Concurrence of superpositions

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Bounds on the concurrence of the superposition state in terms of the concurrences of the states being superposed are found in this paper. The bounds on concurrence are quite different from those on the entanglement measured by von Neumann entropy [Linden *et al.*, Phys. Rev. Lett. **97**, 100502 (2006)]. In particular, a nonzero lower bound can be provided if the states being superposed are properly constrained.

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Most recently, Linden *et al.* [1] have raised a problem, i.e., for two given bipartite (A and B) states $|\Psi\rangle$ and $|\Phi\rangle$, what is the relation between their entanglement and that of their superposed state $|\Gamma\rangle=\alpha|\Psi\rangle+\beta|\Phi\rangle$, with $|\alpha|^2+|\beta|^2=1$. They found upper bounds on the entanglement $|\Gamma\rangle$ in terms of the entanglement of $|\Psi\rangle$ and $|\Phi\rangle$, where the entanglement measure they employed is the von Neumann entropy of the reduced state of either of the parties [2] defined by

$$E(\Psi)=S(\mathrm{Tr}_A|\Psi\rangle\langle\Psi|)=S(\mathrm{Tr}_B|\Psi\rangle\langle\Psi|).$$

Since the entanglement measure for pure states is not unique, it is natural to ask whether the bounds obtained in Ref. [1] only exist for von Neumann entropy. And what is the lower bound on the entanglement of superposition? Motivated by this question, in this paper we employ concurrence [3–5] as the entanglement measure to study how the concurrence of $|\Gamma\rangle$ is bounded in terms of the concurrence of $|\Psi\rangle$ and $|\Phi\rangle$. The result shows that even though the form of the bounds for concurrence are something like those given in Ref. [1], they are quite different. For example, for two biorthogonal states, Ref. [1] has shown an elegant bound on the von Neumann entropy of their superposition, i.e., an equality bound, while we do not find explicit constraints for the two states such that the concurrence of their superposition has equality bounds. It is most important that a *nonzero* lower bound on the concurrence of a superposition state can be provided if the states being superposed satisfy some conditions which include the constraints on the concurrence of the states and their proportions in the superposition state and so on. The paper is organized as follows. First, we introduce a variational but equivalent expression for concurrence; then we study the concurrence of superposition by analogous logic to that of Ref. [1]; the conclusion is drawn finally.

In this paragraph, we first introduce the concurrence and derive the variational form of concurrence which will simplify our presentation. As we know, $|\psi\rangle_{AB}$ of two parties A and B defined in $(n_1\times n_2)$ dimensions can, in general, be considered as a vector, i.e., $|\psi\rangle_{AB}=[a_{00},a_{01},\ldots,a_{0n_2},a_{10},a_{11},\ldots,a_{n_1n_2}]^T$ with the superscript T denoting transpose operation, while throughout the paper we confine all the pure states to matrix notation, i.e.,

$$\psi=\begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0n_2}\\ a_{10} & a_{11} & \cdots & a_{1n_2}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n_10} & a_{n_11} & \cdots & a_{n_1n_2}\end{pmatrix}. \tag{1}$$

With the matrix notation, one can easily find that the reduced density matrix

$$\rho_B=\psi\psi^\dagger. \tag{2}$$

Consider the eigenvalue decomposition of ρ_B, one can have

$$\rho_B=\psi\psi^\dagger=\Psi M\Psi^\dagger, \tag{3}$$

where the columns of Ψ correspond to the eigenvectors of ρ_B and M is a non-negative diagonal matrix with its diagonal entries corresponding to the eigenvalues of ρ_B or the square of the singular values of ψ.

