Implementing a conditional *N*-qubit phase gate in a largely detuned optical cavity

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We propose a scheme to implement an *N*-qubit conditional phase gate with *N* neutral atoms which are trapped in a large detuning optical cavity. In our scheme, since the cavity field is only required to be virtually excited, the phase gate operation is insensitive to the cavity decay and the gate can be implemented with a high fidelity.

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Quantum computation researches the principle of coherent superposition and quantum entanglement and solves certain problems much faster than on a conventional classical computer [1,2]. This matter of fact has triggered in the past years a lot of studies on the theoretical and practical aspects of quantum computing. It is well known that a universal quantum computer can be built from only two kinds of gates: namely, one-qubit and two-qubit unitary gates, which are basic building blocks to construct a quantum computer. In other words, any multiqubit gates, which are required in both the quantum algorithms [1,2] and the quantum error-correction protocols [3], can be decomposed into these elementary gates [4] in principle. On the other hand, directly implementing multiqubit gates still plays an important role in quantum computing because the decomposition of multiqubit gates into elementary gates requires a number of physical gate steps and additional qubits [5] which results in a complexity of the *N*-qubit gate operation. For instance, *N*-qubit controlled phase gates are required to realize *N*-qubit Grover's [2] and Fourier's algorithms [6] or generate *N*-qubit entangled states [7]. However, it becomes very complicated if we construct the *N*-qubit controlled phase gate using one- and two-qubit gates because the *N*-qubit phase gate should be decomposed into 2^N-3 two-qubit controlled gates without auxiliary qubits, or $(2N-19)$ two-qubit controlled gates with $(N-2)$additional qubits [4,6]. Therefore the direct implementation of multiqubit phase gates is important to reduce the complexity of physical realization of practical quantum computation and quantum algorithms.

Many schemes of efficient implementations of multiqubit gates have been proposed in different physical systems, including nuclear magnetic resonance [8], linear optics [9], cold trapped ions [10], Josephson charge qubits in superconducting circuit [11], and cavity quantum electrodynamics (QED) [12] in both optical and microwave domains. Among them, optical cavity QED, where atoms are strongly coupled to quantized electromagnetic fields through a dipole interaction inside a high-*Q* cavity, offers an almost ideal system for the generation of entangled states and the implementation of quantum information processing. Recently, several cavity QED proposals were suggested for implementation of multiqubit controlled phase gates [13–15]. In Ref. [13], Goto and Ichimura presented a scheme to realize a multiqubit controlled phase gate using adiabatic passage with a single-mode optical cavity. In Ref. [14], schemes were proposed for implementing *N*-qubit phase gates by preparing a single-photon source and single-photon injection into an optical cavity. Based on dispersive interactions, schemes [15,16] were proposed for realizing three-qubit phase gates, which cannot be generalized to the *N*-qubit case. It is not difficult to note two points: i.e., (i) these schemes require the transfer of quantum information between the atoms and the cavity modes, so that they are sensitive to the cavity decay in principle, and (ii) they also require individually addressing of atoms inside a cavity during the gate operations.

In this paper, we propose a theoretical scheme to implement an *N*-qubit conditional phase gate with *N* three-level atoms trapped in a high-*Q* optical cavity. One the one hand, the present scheme is operated in the large-detuning regime and no quantum information will be transferred from the atoms to the cavity mode since the cavity field is only virtually excited which leads a neglectable population of the cavity. Thus the scheme is insensitive to the cavity decay which releases the requirement of a strict strong-coupling regime at a certain extent. On the second hand, in contrast to previous schemes [13–15], our scheme does not require individual addressing of atoms and this can simplify experimental realization.

The energy-level structure of the identical *N* three-level atoms is shown in Fig. 1, which is a Λ configuration. Such an atomic-level structure has been proposed to implement quantum-controlled phase gates. For an experimental implementation, we can consider single cesium atoms [17] or calcium ions [18] trapped in an optical cavity, whose feasibility in the quantum information context has been widely demonstrated in experiments. The coherence time of the atomic ground levels $|g\rangle$ and $|s\rangle$ are long as that these states can be used to storage quantum information. In this paper, we encode the atomic ground states $|g\rangle$ and $|s\rangle$ as logic zero and one states—i.e., $|g\rangle \equiv |0\rangle$, and $|s\rangle \equiv |1\rangle$.

