Matrix realignment and partial-transpose approach to entangling power of quantum evolutions

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Based on the matrix realignment and partial transpose, we develop an approach to the entangling power and operator entanglement of quantum unitary operators. We demonstrate the approach by studying several unitary operators on qudits, and indicate that these two matrix rearrangements are convenient to use in studying entangling capabilities of quantum operators.

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Given a unitary operator, in the context of quantum information [1], one may ask how much entanglement capability the operator has. The entangling unitary operator can be considered as a resource for quantum-information processing, and it becomes important to quantitatively describe unitary operators. Recently, there is increasing interest in the entanglement capabilities of quantum evolution and Hamiltonians [2–10]. The entangling power based on the linear entropy [2] is a valuable, and relatively easy to calculate, measure of the entanglement capability of an operator. The entangling power for two qudits can be expressed in terms of operator entanglement [3,7] (also called Schmidt strength [11]). Both entangling power and operator entanglement have been applied to the study of quantum chaotic systems [12–15]. Moreover, the concept of entangling power has been extended to the case with ancillas [16], the case of entanglement-changing power [17], and the case of disentangling power [18].

Let us start by introducing some basics of entanglement of quantum states, the operator entanglement, and the entangling power. For a two-qudit pure state $|\Psi\rangle \in \mathcal{H}_d \otimes \mathcal{H}_d$, one can quantify entanglement by using the linear entropy

$$E(|\Psi\rangle) := 1 - \mathrm{Tr}\rho_1^2, \tag{1}$$

where $\rho_1 = \mathrm{Tr}_2(|\Psi\rangle\langle\Psi|)$ is the reduced density matrix. The linear entropy satisfies the inequalities $0 \leqslant E(|\Psi\rangle) \leqslant 1 - 1/d$, where the lower (upper) bound is reached if and only if $|\Psi\rangle$ is a product state (maximally entangled state).

In the local orthogonal basis $\{|1\rangle, \ldots, |d\rangle\}$, the state $|\Psi\rangle$ is written as

$$|\Psi\rangle = \sum_{i,j=1}^{d} A_{ij}|i\rangle \otimes |j\rangle, \tag{2}$$

where A_{ij} are the coefficients, and A can be considered as a $d\times d$ matrix. After direct calculations, one finds that the reduced density matrix $\rho_1 = AA^\dagger$. Substituting it in Eq. (1) leads to another expression of the linear entropy:

$$E(|\Psi\rangle) = 1 - \mathrm{Tr}(AA^\dagger AA^\dagger). \tag{3}$$

An operator can increase the entanglement of a state, but an operator can also be considered to be entangled because operators themselves inhabit a Hilbert space. The entanglement of quantum operators is introduced [3] by noting that the linear operators over $\mathcal{H}_d$ span a d^2-dimensional Hilbert space with the scalar product between two operators X and Y given by the Hilbert-Schmidt product $\langle X, Y\rangle := \mathrm{Tr}(X^\dagger Y)$, and $\|X\|_{\mathrm{HS}} := \sqrt{\mathrm{Tr}(X^\dagger X)}$. We denote this d^2-dimensional Hilbert space as $\mathcal{H}_{d^2}^{\mathrm{HS}}$. Thus, the operator acting on $\mathcal{H}_d \otimes \mathcal{H}_d$ is a state in the composite Hilbert space $\mathcal{H}_{d^2}^{\mathrm{HS}} \otimes \mathcal{H}_{d^2}^{\mathrm{HS}}$, and the entanglement of an operator X is defined [3]. We use the linear entropy to quantify the entanglement of a unitary operator U, and denote the amount of entanglement by $E(U)$ in the following discussions.

The entangling power quantifies the entanglement capability of a unitary operator U. It is defined as [2]

$$e_{\mathrm{p}}(U) := \overline{E(U|\psi_1\rangle \otimes |\psi_2\rangle)}, \tag{4}$$

where the overbar stands for the average over all product states. It tells us how much entanglement the operator U produces, on average, when acting on product states. After a suitable average over initial product states, one finds [2]

$$e_{\mathrm{p}}(U) = \left(\frac{d}{d+1}\right)^2 [E(U) + E(US_{12}) - E(S_{12})], \tag{5}$$

where S_{12} is the swapping operator which can be written as

$$S_{12} = \sum_{i,j=1}^{d} |ij\rangle\langle ji|. \tag{6}$$

Thus, the entangling power defined on $d\times d$ systems can be expressed in terms of the entanglement of three operators U, US_{12}, and S_{12}. Therefore, by studying the entanglement of these three operators we can obtain the entangling power of U.

