Coherent control of self-trapping of two weakly coupled Bose-Einstein condensates

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We investigate the effect of an external periodic driving to the self-trapping transitions of two weakly coupled Bose-Einstein condensates in a double-well potential. It is shown that the self-trapping can be effectively controlled by modulating the double-well with a high frequency driving and thus renormalizing the transition parameters. In particular, an exact solution is constructed under the balance condition between the external field and the nonlinear interaction among particles, which allows one to precisely control the self-trapping states.

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Periodic driving forces have become an important tool for coherently controlling the tunneling process [1]. A particle in a double-well driven by a periodical external field is a typical model to demonstrate tunneling control [2,3]. Recent realization of dilute Bose degenerate gas in an optical double-well potential provides a new opportunity to revive the old problem due to the highly controllable environment and long coherence time of the system [4,5]. Unlike single-particle tunneling, in this new system the interaction among the tunneling particles plays a crucial role, which leads to a nonlinear term in the Schrödinger equation. A question now arises whether the interaction between the condensed atoms gives rise to new dynamical regimes [6–9]. The investigation of such a problem has attracted considerable interest over the past few years. Among many findings, macroscopic quantum self-trapping (MQST) is one of the most interesting phenomenon, which is a kind of self-locked population imbalances between two Bose-Einstein Condensates (BECs) [10–12]. The MQST has a quantum nature, involving the coherence of a macroscopic number of atoms in the two condensates. Its dependence upon the system parameters has been obtained in the case without periodic driving. Recently, this novel nonlinear effect has been confirmed in experiment [13], which further stimulates the study of this nonlinear system in the presence of an external field.

The periodic modulation effect on the self-trapping of two weakly coupled BECs in a double-well potential has been studied by several groups [14–18]. Analytic expressions for the dependence of the transition parameters on the modulation parameters are derived for high- and low-frequency modulations, but no exact analytic solutions are reported yet. In this paper, we focus on the case of the driving forces acting on the energy bias in a double-well potential. It is shown that an external ac field modulating the double-well can result in the renormalization of the tunneling frequency and of the relative energy bias in the high frequency limit. Particularly, we find an exact solution under a balance condition, that enable us to precisely prepare and manipulate the self-trapping states.

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Self-trapping without periodic driving. We first give a brief review of the self-trapping transitions in two weakly coupled BECs consisting of N atoms at zero temperature, held in an asymmetric double-well trap without driving. Adopting the well-known two-mode approximation [4,5], the system's second-quantized Hamiltonian reads $\hat{H}=\gamma/2(\hat{a}^{\dagger}\hat{a}-\hat{b}^{\dagger}\hat{b})+c/2(\hat{a}^{\dagger}\hat{b}+\hat{b}^{\dagger}\hat{a})+\nu/2(\hat{a}^{\dagger}\hat{a}-\hat{b}^{\dagger}\hat{b})^2$, where generators and annihilators $\hat{a}^{\dagger},\hat{b}^{\dagger}$ and $\hat{a},\hat{b}$ are for two different wells, γ denotes the static energy bias between the two wells, c is the coupling constant between the two condensates, and ν is the nonlinear parameter describing the interaction. For a double-well system the total wave function can be expressed as the superposition of individual wave functions in each well. The superposition coefficients a and b satisfy the two-mode Gross-Pitaevskii equation (GPE) [10–12]

$$i\frac{d}{dt}\begin{pmatrix} a \\ b \end{pmatrix} = H\begin{pmatrix} a \\ b \end{pmatrix} \tag{1}$$

with the Hamiltonian given by

$$H=\begin{pmatrix} \frac{\gamma}{2}+\frac{\lambda}{2}(|a|^2-|b|^2) & \frac{c}{2} \\ \frac{c}{2} & -\frac{\gamma}{2}-\frac{\lambda}{2}(|a|^2-|b|^2) \end{pmatrix}, \tag{2}$$