The concurrence for an arbitrary dimensional bipartite pure state $|\psi\rangle$ is defined [4] by

$$C(|\psi\rangle)=\sqrt{2[-\mathrm{Tr}(\rho_r^2)]}, \tag{4}$$

which turned out to be the length of the concurrence vector obtained by Wootters [6], where $\rho_r=\mathrm{Tr}_\alpha|\psi\rangle\langle\psi|$ denotes the reduced density matrix tracing over either of the two parties. Substituting Eq. (3) into Eq. (4), we have (up to a constant)

$$C(|\psi\rangle)=\sqrt{1-\sum_i\sigma_i^4} \tag{5}$$

$$=\sqrt{\sum_{i\neq j}\sigma_i^2\sigma_j^2}, \tag{6}$$

where σ_i, $\sum_i\sigma_i^2=1$, is one singular value of ψ where the normalized ψ is always implied. Equations (5) and (6) are the so-called variational forms for concurrence to be used in the paper.

Theorem 1 (biorthogonal states). If two pure states ψ_1 and ϕ_1 defined in $(n\times m)$ dimensions are biorthogonal, i.e., satisfy $\psi_1\phi_1^\dagger=\psi_1^\dagger\phi_1=0$, the concurrence of their superposed states $\gamma_1^+=\alpha\psi_1+\beta\phi_1$ with $|\alpha|^2+|\beta|^2=1$ obeys

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$$\frac{|\alpha|^2C(\psi_1)+|\beta|^2C(\phi_1)}{2}\leqslant C(\alpha\psi_1+\beta\phi_1)\leqslant\frac{|\alpha|^2\widetilde{C}(\psi_1,\alpha)+|\beta|^2\widetilde{C}(\phi_1,\beta)}{2},\tag{7}$$

where

$$\widetilde{C}(\psi_1,\alpha)=\sqrt{C^2(\psi_1)+\frac{|\beta|^4}{|\alpha|^4}+2\frac{|\beta|^2}{|\alpha|^2}},\tag{8}$$

with $|\beta|^2=1-|\alpha|^2$.

That we say ψ_1 and ϕ_1 are defined in the same dimension implies that the two states have been properly adjusted. Note that ψ_1 and ϕ_1 may be defined in the Hilbert space with different dimensions. However, one can always add some zero entries to ψ_1 and ϕ_1 such that ψ_1 and ϕ_1 are defined in the same dimension. What is more, one can find that $\psi_1\phi_1^\dagger=0$ is equivalent to the conditions given in Ref. [1] for biorthogonal states after simple algebra.

Proof. As we know, for any two Hermitian matrices H and K defined in $C^{n\times n}$,

$$\lambda_i(H)+\lambda_1(K)\leqslant\lambda_i(H+K)\leqslant\lambda_i(H)+\lambda_n(K)\tag{9}$$

holds, where $\lambda_i(\cdot)$ denotes the eigenvalues in increasing order [7] (see Theorem 4.3.1 in Ref. [7]).

Since $\psi_1\psi_1^\dagger$ and $\phi_1\phi_1^\dagger$ are both Hermitian and defined in $(n\times n)$ dimensions, one has

$$|\alpha|^2\lambda_i(\psi_1\psi_1^\dagger)+|\beta|^2\lambda_1(\phi_1\phi_1^\dagger)\leqslant\lambda_i(|\alpha|^2\psi_1\psi_1^\dagger+|\beta|^2\phi_1\phi_1^\dagger).\tag{10}$$

Because $\psi_1\phi_1^\dagger=0$,

$$\lambda_i(|\alpha|^2\psi_1\psi_1^\dagger+|\beta|^2\phi_1\phi_1^\dagger)=\lambda_i[\gamma_1^+(\gamma_1^+)^\dagger].\tag{11}$$

Substituting Eq. (11) into eq. (6), we have

$$\begin{aligned}&\left(\sum_{i\neq j}^{n}[|\alpha|^2\lambda_i(\psi_1\psi_1^\dagger)+|\beta|^2\lambda_1(\phi_1\phi_1^\dagger)]\times[|\alpha|^2\lambda_j(\psi_1\psi_1^\dagger)+|\beta|^2\lambda_1(\phi_1\phi_1^\dagger)]\right)^{1/2}\\&=\{|\alpha|^4C^2(\psi_1)+(n-1)|\beta|^2\lambda_1(\phi_1\phi_1^\dagger)\times[2|\alpha|^2+n|\beta|^2\lambda_1(\phi_1\phi_1^\dagger)]\}^{1/2}\\&\leqslant\sqrt{\sum_{i\neq j}^{n}\lambda_i[\gamma_1^+(\gamma_1^+)^\dagger]\lambda_j[\gamma_1^+(\gamma_1^+)^\dagger]}=C(\gamma_1^+).\end{aligned}\tag{12}$$

