Time-dependent hyperspherical studies for a two-dimensional attractive Bose-Einstein condensate

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We apply the hyperspherical (HS) method to study a Bose-Einstein condensate in quasi-two-dimensional free space stabilized and confined under the influence of an oscillating magnetic field. The HS method indeed reproduces stabilized breathing mode solutions qualitatively similar to those previously obtained by the Gross-Pitaevskii (GP) equation. Also, the frequencies of our breathing mode solutions are shown to have functional dependence on the physical parameters in a manner similar to the GP results. However, beats in the breathing mode solutions are revealed in the HS approximation, while they are seemingly absent in the GP descriptions. A supplementary analysis of the stationary state solutions shows that the hyperspherical single-particle density exhibits certain characteristic scaling dependence on energy akin to the Townes soliton [Chiao *et al.*, Phys. Rev. Lett. **13**, 479 (1964)], but also some difference in detail. The Kapitza averaging leads to an effective time-independent potential and shows how continuously distributed hyperspherical "bound" states turn into discrete "bound" states on accounting of the modulating field. The HS method is made subject to the Floquet analysis in order to interpret the beats in the breathing mode as coherent excitation among discrete Floquet states.

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I. INTRODUCTION

The successful generation of trapped atomic Bose-Einstein condensates (BECs) has opened a new domain for atomic physics [1]. Exciting new phenomena in various different directions continue to be explored [2]. The formation and manipulation of BECs still pose considerable challenge, while there are growing interests particularly in control and manipulation for practical purposes and possible applications [3,4]. A remarkable feature of the atomic BEC is that both the strength and sign of interaction between atoms can be tuned by using an external magnetic field near a Feshbach resonance [5,6]. This opens up possibilities of creating and controlling the matter-wave soliton by modulation of the BEC's internal force despite the absence of a trap potential. Indeed, several theoretical predictions have demonstrated that an attractive matter-wave bright soliton can be stabilized in quasi-two-dimensional (2D) free space by making the magnetic field strength oscillate in time so that the scattering length would modulate around a negative value [8–10,12]. A pancake-shaped BEC may be considered as quasi-2D. We refer to this system simply as 2D from here on. Under such circumstances, the scattering length can take the form of an oscillating function, $a(t)=a_0+a_1\cos(\Omega t)$, where a_0 is the negative value, a_1 is the amplitude of the oscillation, and Ω is the oscillating frequency.

Abdullaev *et al.* [9] were among the first to demonstrate theoretically the existence of trapless self-confined 2D condensates. They use the Gross-Pitaevskii (GP) equation [7] for numerical simulations and apply two different approximations to the GP equation for interpretation, namely the variational approximation and the direct averaging of the GP equation, obtaining qualitatively similar results for this problem by both methods. A breathing mode solution emerges from a delicate balance achieved between self-focusing and self-expansion due to the manipulated atom-atom interaction.

Saito and Ueda [8] demonstrated that a matter-wave bright soliton can be stabilized in 2D free space, provided that the amplitude of the oscillating scattering length is larger than the magnitude of the negative constant part, $a_1>|a_0|$, and $a_0<a_{cr}$, where a_{cr} is the critical value for the existence of an attractive condensate in a trap. Based on a variational approximation and a Gaussian ansatz for the wave functions, they derive a formula for the frequency of the breathing mode, showing its dependence on the magnitude and frequency of the oscillating part of the scattering length. Their GP results exhibit qualitative agreement with the variational approximation. They also draw some physical analogy to the inverted pendulum subject to a rapidly oscillating force though the analogy may not be quite complete [12].

Montesinos *et al.* [10] provided a rigorous analysis based on the moment method for the stabilization of a 2D condensate and obtained necessary conditions. In addition, they studied the dependence of the stability on initial conditions. Their numerical simulations indicated that the Townes solitons [11], the dynamically unstable ground state of a static nonlinear Schrödinger (NLS) equation, can be stabilized. In contrast, if the Gaussian wave packet is used for the initial condition, it becomes stabilized by adapting to the form of the Townes soliton. While the necessary conditions for the stabilization were established for the 2D condensate, various studies resulted in disagreement regarding the stabilization of a 3D condensate [8–10,12,13]. It is nevertheless generally agreed that a system with a strong trapping in one direction is effectively two-dimensional, thus as a special case, it can be stabilized.