where $\lambda=N\nu$ is set and the total probability $|a|^2+|b|^2$ is normalized to 1, so $N|a|^2$ and $N|b|^2$ are the numbers of atoms in the left well and right well. We emphasize that the mean-field approximation $N\to\infty$ is taken with constant interaction strength $\lambda=N\nu$. When the model (1) is used to describe two BECs in a double-well potential, limit $N\to\infty$ keeping a constant λ is equivalent to having a larger trap with more atoms in the BECs, or to tuning ν smaller by using the Feshbach resonance technique [19]. The very recent work shows that only 4–8 particles per well are needed to reach this limit [20]. For the experiment with about 1150 atoms [13], an elementary estimate gives the values of parameter in Eq. (2): $\gamma=0$, $c\approx 2\pi\cdot 10.4s^{-1}$ and $\lambda/c\approx 15$ [17,21].

With $a=|a|e^{i\theta_a}, b=|b|e^{i\theta_b}$, the Schrödinger equation can be mapped into a classical Hamiltonian system by introducing population difference $z=|a|^2-|b|^2$ and relative phase $\theta=\theta_b-\theta_a$,

$$H_{cl}(z,\theta) = -\gamma z + \frac{\lambda}{2}z^2 - c\sqrt{1-z^2}\cos\theta \tag{3}$$

with z, θ being the canonically conjugate variables of the classical Hamiltonian system [10,16]. Because the existence of multiple stationary states with $z\neq 0$ is equivalent to the existence of multiple metastable MQST states [12,22,23] in the coupled BECs, one may explore self-trapping behavior by analyzing the change of the number of the corresponding fixed points or stationary states with $z\neq 0$. The fixed points of the classical Hamiltonian are given by the following equations [8,9]:

$$\theta^* = 0,\pi, \quad -\gamma + \lambda z^* - \frac{cz^*}{\sqrt{1-z^{*2}}}\cos\theta^* = 0. \tag{4}$$

For the equal-phase mode $\theta^*=0$ only one stable fixed point exists for $\lambda/c\leqslant 1$. When $\lambda/c>1$, there are two stable fixed points and one unstable fixed point for $(\lambda/c)^{2/3}-(\gamma/c)^{2/3}>1$, and only one stable fixed point exists for $(\lambda/c)^{2/3}-(\gamma/c)^{2/3}<1$. For the antiphase mode $\theta^*=\pi$ the parametric dependence of fixed points and stationary states is very different to reveal. When $\lambda/c\geqslant -1$, only one fixed point appears and it is stable. When $\lambda/c<-1$, two stable and one unstable fixed points exist for $(\lambda/c)^{2/3}-(\gamma/c)^{2/3}>1$ and only one stable fixed point emerges for $(\lambda/c)^{2/3}-(\gamma/c)^{2/3}<1$. Small oscillations around those stationary states with nonzero time-averaged values for the population difference z are MQST states. If both the population difference z and relative phase θ oscillate around fixed points, oscillating-phase-type self-trapping states appear. If z changes with time around fixed points, while the relative phase θ varies with time monotonously, running-phase MQST states appear. In the case of the symmetric double-well, a general criterion for the self-trapping is given by $H_{cl}(z(0),\theta(0))>c$ [10–12].

Periodic modulation of self-trapping. Generally, the dynamical behavior of a system can be controlled by applying a periodic driving field, for example, the tunneling control of a particle in a double-well potential perturbed by an external periodic driving [24–26]. Here we discuss how a periodic modulation affects the nonlinear self-trapping. We assume that the modulation is applied on the static energy bias with amplitude γ_1 and frequency ω, i.e., $\gamma=\gamma_0+\gamma_1\cos(\omega t)$. This periodically driving term can be realized by oscillating the laser barrier position of the double-well [10]. In Ref. [18], the authors have considered the case of the symmetric double-well potential with a sinusoidal-form driving, namely $\gamma=A\sin(\omega t)$. The chaotic states of this time-dependent system have also been investigated in Refs. [14,16]. Here we consider a more general case of the asymmetric double-well potential with asymmetry constant γ_0 and cosinusoidal-form driving $\gamma_1\cos(\omega t)$. The former enables us to discuss the renormalization of static bias γ_0 and couple constant c and the inhibition of large amplitude tunneling oscillations. The latter allows an exact periodic solution to demonstrate the ac control of self-trapping analytically.

