Berry's phase for coherent states of Landau levels

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The Berry phases for coherent states and squeezed coherent states of Landau levels are calculated. Coherent states of Landau levels are interpreted as a result of a magnetic flux moved adiabatically from infinity to a finite place on the plane. The Abelian Berry phase for coherent states of Landau levels is an analog of the Aharonov-Bohm effect. Moreover, the non-Abelian Berry phase is calculated for the adiabatic evolution of the magnetic field B.

I. INTRODUCTION

Since the famous work of Berry [1], the geometric phase has been widely investigated and its generalizations have been made in many ways [2–5]. Recently much concern has been concentrated on the geometric phase of entangled states as well as mixed states [6,7]. The coherent state is an important physical concept both theoretically and experimentally [8,9]. It can be generated from an arbitrary reference state, and in this Brief Report Landau levels are chosen to be such reference states. Though coherent states of Landau level have been studied in Refs. [10,11] and Berry's phase for coherent states as well as squeezed coherent states of a one-dimensional harmonic oscillator have been illustrated in Ref. [12], Berry's phase for coherent states of Landau levels, which is highly degenerate and with an additional parameter, i.e., the magnetic field B, is still worthy of further investigation.

This paper is organized as follows. In Sec. II, we show how to get the coherent states of Landau levels, and these states can be regarded as a result of a magnetic flux moved adiabatically from infinity to a finite place on the plane. In Sec. III, we calculate the Abelian and non-Abelian Berry phases for coherent states of Landau levels. The Abelian Berry phase is just like an alternative version of the Aharonov-Bohm (AB) effect; the difference between them is that in our case the cyclic motion of the magnetic flux results in the phase shift. In Sec. IV, we provide the explicit form of the Hamiltonian and Berry's connections for squeezed coherent states of Landau levels. A conclusion and discussion are given in the last section.

II. COHERENT STATES OF LANDAU LEVELS

The motion of a free electron in the two-dimensional xy plane in a static magnetic field along the z direction is described by the following Hamiltonian:

$$H = \frac{\hbar^2}{2\mu}\left[\left(p_x + \frac{e}{c}A_x\right)^2 + \left(p_y + \frac{e}{c}A_y\right)^2\right], \quad (1)$$

where μ is the mass of the electron, $\hbar$ is the Planck constant, $-e$ is the electron charge, c is the speed of light in the

vacuum, p_x and p_y are the linear momenta, and A_x and A_y are vector potentials of the magnetic field satisfying $\partial_x A_y - \partial_y A_x = B$. For simplicity the Zeeman term is not included.

We introduce the following operators:

$$\pi_x = p_x + \frac{e}{c}A_x, \quad \pi_y = p_y + \frac{e}{c}A_y, \quad \pi_\pm = \pi_x \pm i\pi_y, \quad (2)$$

from which one can form a pair of operators $b^\dagger$ and b which satisfy the commutation relation $[b, b^\dagger] = 1$:

$$b = \sqrt{\frac{c}{2\hbar eB}}\pi_-, \quad b^\dagger = \sqrt{\frac{c}{2\hbar eB}}\pi_+. \quad (3)$$

In this case we can rewrite the Hamiltonian in a simpler form,

$$H_0 = \hbar\omega(b^\dagger b + 1/2), \quad (4)$$

where $\omega = eB/c\mu$, and $b^\dagger, b$ are raising and lowering operators between Landau levels with $b^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$, $b|n\rangle = \sqrt{n}|n-1\rangle$. The energy of the Landau levels is $E_n = \hbar\omega(n+1/2)$. One knows that Landau levels are highly degenerate. In the degenerate space of Landau levels we can introduce another pair of raising and lowering operators,

$$a = \sqrt{\frac{c}{2\hbar eB}}\left(-p_x - ip_y + \frac{e}{c}A_x + i\frac{e}{c}A_y\right),$$

$$a^\dagger = \sqrt{\frac{c}{2\hbar eB}}\left(-p_x + ip_y + \frac{e}{c}A_x - i\frac{e}{c}A_y\right).$$