Most theoretical studies on BEC are based on the Gross-Pitaevskii (GP) equation [7], which is a lowest-order mean-field approximation. While the GP equation is widely used to study mean-field properties of BECs and successfully offers fairly good qualitative descriptions for a variety of experiments, an alternative picture based on the adiabatic hyperspherical (HS) method has been applied in studying some aspects of BEC. Bohn *et al.* [14] was the first to apply this approach to study the ground-state properties of a BEC and evaluated the maximum number of atoms that an attractive condensate can accommodate. Sørensen *et al.* [16] applied an adiabatic hyperspherical approach and assumed a Feddeev-like decomposition for the wave function to study correlations in N-boson systems with arbitrary scattering lengths. They later extended their method to study the stability of two-dimensional boson systems [17] and two coupled boson systems in a trap [18]. Kushibe *et al.* [19] extended the formulation to accommodate an anisotropic trapping potential and studied the collective excitation of a BEC using the same zeroth-order hyperspherical harmonic approximation as in Ref. [14]. In those studies, hyperspherical approach has reproduced, semiquantitatively, the main features of the zero-temperature BEC. Nevertheless, there are still subtle differences in the dynamics predicted by the mean-field GP equation and by the adiabatic hyperspherical method. By choosing the hyperspherical coordinates and an appropriate variational trial wave function, the many-atom problem is reduced to a *linear* Schrödinger equation, which is easier to handle than the usual nonlinear GP equation. The spirit of the hyperspherical method as applied at the moment is along the line of the direct diagonalization method, in the sense that if the basis function could cover a sufficiently large portion of the relevant Hilbert space, the method would fully represent the quantum dynamics including fluctuations from the mean field. Blume and Greene [15] applied the HS method to three identical bosons and analyzed how the BEC-like metastable states emerge among many adiabatic states. In the HS approximation in this paper, however, the tight constraint imposed on the channel function does not allow for the fully faithful representation of quantum fluctuations, but the lowest order excitation seems reliable, besides the equation of motion is quantum mechanical. The time-dependent GP treatment is contrasting in that after all the quantum fluctuations are reduced, the collective oscillations are extracted from the time-evolution of the driven system. The Gaussian ansatz [8], leading to an equation of motion for the breadth of the wave packet, is thus capable of representing the collective modes in a classical mechanical sense.

Thus, in this paper, we analyze the stabilization of a BEC initially confined in a quasi-2D axial symmetric trap by solving the linear Schrödinger equation under the framework of the hyperspherical adiabatic method. Direct numerical simulations of the GP equations are also carried out for the purpose of comparison. In addition, due to the linearity of the Schrödinger equation, Floquet analysis can be applied. The frequency beating in the breather solutions is observed. Its implications will be discussed in the concluding section. We also examine the relationship between the hyperspherical and the Townes soliton, namely the stationary solution of the 2D Schrödinger equation with a cubic nonlinearity.

II. THEORY

The hyperspherical method has proven useful in previous applications [14,19] to study the ground state properties of an attractive condensate using the single hyperspherical ground-state channel approximation. We shall refer to this particular approximation as the K-harmonic approximation (KHA) and distinguish it from the adiabatic hyperspherical close coupling (HSCC) method designed for few-body systems [20]. This KHA method has been described in detail in Ref. [19]. Thus we present here only an overview of the KHA method in the first half of this theory section. We will discuss the representation of the problem in the framework of the GP equation next.

A. Time-dependent KHA method

The full Hamiltonian of N identical bosons of mass m confined in a harmonic oscillator potential of a magneto-optical trap reads

$$H = T + V_{\text{trap}} + V_{\text{int}},$$

where

$$T = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\nabla_i^2$$

and the trapping potential V_{trap} is, owing to its harmonicity,

$$V_{\text{trap}} = \frac{1}{2}m\sum_{i=1}^{N}(\omega_x^2x_i^2 + \omega_y^2y_i^2 + \omega_z^2z_i^2),$$

where the origin of the coordinate system is at the minimum of the trap potential. Here, ω_x, ω_y, and ω_z are the trap frequencies in x, y, and z directions, respectively.

The interaction of pairs of atoms is represented by the short-range δ-function potential parametrized by the s-wave scattering length a,

$$V_{\text{int}} = \frac{4\pi\hbar^2}{m}a\sum_{i>j}\delta(\vec{r}_i - \vec{r}_j).$$

For the elliptically anisotropic case of $\omega_x=\omega_y=\omega_\rho\neq\omega_z$, the trapping potential reads

$$V_{\text{trap}} = \frac{1}{2}m\sum_{i=1}^{N}(\omega_\rho^2\rho_i^2 + \omega_z^2z_i^2) = \frac{1}{2}mN(\omega_\rho^2R_\rho^2 + \omega_z^2R_z^2),$$

