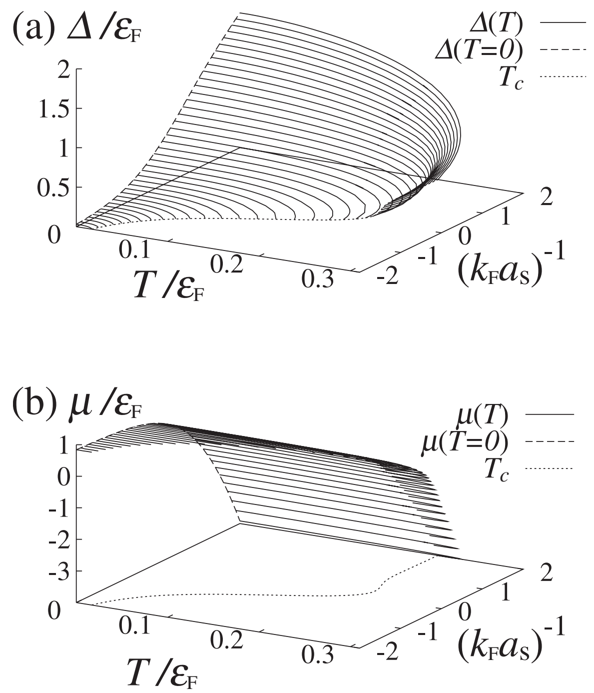

FIG. 1. (a) Off-diagonal mean-field Δ and (b) Fermi chemical potential μ in the BCS-BEC crossover. The pairing interaction is measured in terms of the inverse of the two-body scattering length a_s, normalized by the Fermi momentum k_F. In these panels, the dotted line shows T_c as a function of $(k_F a_s)^{-1}$. In the strong-coupling regime, the apparent first-order behavior of the phase transition is an artifact of the NSR Gaussian treatment of pairing fluctuations (see text).

Figure 1 shows our self-consistent solutions of the coupled equations (2.9) and (2.10) in the BCS-BEC crossover at finite temperatures. We note that these calculations reproduce the NSR results for T_c [3].

When one enters the crossover region, Fig. 2 shows that the order parameter Δ deviates from the weak-coupling BCS result. In the strong-coupling BEC regime, although the superfluid phase transition approaches the value $T_c = 0.218T_F$ [3], $\Delta(T=0)$ continues to increase. As a result, the ratio $2\Delta(T=0)/T_c$ in the BEC regime is larger than the weak-coupling BCS universal constant $2\Delta(T=0)/T_c = 3.54$. We recall that on the BEC side of the crossover, where μ is negative, the energy gap is not equal to Δ [4,14].

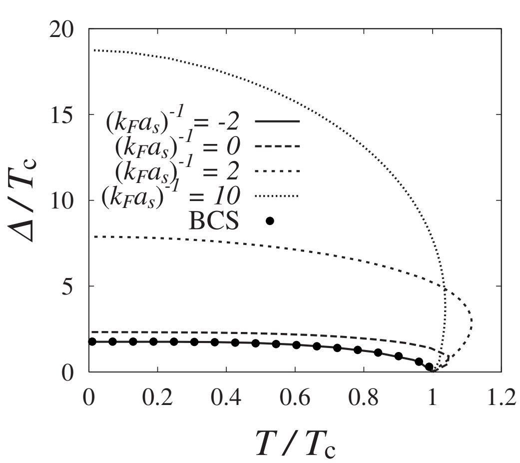

FIG. 2. Calculated values of the superfluid order parameter Δ as a function of temperature. "BCS" labels the weak-coupling BCS limit. The bendover near T_c is an artifact of our NSR Gaussian treatment of fluctuations.

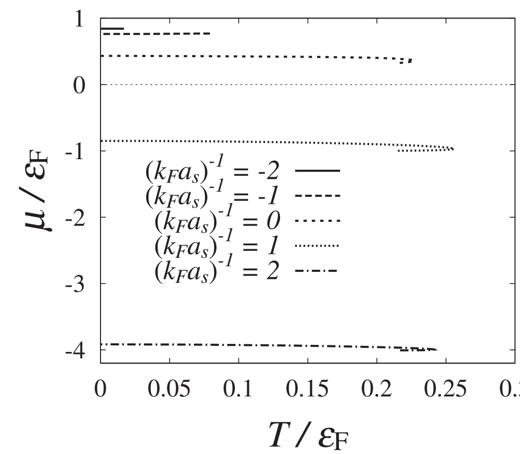

FIG. 3. Fermi chemical potential μ as a function of temperature. For comparison, in the unitarity limit $[(k_F a_s)^{-1} = 0]$, MC results [12,21] give $\mu/\varepsilon_F = 0.44 - 0.49$ and an improved version of NSR theory [22] gives $\mu/\varepsilon_F = 0.4 - 0.47$ just below T_c.

The chemical potential is strongly affected by fluctuations in the Cooper channel and becomes negative in the BEC regime, as shown in Fig. 1(b). In the strong-coupling BEC regime, μ approaches $\mu = -1/2ma_s^2$ [20]. Although μ strongly depends on the magnitude of the interaction, Fig. 3 shows that the temperature dependence of μ is very weak in the entire BCS-BEC crossover. Our results in the unitarity limit are in quite good agreement with quantum Monte Carlo simulations [12,21] as well as a more self-consistent version of NSR theory [22].

