Inclusion of nonidealities in the continuous photodetection model

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Some nonideal effects as nonunit quantum efficiency, dark counts, dead time, and cavity losses that occur in experiments are incorporated within the continuous photodetection model by using the analytical quantum trajectories approach. We show that in standard photocounting experiments the validity of the model can be verified and the formal expression for the quantum jump superoperator can also be checked.

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I. INTRODUCTION

The continuous photodetection model (CPM) was proposed in the early 1980s in order to treat quantum optics situations in which a weak electromagnetic field enclosed in a cavity is continuously measured through the photocounting approach [1]. The theory has received considerable attention in the following years due to its new microscopic interpretation of the photodetection process [2–6], relation to the quantum trajectories approach [7–12], and several proposals for applications. Among them we find studies of photocounts statistics in diverse systems [13–17], quantum nondemolition measurements [18–20], implementation of measurement schemes [21–23], quantum-state preparation [24–28], quantum control via photodetection [29,30], and quantum computation [31].

The CPM is extensively discussed in the literature [3,18,32–34], so we shall mention only its main properties. The model, also referred as a theory, describes the field-state evolution during the photodetection process in a closed cavity and is formulated in terms of two fundamental *operations*, assumed to represent the *only* events taking place at each infinitesimal time interval: (i) The one-count operation, represented by the *quantum jump superoperator* (QJS), describes the detector's action on the field upon a single count, and the trace calculation over the QJS gives the probability per unit time for the occurrence of a detection. (ii) The *no-count* operation describes the field nonunitary evolution in absence of counts.

If one sets the formal expressions for these operations, all possible outcomes of a photocounting experiment can be predicted. For instance, the photocounts [1–3] and the waiting time [35–38] statistics are among the most common quantities to be studied both theoretically and experimentally. Moreover, the CPM conferred a new step in photodetection theories by allowing one to determine the field state after an arbitrary sequence of measurements, thus creating the possibility of controlling the field properties in real-time experiments [16,17,30].

Actually, the QJS is the main formal ingredient within the theory, since it also dictates the form of the no-count superoperator [1]. Two different models for the QJS were proposed *ad hoc*. The first one was proposed by Srinivas and Davies [1], the *SD model*, as

$$\hat{J}\rho = \lambda \hat{a}\rho\hat{a}^{\dagger}, \tag{1}$$

where ρ is the field density matrix, $\hat{a}$ and $\hat{a}^{\dagger}$ are the usual bosonic ladder operators, and λ is roughly the detector's ideal counting rate [1,39]. From the very beginning the authors [1] denounced the presence of some inconsistencies when the QJS (1) is employed for describing a real photodetection process; this point was also addressed in [39]. Nevertheless, this QJS is widely used in the literature [3,4,6,13,16,22,24,26–30].

The other proposal [34,40] assumes for the QJS an expression written in terms of the ladder operators $\hat{E}_{-}=(\hat{a}^{\dagger}\hat{a}+1)^{-1/2}\hat{a}$ and $\hat{E}_{+}=\hat{E}_{-}^{\dagger}$ (also known as *exponential phase operators* [41–45])

$$\hat{J}\rho = \lambda \hat{E}_{-}\rho\hat{E}_{+}. \tag{2}$$

In [39] we called the *E model* such a choice, to differentiate from the SD QJS (1). Besides eliminating the inconsistencies within the SD model, the use of the E model leads to different qualitative and quantitative predictions for several observable quantities. By an analysis of a microscopic model for the detector, it was recently shown that the QJS's (1) and (2) are particular cases of a general time-dependent *transition superoperator*, each one occurring in a particular regime of the detector experimental parameters [46,47]. Moreover, it was pointed out that by manipulating certain detector's accessible parameters one could engineer the form of the QJS, thus changing the dynamics of the photodetection, as well as the field state after a sequence of measurements.

A way to check the validity of the CPM and to decide which QJS better describes the phenomenon in practice can be accomplished through photocount experiments in a high-finesse cavity by comparing the results to the theoretical predictions. However, real detectors and cavities are far from ideal. So our first goal is to include into the CPM the effects of nonideality, such as quantum efficiency (QE), dark counts, detector's dead time, and cavity damping. Our second goal is

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to call attention to the fact that standard photodetection measurements could verify which of the QJS models actually prevails experimentally.