Limit of high frequency. When the driving frequency is large enough such that $\omega\gg c$ is satisfied, the amount of change of the probability amplitudes a and b during a period of oscillation $2\pi/\omega$ can be regarded as being infinitesimal, at most of order c/ω. In the Heidelberg experiment [13], the tunneling frequency is estimated to be $c\approx 2\pi\cdot 10.4s^{-1}$, corresponding to the tunneling times in order of 100 ms. For the limit of high frequency, the modulation frequency ω may be 3–10 times the tunneling frequency c [17,18,21]. In this case, let us make the transformation $a=e^{-i[(\gamma_0 t+\gamma_1/\omega\sin\omega t)/2]}a'$, $b=e^{i[(\gamma_0 t+\gamma_1/\omega\sin\omega t)/2]}b'$. Then combining these with Eqs. (1) and (2) we get

$$i\frac{da'}{dt} = \frac{\lambda}{2}(|b'|^2-|a'|^2)a' + \frac{c}{2}e^{i[(\gamma_0 t+\gamma_1/\omega\sin\omega t)]}b', \tag{5}$$

$$i\frac{db'}{dt} = -\frac{\lambda}{2}(|b'|^2-|a'|^2)b' + \frac{c}{2}e^{-i[(\gamma_0 t+\gamma_1/\omega\sin\omega t)]}a'. \tag{6}$$

Applying the generating function of the Bessel functions [18,25,26] $e^{\pm i\gamma_1/\omega\sin\omega t}=\sum_{m=-\infty}^{\infty}J_m(\gamma_1/\omega)e^{\pm im\omega t}$ to Eqs. (5) and (6) and considering resonance condition $\gamma_0=n\omega$ with integer n, after neglecting the rapidly oscillating terms we easily find that the system is equivalent to the undriven system in a symmetric double-well potential but with an effective coupling constant $c_{eff}=cJ_{-n}(\gamma_1/\omega)$.

Now we consider the small detuning case, i.e., $\gamma_0-n\omega=\delta\ll(\omega,c)$ such that Eqs. (5) and (6) become

$$i\frac{da''}{dt} = \left[\frac{\delta}{2}+\frac{\lambda}{2}(|b''|^2-|a''|^2)\right]a'' + \frac{c_{eff}}{2}b'', \tag{7}$$

$$i\frac{db''}{dt} = -\left[\frac{\delta}{2}+\frac{\lambda}{2}(|b''|^2-|a''|^2)\right]b'' + \frac{c_{eff}}{2}a'', \tag{8}$$

after transformation $a''=a'e^{i\delta/2t}$ and $b''=b'e^{-i\delta/2t}$. In contrast to the original system (1) and (2) without the periodic driving field, from Eqs. (7) and (8) we see that modulating the double-well can lead to the renormalization of couple constant c and of static bias γ_0. Note further that we can set γ_1/ω to make effective coupling constant $c_{eff}=0$, leading to the inhibition of large amplitude tunneling oscillations.

Exactly controlling self-trapping transitions. In the strong driven case, the situation is complicated and exactly solving the system is difficult. However, we shall show that under a balanced condition, an exact solution exists that enables us to strictly control the self-trapping transitions. The balanced condition implies that the external field and the nonlinear interactions among particles obey the relation [27,28],

$$\frac{\gamma_1}{2}\cos(\omega t) + \frac{\lambda}{2}(|b|^2-|a|^2) = \frac{\gamma_1}{2}\cos(\omega t) - \frac{\lambda z(t)}{2} = \frac{\mu}{2}, \tag{9}$$

where parameter μ is determined by the initial population of this system, the amplitude of driving field and the nonlinear interaction, i.e., $\mu=\gamma_1-\lambda z(0)$. So Eq. (9) gives $z(t)=z(0)-2\frac{\gamma_1}{\lambda}\sin^2\frac{\omega t}{2}$, which oscillates below (for $\gamma_1/\lambda>0$) or above (for $\gamma_1/\lambda<0$) the initial state. Under condition (9), driven system (1) obeys the linear Schrödinger equations