Substituting Eq. (11) into Eq. (5), we have

$$\begin{aligned}C(\gamma_1^+)&=\sqrt{1-\sum_i^n\lambda_i^2[\gamma_1^+(\gamma_1^+)^\dagger]}\\&\leqslant\left[|\alpha|^4\left(1-\sum_i^n\lambda_i^2(\psi_1\psi_1^\dagger)\right)+|\beta|^4[1-n\lambda_1^2(\phi_1\phi_1^\dagger)]+2|\alpha|^2|\beta|^2\left(1-\lambda_1\sum_i^n\lambda_i(\psi_1\psi_1^\dagger)\right)\right]^{1/2}\\&=\{|\alpha|^4C^2(\psi_1)+|\beta|^4[1-n\lambda_1^2(\phi_1\phi_1^\dagger)]+2|\alpha|^2|\beta|^2[1-\lambda_1(\phi_1\phi_1^\dagger)]\}^{1/2}.\end{aligned}\tag{13}$$

Simplifying Eqs. (12) and (13) by considering a positive semidefinite $\phi\phi^\dagger$, the two equations can be rewritten by

$$|\alpha|^2C(\psi_1)\leqslant C(\alpha\psi_1+\beta\phi_1)\leqslant\sqrt{|\alpha|^4C^2(\psi_1)+|\beta|^4+2|\alpha|^2|\beta|^2}.\tag{14}$$

Considering the analogous relation to Eq. (9) by exchanging H and K and a positive semidefinite $\psi\psi^\dagger$, based on the above procedure one can also obtain

$$|\beta|^2C(\phi_1)\leqslant C(\alpha\psi_1+\beta\phi_1)\leqslant\sqrt{|\beta|^4C^2(\phi_1)+|\alpha|^4+2|\alpha|^2|\beta|^2}.\tag{15}$$

Therefore, Eqs. (14) and (15) can be given in a symmetric form by

$$\frac{|\alpha|^2C(\psi_1)+|\beta|^2C(\phi_1)}{2}\leqslant C(\alpha\psi_1+\beta\phi_1)\leqslant\frac{|\alpha|^2\widetilde{C}(\psi_1,\alpha)+|\beta|^2\widetilde{C}(\phi_1,\beta)}{2},\tag{16}$$

which completes the proof. ■

Theorem 2 (orthogonal states). If $(n\times m)$-dimensional pure states ϕ_2 and ψ_2 are orthogonal, i.e., $\mathrm{Tr}\psi_2\phi_2^\dagger=0$, the concurrence of their superposed state $\gamma_2^+=\alpha\psi_2+\beta\phi_2$ with rank r satisfies,

$$2\max\{|\alpha|^2l(\alpha,\psi_2,\phi_2),|\beta|^2l(\beta,\phi_2,\psi_2)\}\leqslant C(\gamma_2^+)\leqslant2\min\{|\alpha|^2f(\alpha,\psi_2,\phi_2),|\beta|^2f(\beta,\phi_2,\psi_2)\},\tag{17}$$

where

$$l(\alpha,\psi_2,\phi_2)=\left(\max\left\{0,C^2(\psi_2)-r\frac{|\beta|^4}{|\alpha|^4}\lambda_n^2(\phi_2\phi_2^\dagger)-2\frac{|\beta|^2}{|\alpha|^2}\lambda_n(\phi_2\phi_2^\dagger)+\frac{1-4|\alpha|^4}{4|\alpha|^4}\right\}\right)^{1/2}$$

and

$$f(\alpha,\psi_2,\phi_2)=\left[C^2(\psi_2)+r\frac{|\beta|^2}{|\alpha|^2}\lambda_n(\phi_2\phi_2^\dagger)\times\left(1+(r-1)\frac{|\beta|^2}{|\alpha|^2}\lambda_n(\phi_2\phi_2^\dagger)\right)\right]^{1/2}.$$