Next, we give our approach, and first consider the operator entanglement of a unitary operator. A unitary operator can be written as

$$U = \sum_{ijkl} \langle ij|U|kl\rangle |ij\rangle\langle kl| = \sum_{ijkl} U_{ij,kl}|i\rangle\langle k| \otimes |j\rangle\langle l| = \sum_{ijkl} U_{ij,kl} e_{ik} \otimes e_{jl}, \tag{7}$$

where e_{ik} are orthogonal bases in the space $\mathcal{H}_{d^2}^{\mathrm{HS}}$, and can be considered as states. Now, we define a new matrix U^R as

$$(U^R)_{ij,kl} = U_{ik,jl}. \tag{8}$$

The matrix can be obtained by realignment of matrix U [19]. Note that the above manipulation of the matrix is not restricted to the unitary matrix. Comparing Eqs. (2) and (7), and using Eq. (3), one obtains the operator entanglement of U as

$$E(U) = 1 - \frac{1}{d^4}\mathrm{Tr}[U^R(U^R)^\dagger U^R(U^R)^\dagger]. \tag{9}$$

We see that the operator entanglement is determined by the naturally appearing realigned matrix, which is easy to obtain from the original unitary matrix, and thus our approach is convenient to study operator entanglement.

This matrix realignment is the same as density matrix realignment when studying the separability problem of quantum mixed states [19]. The realignment criterion (also called the cross-norm criterion) is strong to detect many bound entangled states. We see here that the same matrix realignment approach is useful in studying operator entanglement.

There is another matrix rearrangement, called partial transpose [20]. A partial transpose with respect to the first system U^{T_1} is defined as

$$(U^{T_1})_{ij,kl} = U_{kj,il}. \tag{10}$$

The partial transpose method can be used to study entanglement of quantum mixed states. Is it useful in studying entanglement capabilities of quantum operators? We will see that indeed it is.

The entangling power is determined by three operator entanglements $E(U)$, $E(S_{12})$, and $E(S_{12}U)$. The first two can be determined by the realignment method, and the last one of course can be determined by the same method, but with an extra effort to make the matrix multiplication $S_{12}U$. In fact, we have [21]

$$S_{12}(S_{12}U)^R = U^{T_1}.$$

Using the above property and applying Eq. (9) to $S_{12}U$, we obtain

$$E(S_{12}U) = 1 - \frac{1}{d^4}\mathrm{Tr}[U^{T_1}(U^{T_1})^\dagger U^{T_1}(U^{T_1})^\dagger]. \tag{11}$$

Therefore, the operator entanglement of $S_{12}U$ can be written in terms of the partially transposed matrix U^{T_1}.

From Eqs. (5), (9), and (11), we know that the entangling power can be determined by matrix realignment and the partial transpose

$$e_{\mathrm{p}}(U) = \left(\frac{d}{d+1}\right)^2[2 - E(S_{12})] - \frac{1}{(d+1)^2 d^2}\mathrm{Tr}\{[U^R(U^R)^\dagger]^2 + [U^{T_1}(U^{T_1})^\dagger]^2\}. \tag{12}$$

Both these matrix manipulations are useful in the context of separability of quantum states. Here, we find that they naturally emerge in studying operator entanglement and entangling power in quantum-information theory. To illustrate the approach, we consider several examples.

Example 1. The SWAP operator S_{12}. From Eq. (6), it is easy to see that

$$S_{12}^R = S_{12}, \quad S_{12}^\dagger = S_{12}, \quad S_{12}^2 = I.$$

The SWAP operator is invariant under matrix realignment. Then, from Eq. (9), the linear entropy of the SWAP operator is given by

$$E(S_{12}) = 1 - \frac{1}{d^4}\mathrm{Tr}(S_{12}^4) = 1 - \frac{1}{d^2}. \tag{13}$$

From Eq. (5), evidently the entangling power of the SWAP operator is zero.