In Fig. 1, the apparent first-order phase transition in the BEC regime is an artifact of the approximate NSR theory we are using. The reason is as follows. In the BEC regime (where $\mu \ll -\varepsilon_F$), the single-particle BCS excitations $E_{\mathbf{p}} = \sqrt{\xi_{\mathbf{p}}^2 + \Delta^2}$ have a large energy gap given by $E_g \equiv \sqrt{\mu^2 + \Delta^2} \simeq |\mu|$. This energy gap still exists at T_c, where $\Delta = 0$. In this regime, we can set $\tanh(\beta E_{\mathrm{p}}/2) \simeq 1$ in Eq. (2.2). Then, Eq. (2.9) reduces to the expression $\mu = -1/2ma_s^2$ and Eq. (2.10) becomes [10]

$$\frac{N}{2} = N_{c0} - \frac{1}{\beta}\sum_{\mathbf{q},\nu_n} D(\mathbf{q}, i\nu_n) e^{i\delta\nu_n} = N_{c0} + \frac{1}{2}\sum_{\mathbf{q}} \left(\frac{\varepsilon_{\mathbf{q}}^B + U_M N_{c0}}{\omega_{\mathbf{q}}} \coth \frac{\beta}{2}\omega_{\mathbf{q}} - 1 \right) \equiv N_{c0} + N_d. \tag{2.11}$$

Here, $\varepsilon_{\mathbf{q}}^B = q^2/2M$ $(M = 2m)$ and

$$\omega_{\mathbf{q}} = \sqrt{\varepsilon_{\mathbf{q}}^B(\varepsilon_{\mathbf{q}}^B + 2U_M N_{c0})} \tag{2.12}$$

is the Bogoliubov excitation spectrum in an interacting gas of Bose molecules. D in Eq. (2.11) is the Bose Green's function, describing Bogoliubov excitations:

$$D(\mathbf{q},i\nu_n)=\frac{i\nu_n+\varepsilon^B_{\mathbf{q}}+U_M N_{c0}}{(i\nu_n)^2-\omega^2_{\mathbf{q}}}, \quad (2.13)$$

where $U_M=4\pi a_M/M$ (where in our theory $a_M=2a_s$ and $M=2m$) is the effective s-wave repulsive interaction between Cooper pairs. The condensate fraction N_{c0} defined in Eq. (2.11) is given by

$$N_{c0}\equiv\sum_{\mathbf{p}}\frac{\Delta^2}{4\xi^2_{\mathbf{p}}}=\frac{\sqrt{2}m^{3/2}\Delta^2}{16\pi\sqrt{|\mu|}}. \quad (2.14)$$

In Sec. V, we prove that N_{c0} as defined in Eq. (2.14) corresponds precisely to the formal definition for the condensate fraction in a Fermi superfluid in the BEC limit. N_d as defined in Eq. (2.11) is the number of molecules which are not Bose condensed, again in the BEC limit.

Equations (2.11) and (2.12) show that the strong-coupling BEC limit corresponds to the Popov approximation for a weakly interacting molecular Bose gas [23]. As is well known (see, for example, Ref. [24]), the Popov approximation gives a spurious first-order phase transition at T_c. This is the origin of the bendover or first-order phase transition evident in Fig. 1 [25] and other figures in this paper. It is well known how to overcome this problem; namely, one has to include many-body renormalization effects due to the interaction U_M [24,26]. Including such higher-order corrections past the NSR Gaussian fluctuations considered in this paper is also crucial for determining the correct value of effective interaction U_M. The molecular scattering length $a_M=2a_s$ which is obtained in Eq. (2.11) is characteristic of the NSR treatment of fluctuations [3]. The correct result $a_M=0.6a_s$ [27] requires going past the Gaussian approximation [26,28]. In this paper, in contrast, we only treat pairing fluctuations within NSR theory. However, within this approximation, we calculate the superfluid density (and condensate fraction N_c in Sec V) in a consistent manner.

As discussed in Ref. [24], the Popov approximation becomes invalid in the small region close to T_c given by

$$\delta t\equiv\frac{T_c-T}{T_c}\lesssim\left(\frac{1}{6\pi^2}\right)^{1/3}(k_F a_M)=0.26(k_F a_M). \quad (2.15)$$

Although we plot numerical results in the present paper in the whole temperature region for completeness, we emphasize that the restriction in Eq. (2.15) also holds in the BEC regime. We note that the region defined in Eq. (2.15) becomes narrow as one enters deeper into the BEC regime, simply because the molecular scattering length $a_M\propto a_s$ becomes small. Thus, one obtains $\delta t\lesssim 0.26$ at $(k_F a_s)^{-1}=2$ (the case shown in Fig. 2, for example) but $\delta t\lesssim 0.1$ at $(k_F a_s)^{-1}=5$. As Fig. 2 shows, the bendover occurs over an increasingly small region as we go deeper in the BEC region.

Although Eq. (2.11) was obtained in the strong-coupling BEC regime, we note that the condensate fraction N_c in Eq. (2.14) is the mean-field approximation for a Fermi superfluid. This is consistent with the result found in Sec. V that the mean-field expression for the condensate fraction N_c is a good approximation even in the strong-coupling BEC regime (at least within NSR theory). The number of molecules N_d in the *noncondensate* given in Eq. (2.11), in contrast, is due to the pairing fluctuations. This is discussed in more detail in Sec. V.

III. SUPERFLUID DENSITY AND THE SINGLE-PARTICLE GREEN'S FUNCTION

In Ref. [10], our discussion of the superfluid density ρ_s started from the thermodynamic potential $\Omega(v_s)$ in the presence of an imposed superfluid velocity (or phase twist) $\mathbf{v}_s$. Here we give an alternative formulation of ρ_s in terms of how the single-particle Green's function is altered in the presence of a supercurrent. Our numerical calculations in Sec. IV are based on this expression, but it can be proven to be equivalent to the one discussed in Ref. [10]. The result we obtain in this section gives further insight and is convenient for numerical calculations.