Proof. Construct matrix D_2 such that

$$D_2 = |\alpha|^2 \phi_2 \phi_2^\dagger + |\beta|^2 \psi_2 \psi_2^\dagger.$$

The inequality (10) holds in this case, too, i.e.,

$$|\alpha|^2 \lambda_i(\psi_2\psi_2^\dagger) + |\beta|^2 \lambda_1(\phi_2\phi_2^\dagger) \leq \lambda_i(D_2) \leq |\alpha|^2 \lambda_i(\psi_2\psi_2^\dagger) + |\beta|^2 \lambda_n(\phi_2\phi_2^\dagger). \quad (18)$$

D_2 can also be rewritten as

$$D_2 = \frac{1}{2}\gamma_2^+(\gamma_2^+)^\dagger + \frac{1}{2}\gamma_2^-(\gamma_2^-)^\dagger, \quad (19)$$

with $\gamma_2^- = \alpha\psi_2 - \beta\phi_2$. In terms of inequality (9), Eq. (19) implies that

$$\frac{1}{2}\lambda_i[(\gamma_2^+ (\gamma_2^+)^\dagger] + \frac{1}{2}\lambda_1[\gamma_2^-(\gamma_2^-)^\dagger] \leq \lambda_i(D_2) \leq \frac{1}{2}\lambda_i[\gamma_2^+(\gamma_2^+)^\dagger] + \frac{1}{2}\lambda_n[\gamma_2^-(\gamma_2^-)^\dagger]. \quad (20)$$

Comparing Eq. (18) with Eq. (20), one has

$$\frac{1}{2}\lambda_i[\gamma_2^+(\gamma_2^+)^\dagger] + \frac{1}{2}\lambda_1[\gamma_2^-(\gamma_2^-)^\dagger] \leq |\alpha|^2 \lambda_i(\psi_2\psi_2^\dagger) + |\beta|^2 \lambda_n(\phi_2\phi_2^\dagger) \quad (21)$$

and

$$\frac{1}{2}\lambda_i[\gamma_2^+(\gamma_2^+)^\dagger] + \frac{1}{2}\lambda_1[\gamma_2^-(\gamma_2^-)^\dagger] \leq |\alpha|^2 \lambda_n(\psi_2\psi_2^\dagger) + |\beta|^2 \lambda_i(\phi_2\phi_2^\dagger). \quad (22)$$

Due to the positive semidefinite $\gamma_2^-(\gamma_2^-)^\dagger$, $\lambda_1[\gamma_2^-(\gamma_2^-)^\dagger] \geq 0$. Equations (21) and (22) can be rewritten by

$$\frac{1}{2}\lambda_i[\gamma_2^+(\gamma_2^+)^\dagger] \leq |\alpha|^2 \lambda_i(\psi_2\psi_2^\dagger) + |\beta|^2 \lambda_n(\phi_2\phi_2^\dagger) \quad (23)$$

and

$$\frac{1}{2}\lambda_i[\gamma_2^+(\gamma_2^+)^\dagger] \leq |\alpha|^2 \lambda_n(\psi_2\psi_2^\dagger) + |\beta|^2 \lambda_i(\phi_2\phi_2^\dagger). \quad (24)$$

Substitute Eqs. (23) and (24) into Eq. (6), we arrive at

$$\frac{1}{2}C(\gamma_2^+) \leq |\alpha|^2 f(\alpha, \psi_2, \phi_2) \quad (25)$$

and

$$\frac{1}{2}C(\gamma_2^+) \leq |\beta|^2 f(\beta, \phi_2, \psi_2). \quad (26)$$

Rewriting Eqs. (25) and (26) in a symmetric form, it follows that

$$C(\gamma_2^+) \leq 2\min\{|\alpha|^2 f(\alpha,\psi_2,\phi_2), |\beta|^2 f(\beta,\phi_2,\psi_2)\} \leq [|\alpha|^2 f(\alpha,\psi_2,\phi_2) + |\beta|^2 f(\beta,\phi_2,\psi_2)]. \quad (27)$$