We choose the symmetric gauge $A_x = -(y/2)B$ and $A_y = (x/2)B$, and then $a, a^\dagger$ are commutative with $b, b^\dagger$ and $[a, a^\dagger] = 1$. Therefore the Hamiltonian commutes with a and $a^\dagger$. The ground state is defined as $|0,0\rangle = \sqrt{Be/(2\pi c\hbar)}\times\exp[-Be(x^2+y^2)/(4c\hbar)]$, and all other eigenstates of this system can be generated from the $|0,0\rangle$ state with raising operators $|n,m\rangle = (1/\sqrt{n!m!})b^{\dagger n}a^{\dagger m}|0,0\rangle$. The states $|n,m\rangle$ are also orthogonal and normalized bases for this system. States with the same n are in the same energy level, and states with the same n but different m stand for the different degenerate states on the same energy level.

The coherent states of Landau levels are generated in the following way, as in [12]:

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$$|n(\alpha),m\rangle = \exp(\alpha b^\dagger - \alpha^* b)|n,m\rangle,$$

where $\alpha = X_1 + iX_2$. The Hamiltonian for the coherent states is

$$H = D(\alpha)H_0 D^\dagger(\alpha) = \hbar\,\omega\left((b^\dagger - \alpha^*)(b - \alpha) + \frac{1}{2}\right), \quad (5)$$

where $D(\alpha) = \exp(\alpha b^\dagger - \alpha^* b)$, and the eigenstates of this Hamiltonian are always combinations of the degenerate states with the same energy:

$$|n(\alpha)\rangle = \sum_m f_m |n(\alpha),m\rangle, \quad (6)$$

where f_m are arbitrary complex numbers which make $|n(\alpha)\rangle$ normalized. We put Eq. (3) back into Eq. (5); this makes the Hamiltonian easier to understand, and we get

$$H = \frac{\hbar^2}{2\mu}\left[\left(\pi_x - \sqrt{\frac{2\hbar eB}{c}}X_1\right)^2 + \left(\pi_y + \sqrt{\frac{2\hbar eB}{c}}X_2\right)^2\right]. \quad (7)$$

We found that the magnetic vector potential has an added constant vector potential. This can be regarded as a result of a magnetic flux perpendicular to the plane moving adiabatically from infinitely far to a finitely far position on the plane. In the following we would like to show how we get the result.

We can assume that the added magnetic flux is a Gaussian-form magnetic field centered at (x_0, y_0), $B' = (\Phi_0/\pi\Delta^2)\exp\{-[(x-x_0)^2 + (y-y_0)^2]/\Delta^2\}$, where Δ is referred to as the spread or standard deviation for the Gaussian function. We may choose the symmetric gauge with respect to (x_0, y_0), i.e., $\nabla \cdot \mathbf{A}|_{x=x_0, y=y_0} = 0$. The nonsingular vector potential for this added magnetic field is

$$A'_x = \frac{\Phi_0\left\{\exp\left[-\dfrac{(x-x_0)^2 + (y-y_0)^2}{\Delta^2}\right] - 1\right\}(y-y_0)}{2\pi[(x-x_0)^2 + (y-y_0)^2]},$$

$$A'_y = -\frac{\Phi_0\left\{\exp\left[-\dfrac{(x-x_0)^2 + (y-y_0)^2}{\Delta^2}\right] - 1\right\}(x-x_0)}{2\pi[(x-x_0)^2 + (y-y_0)^2]}. \quad (8)$$

One may observe that B' has nothing to do with the Hamiltonian (1) when $x_0, y_0 \to \infty$. Now we assume that the electron in the plane is in a certain eigenstate, for example, $|0,0\rangle$. In this case the electron is localized near the origin because $\langle x\rangle = \langle 0,0|x|0,0\rangle = \langle y\rangle = 0$, $\langle x^2+y^2\rangle = 2c\hbar/Be$. Let $\Delta > \langle\sqrt{x^2+y^2}\rangle$, so when the flux moves adiabatically to a place (x_0, y_0) which is finitely far from the electron (i.e., $x_0, y_0 \gg \langle\sqrt{x^2+y^2}\rangle \simeq \sqrt{2c\hbar/Be}$), we assume that the electron is still distributed around the origin and A'_x, A'_y near the origin of the plane can be regarded as constants. Then the Hamiltonian for the electron will be of the form (7) with

$$X_1 = \sqrt{\frac{e}{2\hbar Bc}}\frac{-\Phi_0 y_0\left[\exp\left(-\dfrac{x_0^2 + y_0^2}{\Delta^2}\right) - 1\right]}{2\pi(x_0^2 + y_0^2)},$$

$$X_2 = \sqrt{\frac{e}{2\hbar Bc}}\frac{\Phi_0 x_0\left[\exp\left(-\dfrac{x_0^2 + y_0^2}{\Delta^2}\right) - 1\right]}{2\pi(x_0^2 + y_0^2)}. \quad (9)$$