When a supercurrent flows in the z direction with the superfluid velocity $v_s=Q_z/2m$, the supercurrent density J_z is given by

$$J_z=\sum_{\mathbf{p},\sigma}\frac{p_z}{m}\langle c^\dagger_{\mathbf{p}\sigma}c_{\mathbf{p},\sigma}\rangle=nv_s+\frac{1}{\beta}\sum_{\mathbf{p},\omega_m}\frac{p_z}{m}\mathrm{Tr}[\hat{g}(\mathbf{p},i\omega_m)]. \quad (3.1)$$

Here $\hat{g}(\mathbf{p},i\omega_m)$ is the (2×2)-matrix single-particle thermal Green's function in the presence of v_s. When the second term in Eq. (3.1) is expanded to $O(v_s)$, it can be written as $J_z=\rho_s v_s$, where the superfluid density ρ_s is defined by

$$\rho_s=n+\frac{2}{\beta}\sum_{\mathbf{p},\omega_m}p_z\frac{\partial}{\partial Q_z}\mathrm{Tr}[\hat{g}(\mathbf{p},i\omega_m)]_{Q_z\to 0}\equiv n-\rho_n. \quad (3.2)$$

The second line defines the normal fluid density ρ_n.

In the mean-field approximation, the supercurrent state is described by the (2×2)-matrix single-particle thermal Green's function given by [29]

$$\hat{g}_0(\mathbf{p},i\omega_m)=\frac{1}{(i\omega_m-\alpha_{\mathbf{p}})-\tilde{\xi}_{\mathbf{p}}\tau_3+\Delta\tau_1}, \quad (3.3)$$

where the superfluid order parameter for the current-carrying state, $\Delta(\mathbf{r})=\Delta e^{i\mathbf{Q}\cdot\mathbf{r}}$ [$\mathbf{Q}=(0,0,Q_z)$], has been used. The effects of the supercurrent v_s appear in the Doppler shift term $\alpha_{\mathbf{p}}\equiv\mathbf{Q}\cdot\mathbf{p}/2m$ and in $\tilde{\xi}_{\mathbf{p}}\equiv\varepsilon_{\mathbf{p}}-\tilde{\mu}$, with $\tilde{\mu}\equiv\mu-Q^2/8m$. However, we do not have to take this dependence of $\tilde{\xi}_{\mathbf{p}}$ on μ into account in calculating ρ_n in Eq. (3.2) because it is second order $O(v_s^2)$.

Substituting Eq. (3.3) into Eq. (3.2), one obtains the well-known mean-field result

$$\rho_n^F=-\frac{2}{3m}\sum_{\mathbf{p}}p^2\frac{\partial f(E_{\mathbf{p}})}{\partial E_{\mathbf{p}}}. \quad (3.4)$$

In addition to ρ_n^F as given by Eq. (3.4), the correction to $\hat{g}_0$ to first order in v_s gives rise to an additional fluctuation contribution ρ_n^B to the normal fluid density. This should be consistent with the number equation (2.3), for the $v_s=0$ state,

$$N = \sum_{\mathbf{p}} 1 + \frac{1}{\beta}\sum_{\mathbf{p},\omega_m} \mathrm{Tr}[\tau_3 \hat{G}(\mathbf{p},i\omega_m)], \tag{3.5}$$

where the renormalized single-particle Green's function $\hat{G}$,

$$\hat{G}(\mathbf{p},i\omega_m) = \hat{G}_0(\mathbf{p},i\omega_m) + \hat{G}_0(\mathbf{p},i\omega_m)\hat{\Sigma}(\mathbf{p},i\omega_m)\hat{G}_0(\mathbf{p},i\omega_m), \tag{3.6}$$

involves the correction from the lowest-order quasiparticle self-energy due to coupling with collective fluctuations:

$$\begin{aligned}\hat{\Sigma}(\mathbf{p},i\omega_m) = \frac{U}{\beta}\sum_{\mathbf{q},\nu_n} \frac{1}{\eta(\mathbf{q},i\nu_n)}\{&[1+U\Xi_{11}(\mathbf{q},i\nu_n)]\tau_+ \\ &\times\hat{G}_0(\mathbf{p}+\mathbf{q},i\omega_m+i\nu_n)\tau_- + [1+U\Xi_{22}(\mathbf{q},i\nu_n)]\tau_- \\ &\times\hat{G}_0(\mathbf{p}+\mathbf{q},i\omega_m+i\nu_n)\tau_+ - 2U\Xi_{12}(\mathbf{q},i\nu_n)\tau_+ \\ &\times\hat{G}_0(\mathbf{p}+\mathbf{q},i\omega_m+i\nu_n)\tau_+\}.\end{aligned} \tag{3.7}$$

Here, $\eta(\mathbf{q},i\nu_n)\equiv\det[1+U\hat{\Xi}(\mathbf{q},i\nu_n)]$ and $\tau_\pm\equiv(\tau_1\pm i\tau_2)/2$ and τ_i are Pauli matrices. In obtaining this result, we have carried out the μ derivative on Ξ_{ij} in Eq. (2.3) by using the identity

$$\frac{\partial\hat{G}_0}{\partial\mu} = -\hat{G}_0\tau_3\hat{G}_0. \tag{3.8}$$

For example,

$$\begin{aligned}\frac{\partial\Xi_{11}}{\partial\mu} = -\frac{1}{\beta}\sum_{\mathbf{p},i\omega_m}\{&\mathrm{Tr}[\tau_3\hat{G}_0(\mathbf{p},i\omega_m)\tau_- \\ &\times\hat{G}(\mathbf{p}+\mathbf{q},i\omega_m+i\nu_n)\tau_+\hat{G}_0(\mathbf{p},i\omega_m)] \\ &+\mathrm{Tr}[\tau_3\hat{G}_0(\mathbf{p}+\mathbf{q},i\omega_m+i\nu_n)\tau_+\hat{G}(\mathbf{p},i\omega_m)\tau_- \\ &\times\hat{G}_0(\mathbf{p}+\mathbf{q},i\omega_m+i\nu_n)]\}.\end{aligned} \tag{3.9}$$