According to Eqs. (5) and (23), we have

$$\frac{1}{2}C(\gamma_2^+) = \sqrt{\frac{1}{4} - \frac{1}{4}\sum_i \lambda_i^2[\gamma_2^+(\gamma_2^+)^\dagger]} \geq \sqrt{\max\left\{0, \frac{1}{4} - \sum_i [|\alpha|^2\lambda_i(\psi_2\psi_2^\dagger) + |\beta|^2\lambda_n(\phi_2\phi_2^\dagger)]^2\right\}} = |\alpha|^2 l(\alpha,\psi_2,\phi_2). \quad (28)$$

Analogously, we can also obtain

$$\frac{1}{2}C(\gamma_2^+) \geq |\beta|^2 l(\beta, \phi_2, \psi_2). \quad (29)$$

Hence, one can obtain the symmetric form

$$\frac{1}{2}C(\gamma_2^+) \geq \max\{|\alpha|^2 l(\alpha,\psi_2,\phi_2), |\beta|^2 l(\beta,\phi_2,\psi_2)\} \geq \frac{1}{2}[|\alpha|^2 l(\alpha,\psi_2,\phi_2) + |\beta|^2 l(\beta,\phi_2,\psi_2)]. \quad (30)$$

■

From Eq. (28), it is obvious that if and only if

$$C^2(\psi_2) > 1 - \frac{1}{4|\alpha|^4} - \frac{1}{r},$$

$$\lambda_n(\phi_2\phi_2^\dagger) \in \left(0, -\frac{1}{r}\frac{|\alpha|^2}{|\beta|^2}(1-\Delta)\right) \quad (31)$$

with

$$\Delta = \sqrt{1 + r\left(C^2(\psi_2) + \frac{1}{4|\alpha|^4} - 1\right)},$$

then

$$l(\alpha, \psi_2, \phi_2) > 0.$$

From Eq. (29), one can also obtain a similar constraint to Eq. (31) for $l(\beta,\phi_2,\psi_2) > 0$, which we omit here.

Theorem 3 (arbitrary states). For any two normalized $(n \times m)$-dimensional pure states ψ_3 and ϕ_3 with $|\alpha|^2 + |\beta|^2 = 1$, the concurrence of the superposed state $\gamma_3^+ = \alpha\psi_3 + \beta\phi_3$ with rank r is bounded as

$$\max\{|\alpha|^2\widetilde{l}(\alpha,\psi_3,\phi_3),|\beta|^2\widetilde{l}(\beta,\phi_3,\psi_3)\} \leqslant \frac{\|\gamma_3^+\|^2}{2}C(\gamma_3^+) \leqslant \min\{|\alpha|^2 f(\alpha,\psi_3,\phi_3),|\beta|^2 f(\beta,\phi_3,\psi_3)\} \quad (32)$$

where

$$\widetilde{l}(\alpha,\psi_3,\phi_3) = \left(\max\left\{0, C^2(\psi_3) - r\frac{|\beta|^4}{|\alpha|^4}\lambda_n^2(\phi_3\phi_3^\dagger) - 2\frac{|\beta|^2}{|\alpha|^2}\lambda_n(\phi_3\phi_3^\dagger) + \frac{\|\gamma_3^+\|^4 - 4|\alpha|^4}{4|\alpha|^4}\right\}\right)^{1/2},$$

and $\|\cdot\|$ denotes the l_2 norm [7].