This modification of the Hamiltonian also corresponds to the transformation $(x,y) \to (x+\delta x, y+\delta y)$, where

$$\delta x = \frac{\Phi_0 x_0\left[\exp\left(-\dfrac{x_0^2 + y_0^2}{\Delta^2}\right) - 1\right]}{\pi B(x_0^2 + y_0^2)},$$

$$\delta y = \frac{\Phi_0 y_0\left[\exp\left(-\dfrac{x_0^2 + y_0^2}{\Delta^2}\right) - 1\right]}{\pi B(x_0^2 + y_0^2)}. \quad (10)$$

So the state $\psi_{00}(x,y) = |0,0\rangle$ will become $\psi_{00}(x+\delta x, y+\delta y)$. Since the distribution of the electron is near the origin, one also makes sure that $\delta x \ll x_0$, $\delta y \ll y_0$.

The coherent states of Landau levels are nothing but the shifted eigenstates of Landau levels in the phase space; here we assume such a shift happens in real space $\psi_{00}(x,y) \to \psi_{00}(x+\delta x, y+\delta y)$. When the conditions above are satisfied, this assumption is reasonable. It is B' that causes this small shift. We may see from Eq. (10) that the direction of B', i.e., the sign of Φ_0, is related to the direction of the shift. When $\Phi_0 > 0$ the shift is parallel to the direction of the flux, and vice versa. Interestingly, if the flux circles the electron once, the electron will also circle the origin in a much smaller loop once. This is the reason why the Berry phase emerged as we will show in the next section.

III. BERRY'S PHASE FOR COHERENT STATES OF LANDAU LEVELS

We know that the Landau levels are highly degenerate, and so are the coherent states of Landau levels. Berry's phase for degenerate states was presented in [3] and may have a non-Abelian nature. We calculated the Berry connections as follows:

$$\langle n(\alpha),m|\partial_{X_1}|n'(\alpha),m'\rangle = (-iX_2)\delta_{n,n'}\delta_{m,m'} + (\sqrt{n'+1}\,\delta_{n,n'+1} - \sqrt{n'}\,\delta_{n,n'-1})\delta_{m,m'},$$

$$\langle n(\alpha),m|\partial_{X_2}|n'(\alpha),m'\rangle$$
$$= iX_1\delta_{n,n'}\delta_{m,m'} + (i\sqrt{n'+1}\,\delta_{n,n'_1} + i\sqrt{n'}\,\delta_{n,n'-1})\delta_{m,m'},$$

$$\langle n(\alpha),m|\partial_B|n'(\alpha),m'\rangle$$
$$= \frac{1}{2B}(\alpha\sqrt{m'}\,\delta_{m,m'-1} - \alpha^*\sqrt{m'+1}\,\delta_{m,m'+1})\delta_{n,n'}$$
$$+ \frac{1}{2B}\sqrt{n'm'}\,\delta_{n,n'-1}\delta_{m,m'-1} - \frac{1}{2B}\sqrt{(n'+1)(m'+1)}$$
$$\times \delta_{n,n'+1}\delta_{m,m'+1}. \quad (11)$$

In the degenerate space, i.e., between states with the same n, the Berry connections become

$$A_{X_1}^{m,m'} = -iX_2\delta_{m,m'}, \quad A_{X_2}^{m,m'} = iX_1\delta_{m,m'},$$

$$A_B^{m,m'} = \frac{1}{2B}(\alpha\sqrt{m'}\,\delta_{m,m'-1} - \alpha^*\sqrt{m'+1}\,\delta_{m,m'+1}). \quad (12)$$