In the presence of a supercurrent, the Green's function analogous to Eq. (3.6) is given by

$$\hat{g}(\mathbf{p},i\omega_m) = \hat{g}_0(\mathbf{p},i\omega_m) + \hat{g}_0(\mathbf{p},i\omega_m)\hat{\Sigma}(\mathbf{p},i\omega_m)\hat{g}_0(\mathbf{p},i\omega_m), \tag{3.10}$$

where $\hat{\Sigma}$ is given by Eq. (3.7) but with $\hat{g}_0$ in Eq. (3.4) replacing $\hat{G}_0$. Substituting Eq. (3.10) into Eq. (3.2), we obtain

$$\rho_n = \rho_n^F + \rho_n^B. \tag{3.11}$$

The Fermi contribution is given by Eq. (3.4), while the bosonic fluctuation contribution ρ_n^B is given by

$$\begin{aligned}\rho_n^B &= -\frac{2}{\beta}\sum_{\mathbf{p},\omega_m} p_z\frac{\partial}{\partial Q_z}\mathrm{Tr}[\hat{g}_0(\mathbf{p},i\omega_m)\hat{\Sigma}(\mathbf{p},i\omega_m)\hat{g}_0(\mathbf{p},i\omega_m)]_{Q_z\to 0} \\ &= \frac{2}{\beta}\sum_{\mathbf{p},\omega_m} p_z\frac{\partial}{\partial Q_z}\mathrm{Tr}\left(\hat{\Sigma}(\mathbf{p},i\omega_m)\frac{\partial\hat{g}_0(\mathbf{p},i\omega_m)}{\partial i\omega_m}\right)_{Q_z\to 0},\end{aligned} \tag{3.12}$$

where we have used the identity

$$\frac{\partial\hat{g}_0}{\partial i\omega_m} = -\hat{g}_0\hat{g}_0. \tag{3.13}$$

Substituting Eq. (3.7) (with $\hat{G}_0\to\hat{g}_0$) into Eq. (3.12) and using the identity

$$\frac{\partial\hat{g}_0}{\partial\alpha_{\mathbf{p}}} = -\frac{\partial\hat{g}_0}{\partial i\omega_m}, \tag{3.14}$$

we find

$$\begin{aligned}\rho_n^B &= \frac{2}{\beta^2}\sum_{\mathbf{p},\omega_m}\frac{\partial}{\partial Q_z}\sum_{\mathbf{q},\nu_n} p_-^z\frac{U}{\eta(\mathbf{q},i\nu_n)} \\ &\quad\times\mathrm{Tr}\Bigg([1+U\Xi_{11}(\mathbf{q},i\nu_n)]\tau_+\hat{g}_0(\mathbf{p}_+,i\omega_m+i\nu_n)\tau_- \\ &\quad\times\frac{\partial\hat{g}_0(\mathbf{p}_-,i\omega_m)}{\partial i\omega_m} \\ &\quad+[1+U\Xi_{22}(\mathbf{q},i\nu_n)]\tau_-\hat{g}_0(\mathbf{p}_+,i\omega_m+i\nu_n)\tau_+\frac{\partial\hat{g}_0(\mathbf{p}_-,i\omega_m)}{\partial i\omega_m} \\ &\quad-2U\Xi_{12}(\mathbf{q},i\nu_n)\tau_+\hat{g}_0(\mathbf{p}_+,i\omega_m+i\nu_n)\tau_+\frac{\partial\hat{g}_0(\mathbf{p}_-,i\omega_m)}{\partial i\omega_m}\Bigg)_{Q_z\to 0} \\ &= -\frac{2m}{\beta}\frac{\partial}{\partial Q_z}\sum_{\mathbf{q},\nu_n}\frac{U}{\eta(\mathbf{q},i\nu_n)}\Bigg([1+U\Xi_{11}(\mathbf{q},i\nu_n)]\frac{\partial\Xi_{22}(\mathbf{q},i\nu_n)}{\partial Q_z} \\ &\quad+[1+U\Xi_{22}(\mathbf{q},i\nu_n)]\frac{\partial\Xi_{11}(\mathbf{q},i\nu_n)}{\partial Q_z} \\ &\quad-2U\Xi_{12}(\mathbf{q},i\nu_n)\frac{\partial\Xi_{12}(\mathbf{q},i\nu_n)}{\partial Q_z}\Bigg)_{Q_z\to 0},\end{aligned} \tag{3.15}$$

where $\mathbf{p}_\pm\equiv\mathbf{p}\pm\mathbf{q}/2$. The correlation functions $\Xi_{ij}(\mathbf{q},i\nu_n)$ appearing in Eq. (3.15) are defined in Eq. (2.5) in terms of the single-particle Green's functions $\hat{g}_0$ in the presence of a supercurrent. In Eq. (3.15), the Q_z derivative only acts on Q_z in the Doppler shift term $\alpha_{\mathbf{p}}=\mathbf{Q}\cdot\mathbf{p}/2m$ in $\hat{g}_0$ in Eq. (3.3). It does *not* act on the Q_z in the shifted chemical potential $\tilde{\mu}=\mu-Q^2/8m$.

To summarize, the total normal fluid density ρ_n associated with fermionic and bosonic degrees of freedom is given by the sum of their contributions:

$$\begin{aligned}\rho_n = -\frac{2}{3m}\sum_{\mathbf{p}} p^2\frac{\partial f(E_{\mathbf{p}})}{\partial E_{\mathbf{p}}} - \frac{2m}{\beta}\frac{\partial}{\partial Q_z}\sum_{\mathbf{q},\nu_n}\frac{U}{\eta(\mathbf{q},i\nu_n)}\Bigg(&[1 \\ +U\Xi_{11}(\mathbf{q},i\nu_n)]\frac{\partial\Xi_{22}(\mathbf{q},i\nu_n)}{\partial Q_z}+[1& \\ +U\Xi_{22}(\mathbf{q},i\nu_n)]\frac{\partial\Xi_{11}(\mathbf{q},i\nu_n)}{\partial Q_z}& \\ -2U\Xi_{12}(\mathbf{q},i\nu_n)\frac{\partial\Xi_{12}(\mathbf{q},i\nu_n)}{\partial Q_z}\Bigg)_{Q_z\to 0}&.\end{aligned} \tag{3.16}$$

We note that the superfluid density ρ_s can be also obtained from a current correlation function [30,31]. The present derivation based on calculating the change in the single-particle Green's function from Eqs. (3.2) and (3.10) is equivalent to

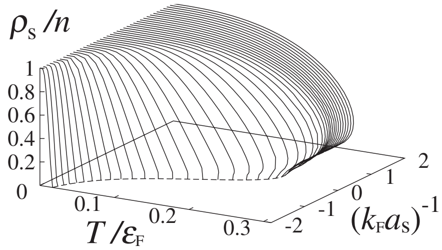

FIG. 4. Calculated superfluid density ρ_s in the BCS-BEC crossover. The self-consistent solutions for Δ and μ shown in Fig. 1 are used. The dashed line shows T_c, where ρ_s vanishes.