Proof. Analogous to Theorem 2, consider the matrix

$$D_3 = |\alpha|^2\psi_3\psi_3^\dagger + |\beta|^2\phi_3\phi_3^\dagger. \quad (33)$$

D_3 can be rewritten as

$$D_3 = \frac{\|\gamma_3^+\|^2}{2}\widetilde{\gamma}_3^+(\widetilde{\gamma}_3^+)^\dagger + \frac{\|\gamma_3^-\|^2}{2}\widetilde{\gamma}_3^-(\widetilde{\gamma}_3^-)^\dagger, \quad (34)$$

with $\gamma_3^\pm = \alpha\psi_3 \pm \beta\phi_3$ and $\widetilde{\gamma}_3^\pm = \gamma_3^\pm/\|\gamma_3^\pm\|$. Based on Eq. (9), we have

$$\frac{\|\gamma_3^+\|^2}{2}\lambda_i[(\widetilde{\gamma}_3^+(\widetilde{\gamma}_3^+)^\dagger] + \frac{\|\gamma_3^-\|^2}{2}\lambda_1[\widetilde{\gamma}_3^-(\widetilde{\gamma}_3^-)^\dagger] \leqslant \lambda_i(D_3) \leqslant |\alpha|^2\lambda_i(\psi_3\psi_3^\dagger) + |\beta|^2\lambda_n(\phi_3\phi_3^\dagger). \quad (35)$$

Following a similar procedure to that of Theorem 2, based on Eq. (35) one can obtain

$$\frac{\|\gamma_3^+\|^2}{2}C(\gamma_3^+) \leqslant \min\{|\alpha|^2 f(\alpha,\psi_3,\phi_3),|\beta|^2 f(\beta,\phi_3,\psi_3)\} \leqslant \frac{1}{2}[|\alpha|^2 f(\alpha,\psi_3,\phi_3) + |\beta|^2 f(\beta,\phi_3,\psi_3)]. \quad (36)$$

Note that $C(\gamma_3^+)$ means the concurrence of the normalized $\alpha\psi_3+\beta\phi_3$, i.e., $C(\widetilde{\gamma}_3^+)$. From Eqs. (5) and (35) again, one has

$$\frac{\|\gamma_3^+\|^2}{2}C(\gamma_3^+) = \sqrt{\frac{\|\gamma_3^+\|^4}{4} - \frac{\|\gamma_3^+\|^4}{4}\lambda_i^2[(\widetilde{\gamma}_3^+(\widetilde{\gamma}_3^+)^\dagger]} \geqslant \sqrt{\max\left\{0, \frac{\|\gamma_3^+\|^4}{4} - \sum_i[|\alpha|^2\lambda_i(\psi_3\psi_3^\dagger) + |\beta|^2\lambda_n(\phi_3\phi_3^\dagger)]^2\right\}} = |\alpha|^2\widetilde{l}(\alpha,\psi_3,\phi_3). \quad (37)$$

In a symmetric form, the bound on concurrence can be given by

$$\frac{1}{2}[|\alpha|^2\widetilde{l}(\alpha,\psi_3,\phi_3) + |\beta|^2\widetilde{l}(\beta,\phi_3,\psi_3)] \leqslant \max\{|\alpha|^2\widetilde{l}(\alpha,\psi_3,\phi_3),|\beta|^2\widetilde{l}(\beta,\phi_3,\psi_3)\} \leqslant \frac{\|\gamma_3^+\|^2}{2}C(\gamma_3^+). \quad (38)$$

Equations (35) and (36) complete the proof. ■

Analogously, if and only if

$$C^2(\psi_3) > 1 - \frac{\|\gamma_3^+\|^4}{4|\alpha|^4} - \frac{1}{r},$$

$$\lambda_n(\phi_3\phi_3^\dagger) \in \left(0, -\frac{1}{r}\frac{|\alpha|^2}{|\beta|^2}(1-\Delta)\right) \quad (39)$$

with

$$\Delta = \sqrt{1 + r\left(C^2(\psi_3) + \frac{\|\gamma_3^+\|^4}{4|\alpha|^4} - 1\right)},$$

then

$$\widetilde{l}(\alpha,\psi_3,\phi_3) > 0.$$

The constraints on $\widetilde{l}(\beta,\phi_2,\psi_2) > 0$ are similar.