where $\rho_i^2=x_i^2+y_i^2$, and the hyperradii are defined by

$$R_\rho^2 = \frac{1}{N}\sum_{i=1}^{N}\rho_i^2,$$

$$R_z^2 = \frac{1}{N}\sum_{i=1}^{N}z_i^2.$$

The kinetic energy operator reads

$$T = -\frac{\hbar^2}{2mN}\left(\frac{1}{R_\rho^{2N-1}}\frac{\partial}{\partial R_\rho}R_\rho^{2N-1}\frac{\partial}{\partial R_\rho} - \frac{\Lambda_\rho^2}{R_\rho^2} + \frac{1}{R_z^{N-1}}\frac{\partial}{\partial R_z}R_z^{N-1}\frac{\partial}{\partial R_z} - \frac{\Lambda_z^2}{R_z^2}\right),$$

where Λ_ρ and Λ_z are the grand angular momentum operators associated with the rotational degrees of freedom on the R_ρ and R_z hyperspheres, respectively. We consider the hyperspherical harmonics, namely the eigenvectors of the grand angular momentum operator as a basis set for expansion. The solution $\Psi(R_\rho,R_z;\Omega_\rho,\Omega_z)$ of the Schrödinger equation may be expressed in terms of the sum of direct products

$$\Psi(R_\rho,R_z;\Omega_\rho,\Omega_z) = N^{-3N/4}R_\rho^{-(2N-1)/2}R_z^{-(N-1)/2}\sum_{\{\lambda_\rho\},\{\lambda_z\}}\mathcal{F}_{\{\lambda_\rho\},\{\lambda_z\}} \times(R_\rho,R_z)\mathcal{Y}^{\rho}_{\{\lambda_\rho\}}(\Omega_\rho)\mathcal{Y}^{z}_{\{\lambda_z\}}(\Omega_z),$$

where $\{\lambda_\rho\}$ and $\{\lambda_z\}$ represent the quantum numbers symbolically, and $\mathcal{Y}$ refers to the hyperspherical harmonics.

In the KHA approximation, we retain only the hyperspherical harmonic with zero grand angular momentum. As a result, the hyperradial equation is then expressed as

$$\left[-\frac{\hbar^2}{2mN}\left\{\frac{\partial^2}{\partial R_\rho^2}+\frac{\partial^2}{\partial R_z^2}\right\}+V_{\text{eff}}(R_\rho,R_z)\right]\mathcal{F}(R_\rho,R_z) = i\frac{\partial}{\partial t}\mathcal{F}(R_\rho,R_z),$$

where $\mathcal{F}(R_\rho,R_z)$ refers to $\mathcal{F}_{\{\lambda_\rho=0\},\{\lambda_z=0\}}(R_\rho,R_z)$ and the effective hyperspherical potential energy $V_{\text{eff}}(R_\rho,R_z)$ is the sum of the pseudocentrifugal potential, V_{trap} and V_{int}, namely,

$$V_{\text{eff}}(R_\rho,R_z) = \frac{\hbar^2}{8mN}\left\{\frac{(2N-1)(2N-3)}{R_\rho^2}+\frac{(N-1)(N-3)}{R_z^2}\right\} + \frac{1}{2}mN(\omega_\rho^2R_\rho^2+\omega_z^2R_z^2)+\frac{G_0^{\rho z}}{R_\rho^2R_z}.$$

Note that the interaction constant $G_0^{\rho z}$ can be reduced to (cf. Appendix in [19])

$$G_0^{\rho z} = \frac{g}{(2\pi)^{3/2}}\frac{N(N-1)}{2}\left(\frac{\Gamma(N)}{N\Gamma(N-1)}\frac{\Gamma\left(\frac{N}{2}\right)}{N^{1/2}\Gamma\left(\frac{N-1}{2}\right)}\right),$$

where $g=4\pi\hbar^2a/m$.

We consider in this paper a BEC initially confined in a quasi-2D axially symmetric trap, assuming that the condensate wave function is always in the ground state of the harmonic potential with respect to the z axis. Therefore the equation for the radial wave function $F(r)$ can be reduced to the following form:

$$\left[-\frac{1}{2}\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\left(\frac{(2N-1)(2N-3)}{8}+G(N-1)^2\right) + \frac{1}{2}\omega_\perp^2(\tau)r^2\right]F(r) = i\frac{\partial}{\partial\tau}F(r), \tag{1}$$

where

$$G = \frac{1}{4\pi}\left(\frac{8\pi m\omega_z}{\hbar}\right)^{1/2}Na, \quad r = R_\rho\bigg/\left(\frac{\hbar}{Nm\omega_\rho}\right)^{1/2},$$

and $\tau=t/\omega_\rho^{-1}.\omega_\perp^2$ should be a unity according to the usual convention, but is introduced here as a function of time to describe the change in trapping frequency (see below). It is important to note that KHA reduces a Schrödinger equation of N identical bosons to a simple, one-dimensional problem with an effective static potential. Also, there are two independent parameters, N and G, in contrast to only one parameter characterizing the atomic interaction in the GP equation. Under an oscillating magnetic field, the scattering length and the interaction parameter G undergo sinusoidal oscillation about their respective field-free values.