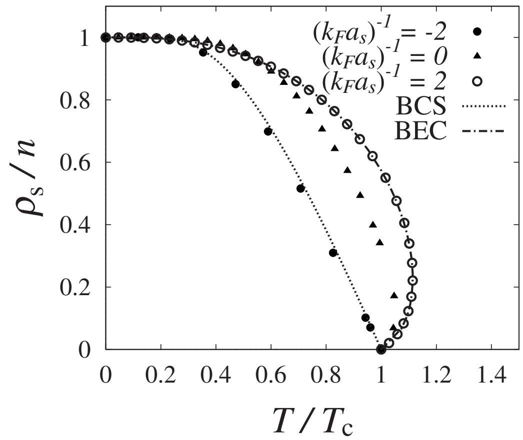

FIG. 5. Superfluid density ρ_s as a function of temperature in the BCS region (solid circles), unitarity limit (solid triangles), and BEC regime (open circles). "BCS" labels the mean-field BCS result, given by $\rho_s = n - \rho_n^F$ with $\mu = \varepsilon_F$. "BEC" gives ρ_s for a dilute Bose gas with $N/2$ bosons described by the excitation spectrum in Eq. (2.12).

calculating the current correlation function to first order in v_s, taking into account both self-energy and current vertex corrections. In Eq. (3.2), the Q_z derivative of the first term in Eq. (3.10) gives the bare current response function. The Q_z derivative of $\hat{g}_0$ in the second term in Eq. (3.10) gives the self-energy corrections, while the Q_z derivative of $\hat{\Sigma}$ gives the vertex corrections. See also Appendix A of Ref. [10] for further discussion.

IV. NUMERICAL RESULTS FOR SUPERFLUID DENSITY

In this section, we present numerical results for the superfluid density $\rho_s = n - \rho_n$, starting from the expression for ρ_n given in Eq. (3.16). As we have noted earlier, the expression for $\rho_s = n\rho_n$ derived in Ref. [10] would give identical results. We emphasize that these numerical results for ρ_s use the renormalized values of both Δ and μ [4] which determine the BCS quasiparticle spectrum over the entire BCS-BEC crossover.

Figure 4 shows the calculated superfluid density ρ_s in the BCS-BEC crossover. The spurious first-order behavior near T_c in the strong-coupling regime is also seen in the self-consistent solutions for Δ and μ in Fig. 1. As discussed in Sec. II, this behavior near T_c would be removed through a more sophisticated treatment of fluctuations, which could lead to the correct second-order phase transition. We plot our calculated results for ρ_s close to T_c, in spite of this problem. We note that the predicted value of T_c in the NSR theory of the BCS-BEC crossover is in good agreement with better theories [22]. The NSR bendover is much less in evidence in the case of a narrow Feshbach resonance, as considered in Ref. [4]. One can prove analytically that our expression in Eq. (3.16) gives $\rho_s = n$ at $T=0$ and also that ρ_s vanishes as $\Delta \to 0$ (normal phase). Getting these two limits correctly (as shown in Fig. 4) is very important in any theory of the superfluid density.

Figure 5 shows ρ_s as a function of temperature in the BCS regime, unitarity limit, and the BEC regime. We note that ρ_s in the BEC regime $[(k_F a_s)^{-1} = 2]$ is in good agreement with the superfluid density ρ_s of a weakly interacting gas of $N/2$ Bose molecules described by Bogoliubov-Popov excitations in Eq. (2.12), as one expects in the extreme BEC limit. More precisely, in this limit one can show (see Ref. [10] for details) that

$$\rho_s = n - \rho_n \simeq n - \rho_n^B, \tag{4.1}$$

where ρ_n^B is given by the Landau formula for the normal fluid of an interacting Bose gas:

$$\rho_n^B = -\frac{2}{3M} \sum_{\mathbf{q}} q^2 \frac{\partial n_B(\omega_{\mathbf{q}})}{\partial \omega_{\mathbf{q}}}. \tag{4.2}$$

Here, $M = 2m$ is the Cooper-pair mass and $n_B(\omega_{\mathbf{q}})$ is the Bose distribution function. The excitation energy $\omega_{\mathbf{q}}$ is given by Eq. (2.12), calculated with the correct values of Δ and μ. The curve labeled by BEC in Fig. 5 corresponds to the result obtained using Eqs. (4.1) and (4.2). This shows the importance of a consistent treatment of fluctuation effects in calculating ρ_s, Δ, and μ.

As one approaches the weak-coupling BCS regime, pair fluctuations become weak, so that in this limit the fermionic contribution ρ_n^F becomes dominant. Thus one has

$$\rho_s = n - \rho_n \simeq \rho_n^F, \tag{4.3}$$

where ρ_n^F is given by the Landau formula for the normal fluid in Eq. (3.4) with the BCS quasiparticle energies $E_{\mathbf{p}}$. The curve labeled by BCS in Fig. 5 corresponds to the result obtained using Eqs. (4.3) and (3.4).