We found that A_{X_1} and A_{X_2} are Abelian. With the adiabatic theorem for degenerate states proved in [13] and to a higher order in [14], we know that the $|f_m|$ in Eq. (6) will not change during the arbitrarily slow evolution of X_1 and X_2. Also, because the Berry's connections of X_1 and X_2 are Abelian, the f_m will gain a Berry phase factor, which is the same for all m, after an adiabatic evolution in the X_1-X_2 plane. The Berry phase is

$$\gamma_n = i\oint_C \langle n(\alpha)|\partial_{X_1}|n(\alpha)\rangle dX_1 + \langle n(\alpha)|\partial_{X_2}|n(\alpha)\rangle dX_2$$

$$= \oint_C X_2 dX_1 - X_1 dX_2 = -2S, \quad (13)$$

where C is the path of the adiabatic evolution of (X_1,X_2) in the X_1-X_2 plane and S is the area of C. This result appeared in [12] for nondegenerate coherent states. The result is also the same as the phase in [15]. However, in our case it is the moving magnetic flux that moves the electron instead of a moving potential well.

With the interpretation in the above section, we can see from Eq. (9) that, when the magnetic flux circles the electron for one loop, the (X_1,X_2) will also enclose an area, and this gives the Berry phase. For example, we let (x_0,y_0) move around the origin in a circle with the radius R for one loop. The Berry phase will be

$$\gamma' = -2S' = -\frac{e\Phi_0^2(1-e^{-R^2/\Delta^2})^2}{4\pi\hbar\,cBR^2}. \quad (14)$$

This can be viewed as an alternative version of the AB effect. The different between them is that we move the magnetic flux instead of the electron.

One may see from Eq. (11) that $A_B^{m,m'}$ is non-Abelian, so the change of B will give a non-Abelian Berry phase. As the non-Abelian Berry phase in such a system has not been shown in the literature before, in the following we would like to give a simple example to illustrate it. We assume $X_1=0$ during the evolution and the other two parameters undergo the loop in Fig. 1. We can get the eigenvalues of matrix A_B as $\varepsilon_\xi/(2B)$ and the corresponding eigenstates $|n(\alpha),\xi\rangle$. The states before evolution, Eq. (6), can be rewritten in the new base as

$$|n(iX_{21},t=0)\rangle = \sum_\xi \langle n(iX_{21}),\xi|n(iX_{21})\rangle|n(iX_{21}),\xi\rangle. \quad (15)$$

After the system undergoes an evolution as shown in Fig. 1, the state may become

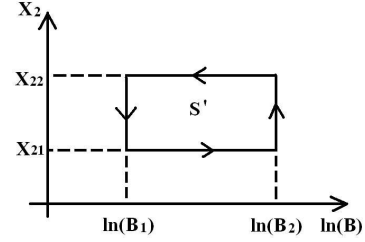FIG. 1. Loop of the adiabatic evolution in the X_2-$\ln(B)$ plane. The magnetic field B appears in Berry's phase in the logarithmic form.

$$|n(iX_{21},t=\tau)\rangle = \sum_\xi e^{-i\varepsilon_\xi S'}\langle n(iX_{21}),\xi|n(iX_{21})\rangle|n(iX_{21}),\xi\rangle,$$

$$(16)$$

where S' is the area enclosed by X_2 and $\ln(B)$ as in Fig. 1. However, if $X_1\neq 0$, the calculation will involve a path-ordered integral and become very complicated.

IV. BERRY'S PHASE FOR SQUEEZED COHERENT STATES OF LANDAU LEVELS

The Hamiltonian for squeezed coherent state is

$$H = D(\alpha)S(\beta)H_0 S^\dagger(\beta)D^\dagger(\alpha), \quad (17)$$

where H_0 is defined in Eq. (4), $\beta = re^{i\theta}$, and

$$S(\beta) = \exp\left(\frac{1}{2}\beta b^{+2} - \frac{1}{2}\beta^* b^2\right). \quad (18)$$