Additional terms can be introduced in an *ad hoc* manner into this equation to account for reaction processes that cause atom losses [21]. We also carried out calculations including loss terms. However, since the wave function is prevented from collapse at the center of the trap in the stabilized cases, the loss mechanism is not important. Therefore it is justified to neglect such terms in our subsequent discussions. The same comment applies to the GP equation.

B. Representation of the ramping and modulation of the interaction

Let us turn to the representation of the modulated coupling strength, keeping the connection between the hyperspherical and the GP equation in mind. As previously shown by Saito and Ueda [8], the corresponding GP equation can be written as

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\nabla^2\psi+\frac{\omega_\perp^2(t)}{2}r^2\psi+g(t)|\psi|^2\psi, \tag{2}$$

where $r\equiv\sqrt{x^2+y^2}$ and $g(t)=(8\pi m\omega_z/\hbar)^{1/2}Na(t)$. So the value of G in Eq. (1) multiplied by 4π is exactly equal to the value of g used in the GP equation. Applying the variational principle with the Gaussian ansatz, Saito and Ueda derive the frequency of the breathing mode with a small oscillation as

$$\omega_{br}^2 = \frac{8\Omega^2}{3g_1^2}(g_0+2\pi)^2,$$

where g_0 is the constant part of $g(t)$ and g_1 is the amplitude of the oscillating part of $g(t)$. The above formula in turn gives the critical value for the constant part of $g(t)$, $g_{cr}=-2\pi$, for stabilization.

We assume the same ramping scheme for the trapping potential and the attractive interaction term as in [8].

$$G(\tau) = f(\tau)(G_0+G_1\sin\Omega\tau),$$

$$\omega_\perp^2 = 1-f(\tau),$$

$$f(\tau) = \begin{cases}\tau/\Gamma & (0\leq\tau\leq\Gamma)\\ 1 & (\tau>\Gamma).\end{cases}$$

Parameters include G_0, the constant part of the coefficient for the interaction term; G_1, the amplitudes of the oscillating part of the coefficient for the interaction term; Ω, the oscillation frequency, and the ramping time Γ. G_0 is chosen to be smaller than -0.5, corresponding to the condition $g_0<g_{cr}$, while the critical value for the nonlinearity in the GP equation is -5.85. Also, G_1 is chosen to be larger than $|G_0|$, thus

satisfying the necessary condition for stabilization. The strength of interaction is gradually increased and the trapping potential is gradually turned off according to the linear ramp. Using the ground state of the confinement potential as the initial state, we numerically solve the Schrödinger equation, Eq. (1), using the Crank-Nicholson scheme with finite difference grids.

C. Floquet states

The Hamiltonian becomes periodic in time after the trapping potential is ramped off. The linear radial Schrödinger equation of the hyperspherical method [cf. Eq. (1)] for $\tau > \Gamma$ becomes

$$\left[-\frac{1}{2}\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\left(\frac{(2N-1)(2N-3)}{8}+G(N-1)^2\right)\right]F(r) = i\frac{\partial}{\partial \tau}F(r). \quad (3)$$

Taking advantage of the linearity of the Schrödinger equation under KHA description and the periodicity of the Hamiltonian without trapping potential, one can apply the Floquet analysis [22] to shed light on the frequency components of the hyperspherical breathing mode solutions. Floquet states $|\phi_n^F\rangle$ of the system are here defined through the eigenvalue equation,

$$U(T,0)|\phi_n^F\rangle = e^{-iE_nT}|\phi_n^F\rangle,$$

where $U(T,0)$ is the period-one evolution operator, T is the period equal to $2\pi/\Omega$, and E_n is the quasieigenenergies of the Floquet state. The eigenstate $|\phi_n^F\rangle$ is obtained by diagonalizing the time-evolution operator $U(T,0)$ expanded by the finite difference basis. The wave function at the end of ramping can thus be expressed as a linear combination of the Floquet eigenstates. Therefore the time evolution of the wave function for $\tau > \Gamma$ is regular in our hyperspherical approach.