As shown in Fig. 1, the atomic transition $|g\rangle \leftrightarrow |e\rangle$ is nonresonantly coupled to a single quantized cavity mode and a classical field with the coupling constants g_c and Ω, respectively, while the atomic ground level $|s\rangle$ is not affected by the cavity and classical fields. Δ_1 and Δ_2 represent the atom-cavity and atom-laser detunings, respectively, which are associated with the corresponding atomic transition $|e\rangle \leftrightarrow |g\rangle$. In the absence of the classical field, the system interaction Hamiltonian can be given by

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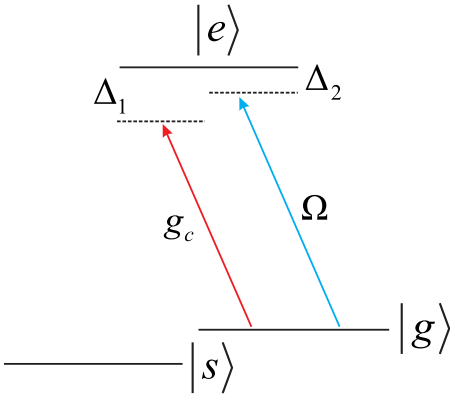

FIG. 1. (Color online) Relevant energy-level structure of the trapped atoms. The atomic transition $|e_i\rangle \leftrightarrow |g_i\rangle$ is nonresonantly coupled to the quantized cavity mode and the classical field with the coupling strengths g_c and Ω, respectively, while the ground state $|s\rangle$ is beyond the system dynamical evolution.

$$H_0 = \sum_{j=1}^{N} [g_c(a|e_j\rangle\langle g_j|e^{-i\Delta_1 t} + a^\dagger|g_j\rangle\langle e_j|e^{i\Delta_1 t}), \tag{1}$$

where the subscript j represents the jth trapped atom and $a^\dagger$ and a are the creation and annihilation operators associated with the quantized cavity mode, respectively. For simplicity, we assume that Δ_1, Δ_1, and g_c are real in the following.

In this paper, we consider the large atom-cavity detuning condition $\Delta_1 \gg g_c$. In this case, we can obtain an effective Hamiltonian

$$H_1 = \frac{g_c^2}{\Delta_1}(a^\dagger a S_- S_+ - a a^\dagger S_+ S_-), \tag{2}$$

where $S_+ = \Sigma_{j=1}^N |e_j\rangle\langle g_j|$ and $S_- = \Sigma_{j=1}^N |g_j\rangle\langle e_j|$. A consequence of the strong detuning between the cavity and atoms is that the population in the cavity mode will be very small if it is not initially populated, so that photons play a negligible role in the temporal evolution. Therefore, if the cavity field is in the vacuum state at the beginning, then no energy exchanges between the atoms and the cavity will happen, and the effective Hamiltonian in Eq. (2) can be reduced to [19]

$$H_2 = -\frac{g_c^2}{\Delta_1} S_+ S_-. \tag{3}$$

Then we take account into the weak external driving field which plays a central role in our scheme. In the presence of the classical field, the full Hamiltonian for the dynamics of the whole system can be redefined as

$$H_f = H_2 + H_3, \tag{4}$$

with

$$H_3 = \Omega(S_+ e^{-i\Delta_2 t} + S_- e^{i\Delta_2 t}), \tag{5}$$

which describes the interaction between the classical field and the atoms.

Based on the above basic model we now analyze how to adjust the atom-laser coupling strength (Ω) and detuning (Δ_2) to realize an N-qubit phase gate of the N trapped neutral atoms. For this, we denote that the atoms are initially prepared in the computational basis states $\{|i_1, i_2, \ldots, i_{N-1}, i_N\rangle\}$ where $i = g, s$. It is obvious that if the all atoms are in the state $|s\rangle$, the state $|s_1, s_2, \ldots, s_N\rangle|0\rangle_c$ does not evolve with the time since

$$H_f|s_1, s_2, \ldots, s_N\rangle|0\rangle_c = 0. \tag{6}$$

Therefore, we only need to consider the temporal evolution of the states with the form

$$|\psi\rangle_{i_1, i_2, \ldots, i_m} = |\cdots g_{i_1} \cdots g_{i_2} \cdots g_{i_m} \cdots\rangle, \tag{7}$$

where the qubits $i_1, i_2, \ldots, i_m (1 \le m \le N)$ are initially in the state $|g\rangle$ and other states are in the state $|s\rangle$.