The eigenstates for this Hamiltonian, i.e., the squeezed coherent states, are

$$|n(\alpha,\beta),m\rangle = D(\alpha)S(\beta)|n,m\rangle. \quad (19)$$

In the same way, we put Eq. (3) into Eq. (17), and we can get

$$H = \frac{1}{2\mu}\{e^{-2r}[\cos(\theta/2)\pi_x' + \sin(\theta/2)\pi_y']^2$$

$$+ e^{2r}[-\sin(\theta/2)\pi_x' + \cos(\theta/2)\pi_y']^2\}, \quad (20)$$

where $\pi_x' = \pi_x - \sqrt{2\hbar eB/c}X_1$, $\pi_y' = \pi_y + \sqrt{2\hbar eB/c}X_2$. For $r=0$, Eq. (20) reduces to Eq. (7). For $r\neq 0$, one can see from this Hamiltonian that the squeezing operation $S(\beta)$ caused an anisotropy in the plane. More clearly, if we set $\theta=0$ Eq. (20) will become $H = (e^{-2r}\pi_x'^2 + e^{2r}\pi_y'^2)/2\mu$; in other words, the kinetic energies $\pi_x'^2/2\mu$ and $\pi_y'^2/2\mu$ are squeezed by the factors e^{-2r} and e^{2r}, respectively.

Now we consider the Berry connections of squeezed coherent states:

$$\left\langle n(\alpha,\beta),m \left| \frac{\partial}{\partial r} \right| n'(\alpha,\beta),m' \right\rangle$$

$$= \left(-\frac{1}{2}e^{-i\theta}\sqrt{n'(n'-1)}\,\delta_{n,n'-2} \right.$$

$$\left. + \frac{1}{2}e^{i\theta}\sqrt{(n'+1)(n'+2)}\,\delta_{n,n'+2} \right)\delta_{m,m'},$$

$$\left\langle n(\alpha,\beta),m \left| \frac{\partial}{\partial \theta} \right| n'(\alpha,\beta),m' \right\rangle$$

$$= \frac{i \sinh(2r)}{4} \left[e^{-i\theta}\sqrt{n'(n'-1)}\,\delta_{n,n'-2} \right.$$

$$+ \left. e^{i\theta}\sqrt{(n'+1)(n'+2)}\,\delta_{n,n'+2} \right]\delta_{m,m'}$$

$$+ \frac{i \sinh^2 r}{2}(2n'+1)\,\delta_{n,n'}\,\delta_{m,m'},$$

$$\left\langle n(\alpha,\beta),m \left| \frac{\partial}{\partial B} \right| n'(\alpha,\beta),m' \right\rangle$$

$$= \frac{1}{2B}\left(2\sinh^2\frac{r}{2}\sqrt{n'm'}\,\delta_{n,n'-1}\delta_{m,m'-1} \right.$$

$$- 2\sinh^2\frac{r}{2}\sqrt{(n'+1)(m'+1)}\,\delta_{m,m'+1}\delta_{n,n'+1}$$

$$+ e^{i\theta}\sinh r\sqrt{m'(n'+1)}\,\delta_{n,n'+1}\delta_{m,m'-1}$$

$$- \left. e^{-i\theta}\sinh r\sqrt{(m'+1)n'}\,\delta_{m,m'+1}\delta_{n,n'-1} \right)$$

$$+ \frac{1}{2B}(\alpha\sqrt{m'}\,\delta_{m,m'-1} - \alpha^*\sqrt{m'+1}\,\delta_{m,m'+1})\,\delta_{n,n'}. \tag{21}$$

To our knowledge, the Berry connection with respect to B has not appeared in the literature before. With these results, the Berry phase is not hard to obtain. If anisotropy existed in two-dimensional electron gas systems, its Hamiltonian would be of the form of Eq. (20).

V. CONCLUSION AND DISCUSSION

In this Brief Report, we have calculated the Berry's phase for coherent states as well as squeezed coherent states of Landau levels. The Hamiltonian of the coherent states of Landau levels is interpreted as a result of a magnetic flux perpendicular to the plane, and it is moved adiabatically from infinity to a distance away from the electron so that some approximations are satisfied. The cyclic adiabatic motion of this magnetic flux caused the Berry phase of coherent states of Landau levels. This is an analog of the AB effect. The non-Abelian phase is also of interest; the magnetic field B appears in the Berry phase in the form of $\ln(B)$. So when $B \to 0$ the Berry phase will be very sensitive to B and become indefinite, and the reversion of the magnet field is prohibited if we want to get this phase. Reference [16] also states this phenomenon that near the level crossing point the Berry phase sometimes vanishes.

ACKNOWLEDGMENTS

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