$$i\frac{da}{dt}=\left(\frac{\gamma_0+\mu}{2}\right)a+\frac{c}{2}b, \quad (10)$$

$$i\frac{db}{dt}=-\left(\frac{\gamma_0+\mu}{2}\right)b+\frac{c}{2}a \quad (11)$$

which admits the exact solutions of the form [29]

$$a(t)=a(0)\left[\cos\frac{\Omega t}{2}-\frac{i(\gamma_0+\mu)}{\Omega}\sin\frac{\Omega t}{2}\right]-\frac{cb(0)}{\Omega}\sin\frac{\Omega t}{2}, \quad (12)$$

$$b(t)=b(0)\left[\cos\frac{\Omega t}{2}+\frac{i(\gamma_0+\mu)}{\Omega}\sin\frac{\Omega t}{2}\right]-\frac{ca(0)}{\Omega}\sin\frac{\Omega t}{2} \quad (13)$$

with parameter $\Omega=\sqrt{c^2+(\gamma_0+\mu)^2}$. Using Eqs. (12) and (13) to the definition of $z(t)$ yields

$$z(t)=z(0)\cos^2\frac{\Omega t}{2}+\left(z(0)\frac{(\gamma_0+\mu)^2-c^2}{\Omega^2}+\frac{2(\gamma_0+\mu)c}{\Omega^2}\sqrt{1-z^2(0)}\cos\theta(0)\right)\sin^2\frac{\Omega t}{2}, \quad (14)$$

where formula $a(0)b^*(0)+a^*(0)b(0)=2\sqrt{1-z^2(0)}\cos\theta(0)$ has been used. Substituting $z(t)$ of Eq. (14) into Eq. (9), we find that the self-consistence between balanced condition (9) and linear Schrödinger equations (10) and (11) requires

$$\omega=\Omega=\sqrt{c^2+(\gamma_0+\mu)^2}, \quad (15)$$

$$z(0)\frac{c^2}{\Omega^2}-\frac{(\gamma_0+\mu)c}{\Omega^2}\sqrt{1-z^2(0)}\cos\theta(0)=\frac{\gamma_1}{\lambda} \quad (16)$$

which is the central result in this paper. It follows from Eq. (15) that $\omega\geqslant c$, which means that the exact solution may be valid both in the high frequency limit and in the strong resonating region [18].

Averaging the population during a time period of $2\pi/\omega$ in Eq. (9), one has an average population difference

$$\langle z(t)\rangle=z(0)-\gamma_1/\lambda. \quad (17)$$

The dependence of the above equation on external parameter γ_1 implies that one may control self-trapping transitions by an external ac field. We shall take the following two cases as examples to demonstrate this.

Case 1: with initial population $z(0)=1$ and any fixed initial relative phase $\theta(0)$. Inserting the initial condition into Eqs. (15) and (16), one easily arrives at the relations: $\omega=\sqrt{c^2+(\gamma_0+\mu)^2}$ and $c^2/\omega^2=\gamma_1/\lambda$, which can be further reduced to the simple form

$$\left(\frac{\gamma_0}{\lambda}+\frac{\gamma_1}{\lambda}-1\right)^2=\left(\frac{1}{\gamma_1/\lambda}-1\right)\left(\frac{c}{\lambda}\right)^2, \quad (18)$$

by eliminating ω and μ. We also note that for Eqs. (17) and (18) to hold true, γ_1/λ has to be larger than zero but less than or equal to one. Thus it follows that $\langle z(t)\rangle$ changes only in region [0, 1). Taking parameters $\gamma_0=0$ and $c/\lambda\approx 1/15$ as in Ref. [13], we easily obtain amplitude of the modulation, $\gamma_1/\lambda=[1\pm\sqrt{1-4(c/\lambda)^2}]/2$, and driving frequency $\omega=\sqrt{c^2+(\gamma_1-\lambda)^2}$. From the preceding analysis, we know that without driving force, the system lies in a running-phase self-trapping state in the parameter region [13]. But when the driving field with $\gamma_1/\lambda=[1+\sqrt{1-4(1/15)^2}]/2\approx 1.0$ and $\omega/c\approx 1.0$ is turned on, an almost complete exchange of populations between two well appears, $z(t)\approx\cos(\omega t)$.