The plan of the paper is as follows. In Sec. II, we present a simple model, which enables us to include the effects of nonideality—QE and dark counts—into the CPM using the quantum trajectory approach. Then we calculate the main quantities characterizing the photodetection process—the photocounting and waiting time distributions. In Sec. II A we do this using the QJS (1), and in Sec. II B we repeat the same procedure for the E model. In Sec. III, we analyze the behavior of the lower moments of the above distributions in realistic situations and point out how one could decide about a QJS from experimental data. Section IV contains the conclusions. In Appendix A we treat the effects of dead time and cavity damping: we show that (i) cavity losses are not significant compared to the nonunit QE effect (ii) the dead-time effect leads to mathematical inconsistencies in the SD model, yet it is free of them in the E model, being, however, quite small compared to the QE effect. Appendix B contains some mathematical details concerning the evaluation of quantities of interest for different quantum states.

II. MODELS OF NONIDEAL PHOTODETECTORS

A. SD model

We consider a free electromagnetic monomodal field of frequency ω, enclosed in an ideal cavity together with a photodetector (in Appendix A we show that the cavity damping is not crucial if the detector has nonunit QE). The *unconditioned time evolution* (UTE) of the field in the presence of the detector—i.e. the evolution when the detector is turned on but the outcomes of the measurements are disregarded (not registered)—is described by the master equation [18,22,34]

$$\dot{\rho} = -i\omega(\hat{n}\rho - \rho\hat{n}) - \frac{\lambda}{2}(\hat{n}\rho + \rho\hat{n} - 2\hat{A}\rho), \tag{3}$$

where $\hat{A}\rho \equiv \hat{a}\rho\hat{a}^\dagger$ is a superoperator and $\hat{n}=\hat{a}^\dagger\hat{a}$ is the number operator. The first term stands for the free field evolution while the second describes the effect of the detector on the field due to their mutual interaction. The parameter λ is the field-detector coupling constant, roughly equal to the ideal counting rate [46,47].

To describe photocounting with QE η and finite dark counts rate λd (d is the ratio between the dark counts rate and the ideal photon counting rate), we assume the following expression for the QJS (cf. the expression resulting from the microscopic model in [47]):

$$\hat{J}\rho = \lambda(\eta\hat{A} + d)\rho. \tag{4}$$

It describes the action of the detector on the field upon a photodetection, and its trace gives the probability per unit time of the click. Actually, the microscopic model [46] suggests that Eq. (4) has a diagonal form in the Fock basis, but this will not be important here, since we shall be interested only in diagonal elements. The first term within the parentheses describes the absorption of a photon from the field with probability per unit time $\mathrm{Tr}[\eta\lambda\hat{A}\rho] = \eta\lambda\bar{n}$, where $\bar{n}$ is the field mean photon number—this means that the detector "sees" all the photons. The second term describes the occurrence of a dark count with field-independent probability density λd, and this event by itself does not modify the field state [the field state after a single dark count is $\lambda\rho d/\mathrm{Tr}(\lambda\rho d) = \rho$]. However, when both terms are present, the field state upon a detector's click becomes a mixture of both outcomes.

From the quantum trajectory approach and CPM [1,2,7], all the quantities related to photodetection can be calculated provided the complementary no-count superoperator $\hat{S}_t$ is known ($\hat{S}_t$ describes the action of the detector on the field during the time interval t without registered counts). Acting $\hat{S}_t$ on the initial field state ρ_0, the no-count state $\rho_S \equiv \hat{S}_t\rho_0$ obeys Eq. (3) when one subtracts the term (4) on the right-hand side (RHS) (see [7,8]). Moreover, as we are interested in calculating probabilities, we shall disregard phase factors $\exp(\pm i\omega\hat{n}t)$, since they are canceled in any trace evaluation. So the evolution equation of ρ_S is

$$\dot{\rho}_S = -\frac{\lambda}{2}(\hat{n}\rho_S + \rho_S\hat{n}) + \lambda q\hat{A}\rho_S - \lambda d\rho_S, \quad q \equiv 1-\eta. \tag{5}$$

Setting the transformation

$$\rho_S = e^{-d\lambda t}\hat{U}_t\rho_1, \quad \hat{U}_t\rho = e^{-\lambda t\hat{n}/2}\rho e^{-\lambda t\hat{n}/2} \tag{6}$$

in Eq. (5) we obtain a simple equation for ρ_1,

$$\dot{\rho}_1 = \lambda q e^{-\lambda t}\hat{A}\rho_1, \tag{7}$$

whose solution is

$$\rho_1 = \sum_{l=0}^{\infty}\frac{(q\phi_t)^l}{l!}\hat{a}^l\rho_0(\hat{a}^\dagger)^l \equiv \exp(q\phi_t\hat{A})\rho_0, \tag{8}$$

where

$$\phi_t = 1 - e^{-\lambda t}. \tag{9}$$

Thus the no-count superoperator is

$$\hat{S}_t\rho_0 = e^{-d\lambda t}\hat{U}_t(e^{q\phi_t\hat{A}}\rho_0). \tag{10}$$

The field UTE superoperator $\hat{T}_t$, defined as the solution to Eq. (3), is naturally given by setting $d=\eta=0$ in Eqs. (5) and (10)—i.e.

$$\hat{T}_t = \hat{U}_t(e^{\phi_t\hat{A}}\rho_0). \tag{11}$$

We introduced in Eq. (8) a compact notation for the infinite sum in terms of the exponential superoperator. We can deal with such superoperators as they were common operators, provided we use the "commutation relations"

$$\hat{A}\hat{U}_t = e^{-\lambda t}\hat{U}_t\hat{A}, \quad e^{y\hat{A}}\hat{U}_t = \hat{U}_t\exp(ye^{-\lambda t}\hat{A}), \tag{12}$$

obtained by expanding the superoperators in series.