III. RESULTS AND DISCUSSIONS

A. Stationary states

Let us begin with the static situation, that is without an external driving field. Reference [11], an illustrious paper on self-trapped optical beams, points out that the 2D nonlinear Schrödinger equation

$$\nabla^2 R_\mu - 2\mu R_\mu - 2gR_\mu^3 = 0,$$

subject to

$$\lim_{r\to\infty} R_\mu(r) = 0, R'_\mu(0) = 0,$$

where μ is the chemical potential, permits a stationary solution localized in a finite range of space, namely the so-called "Townes soliton," in which dispersion and contraction balance out [10]. This holds for any value of $g<0$, although it does not necessarily mean that the Townes soliton is stable against a perturbation. An interesting property of the Townes soliton is its scaling via the following transformation which generates solutions of arbitrary widths:

$$r^* = \sqrt{2\mu}r,$$

such that

$$R^*(r^*) = \sqrt{\frac{g}{2\mu}}R_\mu(r).$$

Meanwhile, in the absence of a trapping potential, the hyperspherical effective potential is given simply by [cf. Eq. (1)]

$$V_{\text{eff}}(r) = \frac{1}{r^2}\left(\frac{(2N-1)(2N-3)}{8}+4\pi g(N-1)^2\right) = \frac{\tilde{g}}{2r^2}, \quad (4)$$

so that the stationary state satisfies

$$\left(-\frac{1}{2}\frac{\partial^2}{\partial r^2}+\frac{\tilde{g}}{2r^2}\right)F_E(r) = EF_E(r). \quad (5)$$

This is equivalent to the differential equation defining the Bessel function of an imaginary order. Under the transformation,

$$r^* = \sqrt{2E}r,$$

we find the scaling

$$F_1^*(r^*) = \sqrt{\frac{1}{2E}}F_E(r).$$

For $\tilde{g}<0$, there thus exists a bound solution at any given negative energy, namely the bound state spectrum of this particular system is continuous. (See [23], for example, for a mathematical scaling argument to this effect.) Let us convert this solution into the single-particle density, using Eq. (6) of Ref. [19],

$$n_j(r) = \frac{1}{\pi}\int_{r/\sqrt{N}}^{\infty}\frac{|F_j(\rho)|^2}{\rho^2}\left(1-\frac{r^2}{N\rho^2}\right)^{N-2}d\rho. \quad (6)$$

It then follows that

$$n(r^*) = \sqrt{\frac{1}{2E}}n_E(r). \quad (7)$$

For handling this problem numerically, introduce here an inner cutoff point R_{cut} (0.01 oscillator unit for specific numerical implementations) such that $F_j(r)\equiv 0$ at $r<R_{\text{cut}}$. Figure 1 shows the one-particle density so calculated for $N=100$ and $g=-2.2\pi$. Our result indicates that the KHA single-particle density enjoys a scaling law similar to that of the Townes soliton except for the absence of its manifest dependence on the coupling strength g. The KHA single-particle density differs from the Townes soliton particularly near $r^*=0$. For the hyperspherical, this type of saturation appearing for low values of j results from the fact that the corresponding wave functions are more susceptible to the imposed cutoff radius R_{cut}. This observation suggests that the unsaturated behavior of $n(r^*)$ (KHA) stems from the finiteness of the lowest-order

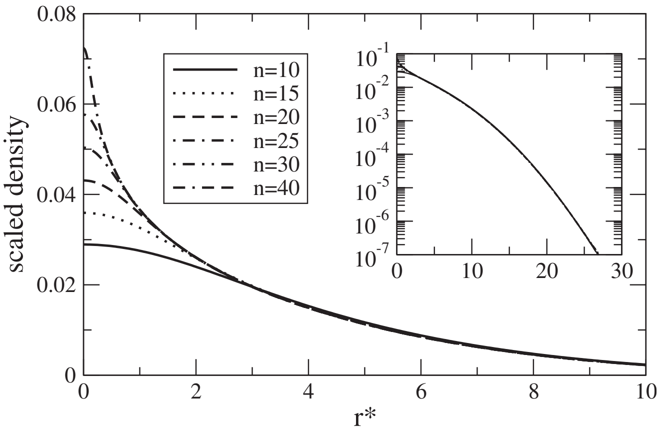

FIG. 1. Illustration of the scaled hyperspherical density $\mu^{-1}n_j(\rho^*)$. Here index j pertains to the enumeration of the hyperspherical eigenvectors obtained with cutoff $R_{\text{cut}}=0.01$(o.u.). The envelope curve would correspond to the limit of $R_{\text{cut}}\to 0$. The lower the eigenenergy E_j is, the more marked departure from the envelope is observed, and the shape is closer to that of the Townes soliton. This suggests that the more improved description would require the hyperspherical channel to represent the vanishingly small amplitude at $r\simeq 0$ for each particle as does the GP wave function.