Since the state $|s\rangle$ does not play any role in the interaction, the above Hamiltonian in Eq. (4) can be spanned by the finite basic states $\{|\phi_{m,n} \equiv \sqrt{n!(m-n)!/m!} J_{i_1, i_2, \ldots, i_m, +}^n |\psi\rangle_{i_1, i_2, \ldots, i_m}\}$, where $J_{i_1, i_2, \ldots, i_m, +} = \Sigma_{j=i_1, i_2, \ldots, i_m} |e_j\rangle\langle g_j|$. The state $|\phi_{m,n}\rangle$ describes that n atoms transfer into the state $|e\rangle$ from $|g\rangle$. As two special cases, $|C_{m,0}\rangle$ is given by Eq. (7) and $|C_{m,m}\rangle \equiv |\cdots e_{i_1} \cdots e_{i_2} \cdots e_{i_m} \cdots\rangle$ describes that the qubits $i_1, i_2, \ldots, i_m (1 \le m \le N)$ occupy the state $|e\rangle$ and other states are in the state $|s\rangle$. The effective interaction can be rewritten as

$$\begin{aligned} H_m = & -\frac{g_c^2}{\Delta_1} \sum_{n=0}^{m} [n(m+1) - n^2] |\phi_{m,n}\rangle\langle\phi_{m,n}| \\ & + \Omega \left(\sum_{n=0}^{m} \sqrt{m + n(m-1) - n^2} |\phi_{m,n+1}\rangle \right. \\ & \left. \times \langle \phi_{m,n}| e^{-i\Delta_2 t} + \text{H.c.} \right). \end{aligned} \tag{8}$$

To implement the N-qubit phase gate, we consider the parameters Ω and Δ_2 to satisfy the following conditions:

$$\Delta_2 = -N\frac{g_c^2}{\Delta_1}, \tag{9a}$$

$$\frac{g_c^2}{\Delta_1} \gg \sqrt{N}\Omega, \tag{9b}$$

which indicates that the effective operation of the present scheme depends on the number of atoms participating in the gate. The physical idea behind these equations is that Eq. (9a) results in the resonance transition between $|\phi_{N,1}\rangle$ and $|\phi_{N,0}\rangle$, while Eq. (9b) make other transitions largely detuned. Through this selective resonant interaction [20] we have the effective interaction

$$H_{eff} = \Omega_1 \sqrt{N} |\phi_{N,1}\rangle\langle\phi_{N,0}| + \text{H.c.} \tag{10}$$

for the initial state that all atoms in the state $|g\rangle$. This interaction offers a square-root improvement of resources when compared to the performance of a multiqubit gate with auxiliary qubits. For other initial states, similar to the above analysis, the effective interaction to the first order is dispersive from Eq. (9b). When we choose the interaction time t

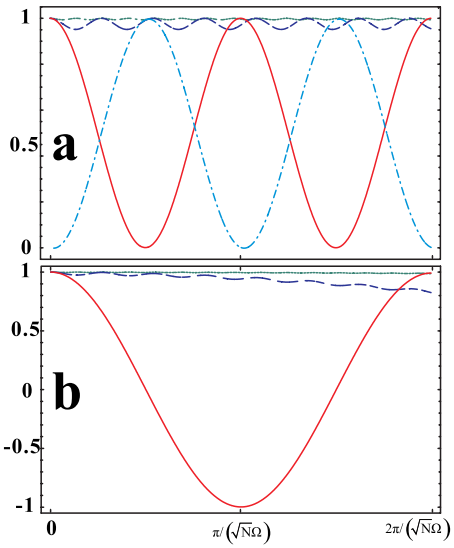

FIG. 2. (Color online) (a) Temporal evolution of the system states. The solid and dash-dotted lines represent the states $|\phi_{3,0}\rangle$ and $|\phi_{3,1}\rangle$, respectively, while the dashed and dotted lines represent the initial states $|g_1g_2s_3\rangle$ and $|g_1s_2s_3\rangle$, respectively. (b) Real parts of the coefficients for different initial system states. The solid, dashed, and dotted lines represent the initial states $|g_1g_2g_3\rangle$, $|g_1g_2s_3\rangle$, and $|g_1s_2s_3\rangle$, respectively. Other common parameters: $N=3$, $\Omega=g_c/500$, $\Delta_1=40g_c$, and $\Delta_2=-3g_c^2$.

$=\pi/(\sqrt{N}\Omega)$, the state $|g_1,g_2,\ldots,g_{N-1},g_N\rangle$ will acquire a $e^{i\pi}$ phase factor—i.e.,

$$|g_1,g_2,\ldots,g_{N-1},g_N\rangle \rightarrow -|g_1,g_2,\ldots,g_{N-1},g_N\rangle, \quad (11)$$

while other states in computational bases remain unchanged—i.e.,

$$|i_1,i_2,\ldots,i_{N-1},i_N\rangle \rightarrow |i_1,i_2,\ldots,i_{N-1},i_N\rangle, \quad (12)$$

where at least one qubit is initially in the state $|s\rangle$.