V. CONDENSATE FRACTION IN THE BCS-BEC CROSSOVER

In this section, we calculate the condensate fraction N_c in the BCS-BEC crossover. The condensate fraction N_c in the superfluid phase is most conveniently defined [8] as the maximum eigenvalue of the two-particle density matrix, $\tilde{\rho}_2(\mathbf{r}, \mathbf{r}', \mathbf{r}'', \mathbf{r}''') \equiv \langle \psi_\uparrow^\dagger(\mathbf{r}) \psi_\downarrow^\dagger(\mathbf{r}') \psi_\downarrow(\mathbf{r}'') \psi_\uparrow(\mathbf{r}''') \rangle$, where $\psi_\sigma(\mathbf{r})$ is a fermion field operator. The condensate fraction N_c is given as the maximum eigenvalue, of order N. When only one eigenvalue is $O(N)$, one finds

$$\tilde{\rho}_2(\mathbf{r}, \mathbf{r}', \mathbf{r}'', \mathbf{r}''') = N_c \phi_0^*(\mathbf{r}, \mathbf{r}') \phi_0(\mathbf{r}'', \mathbf{r}'''), \tag{5.1}$$

where terms of order $O(1)$ have been ignored. Here $\phi_0(\mathbf{r},\mathbf{r}')$ is the (normalized) two-particle eigenfunction of $\tilde{\rho}_2$ with the eigenvalue N_c. The off-diagonal long-range order of a Fermi superfluid [8] is characterized as, for large separation of $(\mathbf{r},\mathbf{r}')$ and $(\mathbf{r}'',\mathbf{r}''')$,

$$\tilde{\rho}_2(\mathbf{r},\mathbf{r}',\mathbf{r}'',\mathbf{r}''') = \langle\psi_\uparrow^\dagger(\mathbf{r})\psi_\downarrow^\dagger(\mathbf{r}')\rangle\langle\psi_\downarrow(\mathbf{r}'')\psi_\uparrow(\mathbf{r}''')\rangle. \quad (5.2)$$

Comparing Eqs. (5.1) and (5.2), the condensate fraction N_c is seen to be the normalization factor of the Cooper-pair wave function, $\Phi(\mathbf{r},\mathbf{r}')\equiv\langle\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r}')\rangle$ (see, for example, Ref. [12]),

$$N_c = \int d\mathbf{r}d\mathbf{r}'|\Phi(\mathbf{r},\mathbf{r}')|^2. \quad (5.3)$$

In physical terms, the maximum eigenvalue N_c describes the occupancy of two-particle states.

In a uniform Fermi superfluid, the BCS mean-field approximation gives

$$\Phi(\mathbf{r},\mathbf{r}') = \sum_{\mathbf{p}}\langle c_{\mathbf{p}\uparrow}^\dagger c_{-\mathbf{p}\downarrow}^\dagger\rangle e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')} = \sum_{\mathbf{p}}\frac{1}{2E_{\mathbf{p}}}\tanh\frac{\beta}{2}E_{\mathbf{p}}e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}. \quad (5.4)$$

In the strong-coupling BEC regime (where $\mu\ll-\varepsilon_F$), we can set $\tanh\beta E_{\mathbf{p}}/2=1$ in Eq. (5.4). In this case, substituting Eq. (5.4) into Eq. (5.3), the mean-field expression for the condensate fraction ($\equiv N_{c0}$) in the BEC regime reduces to

$$N_{c0} = \sum_{\mathbf{p}}\frac{\Delta^2}{4E_{\mathbf{p}}^2} \simeq \sum_{\mathbf{p}}\frac{\Delta^2}{4\xi_{\mathbf{p}}^2}. \quad (5.5)$$

In obtaining this expression, we have used the fact that $|\mu|\gg\Delta$ in the BEC regime [14].

More generally, in terms of the single-particle Green's functions, one can write Eq. (5.3) as

$$N_c = \frac{1}{\beta^2}\sum_{\mathbf{p},\omega_m,\omega_m'}G_{21}(\mathbf{p},i\omega_m)G_{12}(\mathbf{p},i\omega_m'). \quad (5.6)$$

To calculate the strong-coupling effects on N_c, we substitute Eq. (3.6) into Eq. (5.6). Since this Green's function $\hat{G}=\hat{G}_0+\hat{G}_0\hat{\Sigma}\hat{G}_0$ only includes first-order self-energy corrections, we only retain the correction terms to N_c to $O(\hat{\Sigma})$, giving

$$N_0 = N_{c0} + \delta N_c. \quad (5.7)$$

Here, the mean-field component N_{c0} is the BCS Fermi quasiparticle contribution

$$N_{c0} \equiv \frac{1}{\beta^2}\sum_{\mathbf{p},\omega_m,\omega_m'}G_0^{21}(\mathbf{p},i\omega_m)G_0^{12}(\mathbf{p},i\omega_m') = \sum_{\mathbf{p}}\frac{\Delta^2}{4E_{\mathbf{p}}^2}\tanh^2\frac{\beta E_{\mathbf{p}}}{2}. \quad (5.8)$$

The first-order fluctuation contribution δN_c is given by

$$\delta N_c = \frac{1}{\beta^2}\sum_{\mathbf{p},\omega_m,\omega_m'}\left\{G_0^{21}(\mathbf{p},i\omega_m)\mathrm{Tr}[\tau_-\hat{G}_0(\mathbf{p},i\omega_m')\hat{\Sigma}(\mathbf{p},i\omega_m')\hat{G}_0(\mathbf{p},i\omega_m')] + \mathrm{Tr}[\tau_+\hat{G}_0(\mathbf{p},i\omega_m)\hat{\Sigma}(\mathbf{p},i\omega_m)\hat{G}_0(\mathbf{p},i\omega_m)]G_0^{12}(\mathbf{p},i\omega_m')\right\}. \quad (5.9)$$

The correction term δN_c in Eq. (5.9) is not important in the weak-coupling BCS regime, where fluctuation effects clearly can be ignored. Figure 6 shows that δN_c is also negligibly small in the strong-coupling regime. Thus N_c is well approximated by the mean-field expression in Eq. (5.8) over the entire BCS-BEC crossover, at least in our NSR-type approximation. We recall that the same pair fluctuations made a large contribution to ρ_s as we went from the BCS to BEC region. The difference is that, by definition, δN_c in Eq. (5.9) arises from self-energy corrections to the single-particle anomalous Green's function G_{12}. There is no distinct bosonic contribution, such as ρ_n^B in the normal fluid density. Thus it is not unexpected that the fluctuations are a small correction to N_c.