Case 2: with initial conditions $z(0)=0.0$ and $\theta(0)=0.0$. A similar parameter relation is derived as follows:

$$\left(\frac{\gamma_1}{\lambda}+\frac{\gamma_0}{\lambda}\right)\frac{c}{\lambda}\Bigg/\left[\left(\frac{\gamma_1}{\lambda}+\frac{\gamma_0}{\lambda}\right)^2+\left(\frac{c}{\lambda}\right)^2\right]=-\frac{\gamma_1}{\lambda} \quad (19)$$

from Eqs. (15) and (16) for the given conditions. In the symmetric case, $\gamma_0=0.0$, we have $\gamma_1/\lambda=\pm\sqrt{-(c/\lambda)^2-c/\lambda}$ and average population $\langle z(t)\rangle=-\gamma_1/\lambda$ in the parameter regions $-1<c/\lambda<0$ and $-1/2<\gamma_1/\lambda<1/2$. From Eq. (4) we observe that the undriven system with $\gamma_1=0$ can lie in stationary state $(z=0,\theta=0)$ without MQST for this region. But when the ac field is applied, the system periodically oscillates in the exact form of Eq. (14) with average $\langle z(t)\rangle=-\gamma_1/\lambda=\mp\sqrt{-(c/\lambda)^2-c/\lambda}$, indicating a nonstationary MQST state.

It is interesting to investigate the experimental feasibility of the balance state. Clearly, the exact balance solution (14) describes the regular macroscopic quantum state of the system found uniquely yet. Particularly, we have demonstrated that the balance state may be a nonstationary MQST state, which oscillates near a metastable stationary MQST state [12,22,23] of the corresponding undriven system. According to the quantum-mechanical prediction, the BEC system will occupy its possible regular state, the exact nonstationary MQST state, with certain probability P which may be greater compared to the unstable chaotic states of the system [14,16]. Noticing that balance conditions (9), (15), and (16) depend on the initial state, system parameters and external driving parameters, although the initial states and system parameters are not easy to be precisely controlled, we can carefully adjust the external driving parameters to fit the balance conditions. In the mean sense, when the number of the adjustments to the driving parameters arrives at $1/P$, the system could reach the exact balance state in the experiment.

In conclusion, we have investigated the periodic modulation effects on the self-trapping of two weakly coupled BECs in a double-well potential analytically. It is shown that the coherent tunneling can be increased or decreased, depending on the external force parameters, thus the transition parameters of the self-trapping can be effectively adjusted. In particular, we use a balanced condition to derive an exact solution of the system, that allows us to control the self-trapping states with required averaging population.

Finally, we give a brief discussion to the experimental matters associated with our results. Experimentally, BECs are coherently split into a double-well potential generated by two laser beams. The energy offset can be periodically modulated by adjusting the intensity of laser beams. In order to achieve a proper modulation to the double-well experimentally [13,17], one requires tilting the two wells and periodically shifting the focus of the blue-detuned laser carefully. Similarly, based on the experiment setup in Ref. [13], our balance state could be tested by oscillating the laser barrier position of the double-well with adjustable amplitude and frequency [10] such that parameters γ_1 and ω are adjusted. Once the MQST in the form of Eq. (17) is detected [13], Eq. (9) or Eqs. (15) and (16) will be fitted and the system will enter the balance state. Note that the MQST described by Eq. (17) can be easily observed by fixing parameters $z(0), \lambda$ and tuning γ_1, and detecting the proportion relation of Eq. (17) between average population difference $\langle z(t) \rangle$ and driving amplitude γ_1. Our analytical results have shown the dramatic affects of the periodic modulation to the tunneling dynamics and the nonlinear self-trapping states. We hope the results will stimulate further experiments.

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