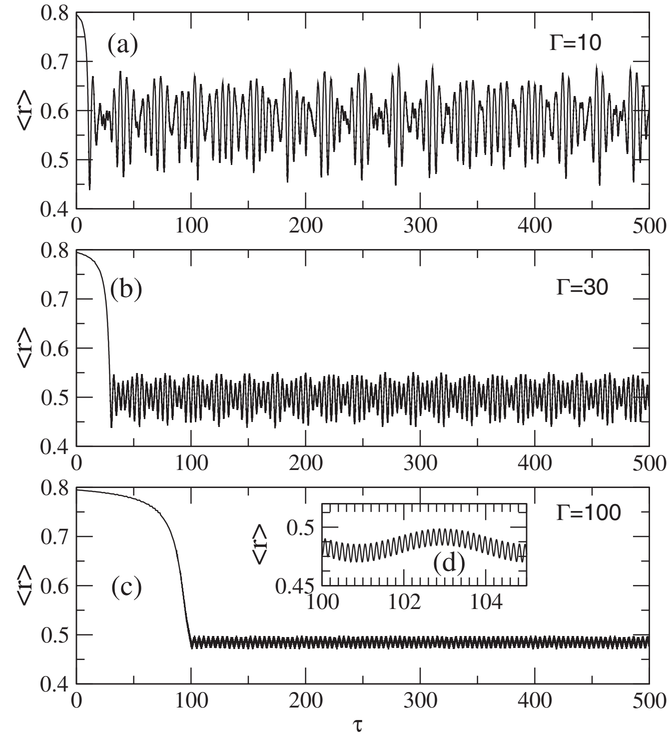

FIG. 2. Demonstration for the beating in the time evolution of the monopole moment $\langle r\rangle=(1/4)\pi\int r|F(r)|^2dr$. In all cases, $g_0=-2.2\pi$, $g_1=8\pi$, and $\Omega=45$. (a) $\Gamma=10$. (b) $\Gamma=30$. (c) $\Gamma=100$.

hyperspherical harmonic at configurations where $r=0$ for every particle. Let us also note that, according to Ref. [24], the GP solution acquires an additional node every time g exceeds a certain threshold value, transcending to a different class of solutions. This feature is certainly absent from the KHA because the KHA constrains the channel function to be completely nodeless on the entire hypersphere.

B. Driven BEC

Now let us move on to examine the response and stability of the BEC under the time-dependent driving field using Eq. (1). Following Ref. [8], Fig. 2 presents some typical results for the time evolution of the monopole moment $\langle r\rangle=(4\pi)^{-1}\int r|F|^2dr$. Parameters used here are $g_0=-2.2\pi$, $g_1=8\pi$, and $\Omega=45$, while the ramp times are varied in the three cases. This figure corroborates the dynamical stabilization of a matter-wave soliton in two-dimensional free space. It also shows that the dynamics under the hyperspherical approximation is qualitatively similar to the results obtained from the GP equation, which predicts breathing mode solutions. In the stabilized cases, the monopole moment undergoes oscillations after the radial trapping potential is turned off. The oscillation can be separated into two parts: a rapid oscillation with a small amplitude [cf. Fig. 2(d) (inset)] superimposed on a slower, smoothly varying oscillation, which is referred as the breathing mode. The breathing mode oscillation effectively prevents the solution from converging to the origin, where collapse occurs. The slow modulation of the monopole moment depends on the ramp time Γ. For a longer ramping time, the modulation of the amplitude is almost sinusoidal. In contrast, the modulation shows beats as the ramping becomes faster.

Saito and Ueda [8] observed from their numerical simulations using the GP equation that the amplitude of the excitation decreases with increasing ramping time Γ. Our results are consistent with this. However, while the breathing mode is a commonly seen behavior from the results of the nonlinear GP equation, their time-evolution depends sensitively on the initial condition and is not easily controllable. On the other hand, the hyper-radial equation being linear, it helps us investigate the dynamics, taking advantage of the periodicity of the driving field. This approach, however, does not account for relaxation of the system because of the single-channel constraint. Admitting such constraints of the KHA approximation, we may nevertheless gain some insight into the system's evolution after the trapping potential is turned off. The wave function can thus be written as a linear combination of the "*Floquet*" eigenstates. The frequency of a particular breathing mode corresponds to the energy difference between a pair of Floquet eigenstates, $\omega_{mn}=(E_m-E_n)/\hbar$.

For a long ramping time, the dynamics is adiabatic and the wave function evolves into the lowest Floquet pair. As a result, the modulation of the monopole moment is almost sinusoidal. For example, the single modulation frequency observed in Fig. 2(c) corresponds to the energy difference between the ground and the first excited Floquet states. As the ramping time decreases, excitations to higher states take place. If more than one excited Floquet states are substantially excited, frequency beating in the evolution of the monopole moment would be observed. For example, in Fig. 2(b), the beating frequency is exactly equal to the difference between $\omega_{21}-\omega_{32}$, which is equal to $[(E_2-E_1)-(E_3-E_2)]/\hbar$. In Fig. 2(a), a second beating can be observed as the ramping time is further decreased. This suggests the

TABLE I. Comparison of the Floquet quasienergy spectrum and the spectrum of the Kapitza-averaged effective potential, V_{kap}. Parameters used: $G_0=-0.55$, $G_1=2.0$, $\Omega=45$, and $N=100$.