We note, however, since fluctuations in the Cooper channel are taken into account in the equation of state in Eq. (2.10), they modify the condensate fraction N_c given by Eq. (5.8). For example, let us consider the BEC regime at $T=0$, where the gap equation gives $\mu=-1/2ma_s^2$. In this limit, the number equation reduces to

$$N = N_{c0} + N_d, \quad (5.10)$$

where N_{c0} is given by Eq. (2.14) and [taking the $T\to0$ limit of Eq. (2.11)]

$$N_d = \frac{1}{2}\sum_{\mathbf{q}}\left(\frac{\varepsilon_{\mathbf{q}}^B + U_M N_{c0}}{E_{\mathbf{q}}^B} - 1\right) \simeq \frac{8}{3\sqrt{\pi}}(N_{c0}a_M)^{3/2} \quad (5.11)$$

gives the quantum depletion from the molecular condensate due to the effective interaction U_M between Cooper pairs. Recently, the mean-field result in Eq. (5.8) has been used to study the condensate fraction N_c in a superfluid Fermi gas at $T=0$ [11]. In this case, the gap and number equations in the mean-field approximation for the BCS-BEC crossover reduce to

$$1 = -\frac{4\pi a_s}{m}\sum_{\mathbf{p}}\left(\frac{1}{2E_{\mathbf{p}}} - \frac{1}{2\varepsilon_{\mathbf{p}}}\right), \quad (5.12)$$

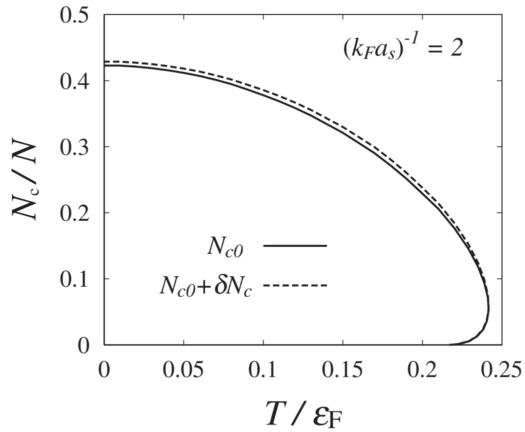

FIG. 6. Condensate fraction N_c in the strong-coupling BEC regime. The solid line shows N_{c0} and the dashed line includes the small correction δN_c from self-energies due to pairing fluctuations.

$$N = \sum_{\mathbf{p}} \left(1 - \frac{\xi_{\mathbf{p}}}{E_{\mathbf{p}}}\right). \tag{5.13}$$

In the BEC regime, while Eq. (5.12) again gives $\mu = -1/2ma_s^2$, Eq. (5.13) reduces to $N/2 = N_{c0}$ in Eq. (2.11). As expected, the depletion N_d at $T=0$ from the condensate due to the interaction between Cooper pairs is omitted when the BCS-BEC crossover is described by mean-field approximation [11].

Figure 7 shows the condensate fraction in the BCS-BEC crossover regime at $T=0$. Because of the omission of the quantum depletion effect, the simple mean-field (MF) result given in Ref. [11] is larger than our result (N_{c0}). In the region $(k_F a_s)^{-1} \gtrsim 1.5$, our results are well described by the condensate fraction for a superfluid molecular Bose gas given by Eqs. (5.10) and (5.11) (labeled BEC in Fig. 7).

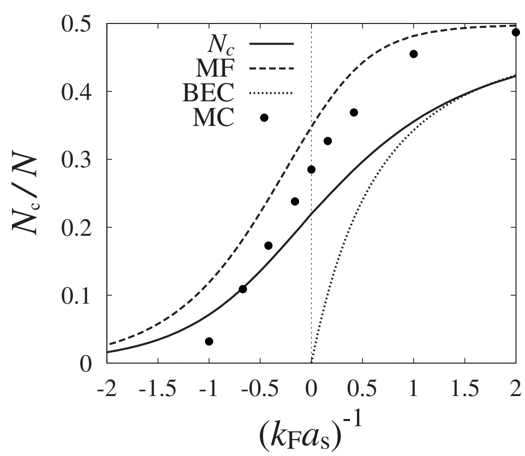

FIG. 7. Calculated condensate fraction N_c in the BCS-BEC crossover at $T=0$ (solid line). In this figure, as well as in Figs. 8 and 9, we only show N_{c0}. "MF" shows the condensate fractionin the case when Δ and μ are determined in the mean-field results in Eqs. (5.12) and (5.13). "BEC" shows the result for a Bose gas described by Eqs. (5.10) and (5.11). The solid circles shows recent Monte Carlo results for N_c [12] for comparison.

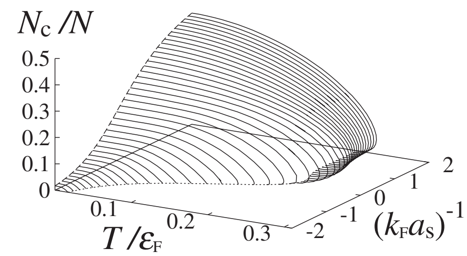

FIG. 8. Condensate fraction N_c as a function of temperature in the BCS-BEC crossover. In this calculation, the self-consistent solutions for Δ and μ in Fig. 1 are used.