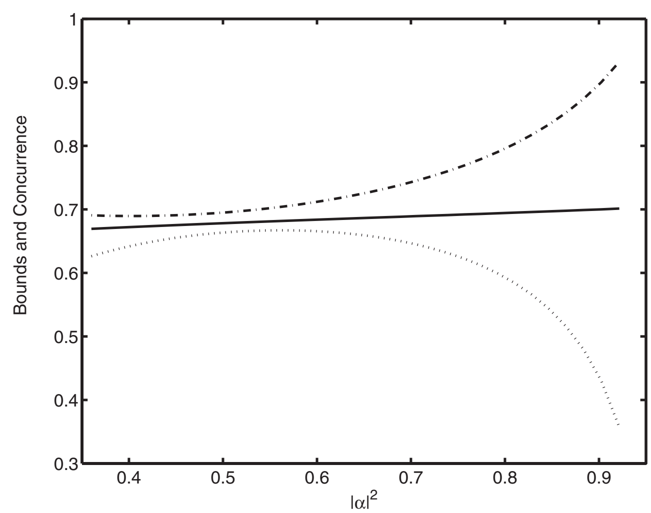

FIG. 1. The upper and lower bounds and concurrence of γ_0 vs $|\alpha|^2$. The dash-dotted line corresponds to the upper bound, and the dotted line corresponds to the lower bound, while the solid line between them is the concurrence of γ_0. Units are dimensionless.

In order to illustrate the bounds intuitionally, we directly provide a simple example for the superposition of two arbitrary states, i.e., that stated in Theorem 3. However, due to the constraint conditions given by the theorems [Eqs. (31) and (39)], we know that not all the pairs of given pure states can lead to good bounds. In this sense, we consider two such states: one is the maximally bipartite entangled state of qubits, i.e., $\phi_0=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$; the other is a bipartite entangled state randomly generated by MATLAB; here $\psi_0=\begin{pmatrix}0.7594 & 0.2067\\ 0.1583 & 0.5962\end{pmatrix}$ is our choice. An intuitional demonstration of the upper and lower bounds and the concurrence of $\gamma_0=\alpha\phi_0+\beta\psi_0$ vs $|\alpha|^2$ is shown in Fig. 1. In fact, we have also studied some superpositions of a pair of higher-dimensional entangled states (a maximally entangled state and a random one); analogous figures to Fig. 1 can also be obtained, but are omitted here.

In summary, we have given the bounds on the concurrence of superposition states, which are very different from those in Ref. [1]. A lower bound can also be provided if the states being superposed are constrained as mentioned. However, it seems to be difficult to present a useful lower bound for two arbitrary states by the current approach. This result can be readily extended to the superposition of more than two terms by repeating our procedure based on Eq. (9). One can easily see that if the current bound on concurrence is converted into that on the square of the concurrence (it is only simple algebra), the generalization to the case of more than two terms will be straightforward. What is more, one will see that if the negativity [8] is employed as the entanglement measure, it is also difficult to find useful (upper and lower) bounds based on the current approach.

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[1] Noah Linden, Sandu Popescu, and John A. Smolin, Phys. Rev. Lett. **97**, 100502 (2006).

[2] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, Phys. Rev. A **53**, 2046 (1996).

[3] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).

[4] Pranaw Rungta, V. Bužek, Carlton M. Caves, M. Hillery, and G. J. Milburn, Phys. Rev. A **64**, 042315 (2001).

[5] K. Audenaert, F. Verstraete, and B. De Moor, Phys. Rev. A **64**, 052304 (2001).

[6] W. K. Wootters, Quantum Inf. Comput. **1**, 27 (2001).

[7] R. A. Horn and C. R. Johnson, *Matrix Analysis* (Cambridge University Press, New York, 1985).

[8] G. Vidal and R. F. Werner, Phys. Rev. A **65**, 032314 (2002).