In order to confirm the effective interaction mentioned above and the validity of Eq. (11) we study numerically the atom-cavity system by using the full Hamiltonian $H=H_0+H_3$. For simplicity but without loss of generality, we consider a situation where only three atoms are trapped in the cavity ($N=3$) and they are initially prepared in the state space $\{|g\rangle,|s\rangle\}$. As depicted in Fig. 2(a), choosing the suitable parameters to satisfy Eqs. (9a) and (9b), we clearly observe Rabi oscillations with a period $\pi/(\sqrt{N}\Omega)$ between the atomic states $|\phi_{3,1}\rangle$ and $|\phi_{3,0}\rangle$ when the initial state is $|\phi_{3,0}\rangle$, while for the other initial states where at least one qubit is initially in the state $|s\rangle$, the system prefers to maintain its initial state. Therefore, on the one hand, the system evolution can be effectively described by the Hamiltonian H_{eff} and Eqs. (11) and (12) are feasible under the above approximations ($\Delta_1 \gg g_c$ and $g_c^2/\Delta_1 \gg \sqrt{N}\Omega$). On the other hand, as shown in Fig. 2(b), only when all atoms are initially prepared in the state $|g\rangle$ will the system acquire a phase π after a time $\pi/(\sqrt{N}\Omega)$, while the system remains unchanged for other initial states.

In addition, under the limits of large atom-cavity detuning and extremely weak external excitation, we note that the probability of having one or more photons in the cavity is smaller than 0.002. Thus one of the decoherence mechanisms, the cavity decay, should play a neglected role in the gate implementation. In fact, it is not difficult to note that the dominant source of decoherence in this proposal is the fact that the state $|\phi_{N,1}\rangle$ can decay spontaneously to $|\phi_{N,0}\rangle$ since the upper level is remarkably populated during the temporal evolution. For a quantificational description, we made some numerical simulations for the full system master equation by only taking into account the atomic spontaneous emission since the cavity decay plays a negligible role in the gate operation. The gate fidelity can be defined as $F \equiv \langle\Psi_{out}|\rho|\Psi_{out}\rangle$ in the present case, where $|\Psi_{out}\rangle$ is the ideal output system state and ρ is the actual output density matrix for a input state $|\Psi_{in}\rangle$. Figure 3 plots how the fidelity depends on the atomic spontaneous emission rate γ_s for a initial input state $|\Psi_{in}\rangle=\Sigma_{i=g,s}|i_1,i_2,\ldots,i_{N-1},i_N\rangle/\sqrt{N}$. We can see that we have a very high gate fidelity of 0.995 for a spontaneous emission rate of $\gamma_s=10^{-5}g_c$. For a set of actual cavity QED parameters [21] $(g_c,\kappa,\gamma_s)/2\pi=(110,14,2.6)$ MHz, $\Omega=g_c/100$, $\Delta_1=10g_c$, and $\Delta_2=-3g_c^2/\Delta_1$, the gate operation time $\pi/(\sqrt{N}\Omega)\sim 0.3$ μs and the fidelity is not very high ($\sim$0.9) due to a large photon loss. However, in future experiments, we may choose a kind of appropriate atom or ion that has a small spontaneous emission rate, leading to a high gate operation fidelity.

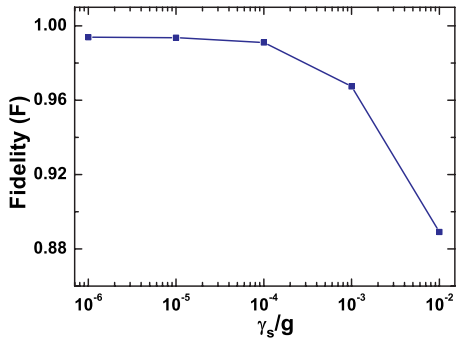

FIG. 3. (Color online) N-qubit-phase-gate fidelity (F) vs the atomic spontaneous emission rate (γ_s). The used parameters are $N=3$, $\Omega=g_c/500$, $\Delta_1=40g_c$, $\Delta_2=-3g_c^2/\Delta_1$, and $\kappa=0.1g$.

In summary, we have proposed a theoretical scheme to implement an N-qubit conditional phase gate with neutral atoms trapped in a large-detuning optical cavity. The present scheme is insensitive to the cavity decay and the N-qubit conditional phase gate can be implemented with a high fidelity because the cavity field is only virtually excited and no quantum information will be exchanged between the atoms and the cavity field. On the other hand, therefore, the scheme has a high feasibility with current or near-future experimental technology.

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