Figure 7 also compares our $T=0$ results with those obtained by quantum MC simulations [12]. The latter calculation gives results consistent with $a_M = 0.6 a_s$ [27]. In contrast, our NSR theory gives the larger mean-field molecular scattering length $a_M = 2a_s$. As a result, we overestimate the magnitude of the depletion and thus our values for N_c are smaller than the MC calculation in the BCS-BEC crossover regime. The measurement of the depletion deep in the BEC regime would be a useful way of determining the magnitude of a_M.

Figure 8 shows the condensate fraction in the BCS-BEC crossover at finite temperatures. In the weak-coupling BCS regime, the condensate fraction N_c is very small even far below T_c, because only atoms very close to the Fermi surface form Cooper pairs which are Bose condensed. In this regime, Fig. 9 shows that the temperature dependence of N_c is very well described by the weak-coupling BCS result. In the crossover region, the temperature dependence of N_c deviates from the simple BCS result, as shown by the case $(k_F a_s)^{-1} = 0$ in Fig. 9. In the BEC limit, Fig. 8 shows that the condensate fraction at $T=0$ approaches $N/2$, reflecting the fact that all atoms form Cooper pairs which are Bose condensed. In this regime, Fig. 9 shows that the temperature dependence of N_c agrees with the condensate fraction for a dilute Bose gas in the Popov approximation given by Eq. (2.11). Figure 7 shows that the BEC picture is a very good approximation

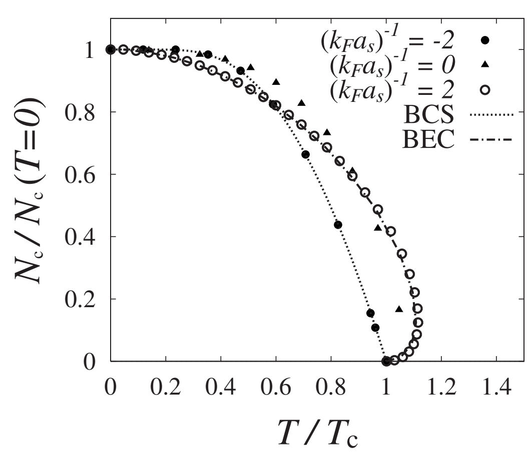

FIG. 9. Condensate fraction N_c as a function of temperature in the BCS regime, unitarity limit, and the BEC regime. The curve "BCS" is the weak-coupling BCS result. The curve labeled "BEC" is the condensate fraction for a Bose superfluid determined by Eq. (2.11). Results are normalized to values at $T=0$.

when $(k_F a_s)^{-1} \gtrsim 1.5$. The fact that N_c agrees with the Popov theory for a weakly interacting molecular Bose gas shows that the superfluid phase transition in this regime is dominated by the thermal depletion of Cooper pair condensate, and not by the dissociation of Cooper pairs characteristic of the weak-coupling BCS regime.

VI. CONCLUSIONS

In this paper, we have calculated the superfluid density ρ_s and condensate fraction N_c in the BCS-BEC crossover regime of a uniform superfluid Fermi gas at finite temperatures. We have included strong-coupling fluctuation effects on both ρ_s and N_c within a Gaussian approximation. The same fluctuation effects were also taken into account in calculating the superfluid order parameter Δ and Fermi chemical potential μ in the BCS-BEC crossover, within the NSR theory [2–4].

The expression we used to calculate ρ_s was derived from the single-particle Green's function in the presence of supercurrent as given by Eq. (3.1), which brings in self-energy corrections due to dynamic pair fluctuations. In this paper, we have concentrated on the explicit numerical calculation of ρ_s within a Gaussian approximation. In contrast, our companion paper [10] uses a different (but equivalent) formulation which exhibits the structure of ρ_s in a more direct fashion, in particular, the relation to collective modes.

Our result for the normal fluid density in Eq. (3.16) naturally separates into a mean-field part associated with fermions ρ_n^F and a bosonic pairing fluctuation contribution ρ_n^B. As discussed in Ref. [10], ρ_n^F is given by the Landau excitation formula in Eq. (3.4) in the whole BCS-BEC crossover. In the strong-coupling BEC regime, ρ_n^F is negligible and ρ_n^B reduces to the Landau formula for the normal fluid of a Bose gas of tightly bound Copper pairs [10]. However, in the region near unitarity, ρ_n^B is not expected to be given by a Landau-type formula because the bosonic pairing fluctuations are strongly damped. It is in this region that the numerical calculations for ρ_s reported in this paper are especially useful.

The superfluid density is a fundamental quantity in two-fluid hydrodynamics [32], and we will use our results in a future study of hydrodynamic modes in the BCS-BEC crossover regime of a Fermi superfluid at finite temperatures.

In contrast to the superfluid density, the mean-field expression for the condensate fraction N_c is a good approximation even in the strong-coupling BEC regime. The fluctuation contribution to N_c gives rise to the noncondensate component. In the BEC regime, we showed that the fluctuation contribution gives the condensate depletion N_d due to the effective interaction U_M between Cooper pairs, which is finite even at $T=0$.

In the BEC regime, the strong-coupling theory presented in this paper reduces to that of a weakly interacting Bose gas of molecules, with an excitation spectrum given by the Bogoliubov-Popov approximation. This is also the origin of the spurious first-order phase transition our theory exhibits (see Figs. 4 and 5). This is a well-known problem in dealing with dilute Bose gases [24]. The recovery of the second-order phase transition in the entire BCS-BEC crossover and the normalized magnitude of the effective interaction between Cooper pairs [27], both require the inclusion of higher-order fluctuations past the NSR Gaussian approximation which we have used. In this regard, we have emphasized that calculating the value of ρ_s is very dependent on using the strong-coupling approximation for Δ and μ as well, quantities which determine the single-particle excitation spectrum. Thus, when we calculate Δ and μ beyond the Gaussian fluctuation level, we also need to improve the microscopic model used to calculate ρ_s. The approach presented in this paper, as well as in Ref. [10], can be a starting point for such improved calculations.

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