Example 2. The unitary operator V generated by the SWAP

$$V = \exp(-itS_{12}) = \cos(t)I - i\sin(t)S_{12}. \tag{14}$$

It is straightforward to check the following identities:

$$I^{T_1} = I, \quad I^R = dP_+, \quad S_{12}^{T_1} = dP_+, \quad S_{12}^R = S, \tag{15}$$

with the projector

$$P_+ = |\Psi_+\rangle\langle\Psi_+|, \quad |\Psi_+\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d}|i\rangle\otimes|i\rangle. \tag{16}$$

From the above identities, we obtain

$$V^R(t) = \cos(t)dP_+ - i\sin(t)S_{12},$$

$$V^{T_1}(t) = \cos(t)I - i\sin(t)dP_+. \tag{17}$$

Then we find

$$V^R(t)[V^R(t)]^\dagger = \cos^2(t)d^2P_+ + \sin^2(t)I,$$

$$V^{T_1}(t)[V^{T_1}(t)]^\dagger = \cos^2(t)I + \sin^2(t)d^2P_+. \tag{18}$$

From the above two equations and Eqs. (9) and (11), we find the linear entropies

$$E(V) = \left(1 - \frac{1}{d^2}\right)(1 - \cos^4 t), \tag{19}$$

$$E(VS_{12}) = \left(1 - \frac{1}{d^2}\right)(1 - \sin^4 t). \tag{20}$$

Substituting Eqs. (19) and (20) into (5) leads to the expression of the entangling power

$$e_{\mathrm{p}} = \frac{d^2 - 1}{2(d+1)^2}\sin^2(2t). \tag{21}$$

From Eqs. (19) and (21), we see that the maximal value of the operator entanglement occurs at $t = \pi/2$; however, at this point the entangling power is zero. This point corresponds to the SWAP operation. The maximal entangling power occurs at $t = \pi/4$, which corresponds to the $\sqrt{\mathrm{SWAP}}$ gate, the square of which is just the SWAP gate. Thus, the $\sqrt{\mathrm{SWAP}}$ gate can be used as an important gate for quantum computing not only in qubit systems [22], but also in qudit systems. Quantitatively, the operator entanglement and entangling power of the $\sqrt{\mathrm{SWAP}}$ gate are given by

$$E(V) = \frac{3}{4}\left(1 - \frac{1}{d^2}\right), \quad e_{\rm p} = \frac{d^2-1}{2(d+1)^2}, \qquad (22)$$

respectively.

Example 3. A general two-qudit controlled-U gate is given by

$$C_U := \sum_{n=1}^{d} |n\rangle\langle n| \otimes U_n. \qquad (23)$$

The controlled-U gate implements the unitary operator U_n on the second system if and only if the first system is in state $|n\rangle$. For the controlled-U operation, it was found that [16]

$$e_{\rm p}(C_U) = \left(\frac{d}{d+1}\right)^2 E(C_U). \qquad (24)$$

Let us prove this via our approach. From Eq. (5), to prove the above identity is equivalent to proving that

$$E(C_U S_{12}) = E(S_{12}). \qquad (25)$$

In fact, we have a more general result that if the partial transpose of a unitary operator U is still a unitary operator, then $E(US_{12})=1-1/d^2=E(S_{12})$. This result immediately follows from Eq. (11). For our operator C_U, from the definition, it is not difficult to see that it is invariant under a partial transpose with respect to the first system. Of course, C_U is unitary, and then Eq. (25) holds. In this case, the entangling power is proportional to the operator entanglement of the controlled-U gate. We see that it is easier to obtain Eq. (25) via our approach.

In conclusion, we have developed a method for studying entangling power and operator entanglement. One only needs to obtain the realigned unitary operator and partially transposed operator to determine the entangling power. Once we have an analytical expression for the unitary matrix, then analytical expressions for entangling power and operator entanglement can be obtained. If we cannot have the analytical expression, it is very convenient to make the matrix rearrangements numerically, and then the entangling power and operator entanglement can be quickly computed.

The matrix realignment and partial transpose play very important roles in the theory of separability of quantum mixed states, and we see here that they naturally appear in the study of entanglement capabilities of quantum evolution. The approach developed here can be applied to investigate entanglement capabilities in many physical systems such as composite quantum chaotic systems.

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