Floquet quasienergy	Energy levels of V_{kap}
−8.7077	−8.703783
−7.1751	−7.172313
−5.9030	−5.901099
−4.8502	−4.849009
−3.9811	−3.980417
−3.2652	−3.264772
−2.6763	−2.676109
−2.1926	−2.192525

ramping time is a measure of nonadiabatic excitation of the Floquet states. Note that the adiabaticity of the excitation depends on Ω and g_1 as well.

C. Remarks on the Kapitza averaging

It may seem curious that while the stationary states have continuous bound spectrum the Floquet eigenvalues are discrete. This point evokes a discussion on the Kapitza averaging. The classical dynamics in a high frequency field can be considered as one consisting of a rapid oscillation imposed on a slowly varying envelope. By integrating over the fast oscillating motion, the time-dependent term reduces to an approximately static interaction potential, which governs the slow part of the motion, namely

$$V_{\text{kap}}(r) = \frac{G_1^2(N-1)^4}{\Omega^2}\frac{1}{r^6}.$$

This is the Kapitza averaging [25] which is related to the Floquet operator in a nontrivial way through a canonical transformation. Note that in this form, the effective potential is an expansion up to Ω^{-2}. For high frequency fields, we found that the Kapitza effective potential gives a quantum spectrum coinciding with the quasienergy spectrum to a few significant digits (cf. Table I). Such observation is consistent with the theory of Rahav *et al.* [26]. In the high frequency limit, one expects that localized quasistationary states exist for $g<g_{cr}$ since the Kapitza-averaging merely introduces a cutoff radius by means of a short-range barrier near $r=0$, and the $1/r^2$ attractive potential supports an infinite number of bound states. For numerical stability, use of cutoff radius helps; the energy levels thus obtained are converged for the range $0.015<r_{\text{cut}}<3.0$ for the parameters used in Table I.

D. Comparison with the GP treatment

In order to obtain the breathing mode frequencies, we solve the time-dependent GP equation separately using finite-difference grids and the Crank-Nicholson scheme. In Fig. 3, the monopole moment exhibits a smoothly varying breathing mode superimposed on a fast oscillation of small amplitude, where the parameters used are $g_0=-2.04\pi$, $g_1=8\pi$, $\Omega=45$, and $\Gamma=10$. In our numerical calculations, these

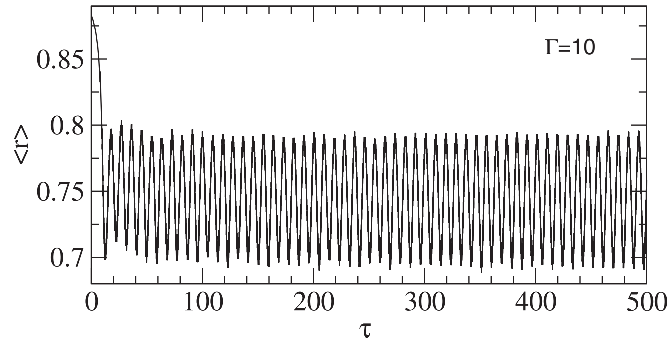

FIG. 3. Time evolution of the monopole moment obtained by direct integration of the GP equation, where the parameters used are $g_0=-2.04\pi$, $g_1=8\pi$, $\Omega=45$, and $\Gamma=10$.

parameters correspond to those used in generating the KHA results shown in Fig. 2. No frequency beating occurs in the framework of the nonlinear GP equation, however.

Incidentally, Sakaguchi and Malomed [27] found beating of the peak amplitude of the wave function as a function of time while studying effects of a periodic modulation of the fundamental soliton in 1D. In their study, the amplitude of the oscillating perturbation is very small. The resulted beating is purely perturbative in nature and occurs only at a frequency of $\omega-\omega_{sol}$, where ω_{sol} is the intrinsic frequency of the soliton dependent on its amplitude. Accounting for the difference between nonlinear and its associated linear dynamics appears of value such as by the multimode expansion of the GP solution, but such a task will not be pursued here.

Figure 4 shows the g_0 dependence of the breathing mode frequencies, obtained from the GP equation and the KHA approach. Gaussian analysis shows that the frequency of the breathing mode has a linear dependence on g_0, given that Ω/g_1 is fixed and the breathing mode amplitude is small. One would have to consider corrections due to the large oscillation amplitude. Indeed, as shown in Fig. 2, the amplitude is affected by the ramping procedure. The critical value under the KHA approach can be obtained by extrapolating the dependence of breathing mode frequency on g_0. Note that the

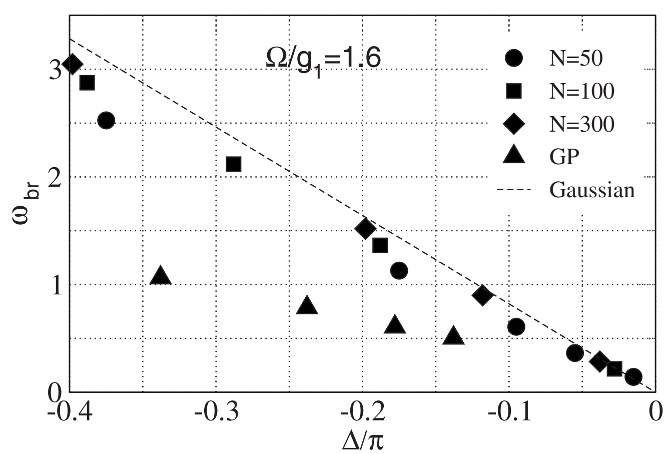

FIG. 4. Δ dependence of the breathing-mode frequency, ω_{br}, where Ω/g_1 is 1.6. Δ is defined as the difference between g_0 and the respective g_{cr}.

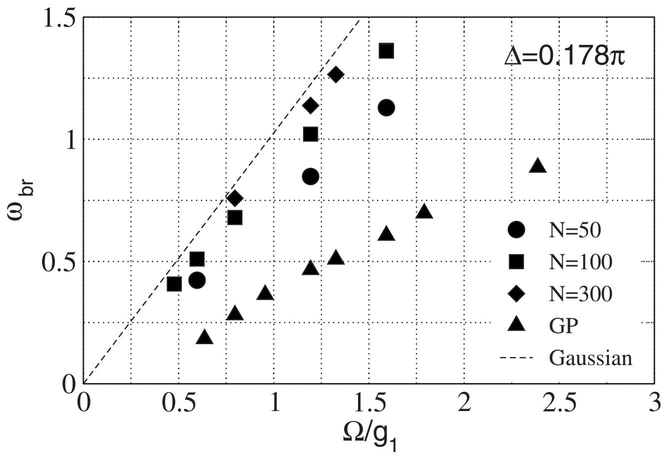

FIG. 5. Ω/g_1 dependence of the breathing-mode frequency, ω_{br}, where $\Delta=0.178\pi$.

critical value thus obtained depends on N. Our results indicate that the critical value increases as the number of bosons increases. Nevertheless, the dependence is very weak and the critical values for N in the range of $N=100-600$ remain close to -2π. Note that the critical value given by the Gaussian approximation is exactly -2π, while a direct numerical simulation of the GP equation gives a value of −5.85.

Also shown in Fig. 4 is that the KHA results tend to those predicted by the Gaussian approximation as N increases which is consistent with the fact that in this limit of large N, the hyperspherical method actually corresponds to the Gaussian approximation to $F(r)$ as noted in Ref. [19]. Figure 5 shows the Ω/g_1 dependence of the breathing mode frequencies for different numbers of atoms. Note that we present here data with a fixed value of $\Delta=g_0-g_{cr}$, which is defined as the difference of g relative to its corresponding critical values. These results clearly show a linear dependence consistent with [8].

IV. CONCLUSIONS

We analyzed the stabilization of a BEC initially confined in a quasi-2D axially symmetric trap by solving the linear Schrödinger equation under the K-harmonic approximation. We compared the scaling law for the stationary solutions in the absence of the modulating field, which for one correspond to the Townes soliton in quantum optics and the other correspond to the Bessel functions of an imaginary order. The single-particle density shows similar characteristics. When driven, the breathing mode solutions emerged, corroborating the assertion that the condensate can be stabilized in free space. In the context of the KH approximation, the stabilization can be understood in terms of the Kapitza averaging and Floquet analysis, the former leading to the discrete quasistationary states and the latter to the coherent excitation among them. Also, the behavior of the breathing modes agreed qualitatively with that obtained from direct integration of the mean-field GP equation.

The breathing mode solutions exhibit characteristic modulation frequencies, which depend on the oscillating frequency, and the magnitudes of the constant and oscillating parts of the interaction parameter g. Our results show a good agreement with the formula derived from a variational approximation using the Gaussian wave packet. In comparison with the results obtained from numerical simulations of the GP equations, the KHA results are close to those obtained from the Gaussian approximation to the GP equation. This is not entirely unexpected since KHA may be regarded as a naturally quantized version of the Gaussian variational method in the sense that the breadth of the Gaussian wave packet plays the role of a dynamical parameter and its conjugate momentum is well-defined. In consequence, results from these two methods converge as $N\to\infty$. That the GP equation has no explicit dependence on N in predicting the beat frequency led us to suspect that a systematic way of incorporating elementary excitations, say in the spirit of the Bogoliubov-type treatment, would be useful for studying the suppression of higher frequency modulation. Further studies are required.

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