Now we can calculate the m-count superoperator $\hat{N}_t(m)$, which describes the field state after m registered counts (whether real or dark ones) in the time interval $(0,t)$ and whose trace gives the probability for this event. It reads

$$\hat{N}_t(m)\rho = \int \cdots \int \hat{h}\rho, \tag{13}$$

where the integrals are evaluated over all the time intervals between the counts,

$$\int \cdots \int \equiv \int_0^t dt_m \int_0^{t_m} dt_{m-1} \cdots \int_0^{t_2} dt_1, \tag{14}$$

and the *conditioned density operator* is

$$\hat{h}\rho \equiv \hat{S}_{t-t_m}\hat{J}\hat{S}_{t_m-t_{m-1}}\hat{J}\cdots\hat{J}\hat{S}_{t_1}\rho. \tag{15}$$

Expanding the QJS (4) in Eq. (15) in terms of $\eta\hat{A}$ and d, one obtains a finite sum whose first term, proportional to d^0, describes the detection of m photons:

$$\hat{h}^{(0)} = (\lambda\eta)^m \hat{S}_{t-t_m}\hat{A}\cdots\hat{A}\hat{S}_{t_1} = (\lambda\eta)^m e^{-\lambda(t_1+t_2+\cdots+t_m)}\hat{S}_t\hat{A}^m. \tag{16}$$

After integrating Eq. (16) we obtain the first term in Eq. (13), describing the field state after the loss by absorption of m photons,

$$\hat{n}_t(m) \equiv \int \cdots \int \hat{h}^{(0)} = \hat{S}_t\frac{(\eta\phi_t\hat{A})^m}{m!}. \tag{17}$$

Calculating in a similar way the contribution of the terms with higher powers in d we arrive at the formula

$$\hat{N}_t(m) = \sum_{k=0}^{m}\frac{(d\lambda t)^k}{k!}\hat{n}_t(m-k) = \hat{S}_t\frac{(d\lambda t + \eta\phi_t\hat{A})^m}{m!}. \tag{18}$$

One can easily verify that the m-count superoperators (18) satisfy identically the fundamental relation [1,7]

$$\sum_{m=0}^{\infty}\hat{N}_t(m) = \hat{T}_t. \tag{19}$$

The factorial moments of the photocounts distribution are easily evaluated as

$$\begin{aligned}\overline{m\cdots(m-l)}_t &= \sum_{m=0}^{\infty} m\cdots(m-l)\mathrm{Tr}[\hat{N}_t(m)\rho] \\ &= \mathrm{Tr}[\hat{S}_t(d\lambda t+\eta\phi_t\hat{A})^{l+1}\exp(d\lambda t+\eta\phi_t\hat{A})\rho] \\ &= \mathrm{Tr}[\hat{U}_t(d\lambda t+\eta\phi_t\hat{A})^{l+1}e^{\phi_t\hat{A}}\rho].\end{aligned} \tag{20}$$

Thus we need to calculate the expression

$$\begin{aligned}\Phi_k(b,x) &\equiv \mathrm{Tr}[\hat{U}_b e^{x\hat{A}}\hat{A}^k\rho] = \sum_{n,l=0}^{\infty}\frac{(n+l+k)!}{n!l!}e^{-\lambda bn}x^l\rho_{n+l+k} \\ &= \sum_{n=k}^{\infty}\rho_n\frac{n!}{(n-k)!}(x+e^{-\lambda b})^{n-k},\end{aligned} \tag{21}$$

where $\rho_n = \langle n|\rho|n\rangle$. Evaluating

$$\Phi_k(t,\phi_t) = \sum_{n=0}^{\infty}\rho_n\frac{n!}{(n-k)!} \tag{22}$$

[see Eq. (9) for the expression of ϕ_t] we obtain general expressions for the lower factorial moments:

$$\bar{m}_t = d\lambda t + \eta\bar{n}\phi_t, \tag{23}$$

$$\overline{m(m-1)}_t = (d\lambda t)^2 + 2\eta\bar{n}d\lambda t\phi_t + (\eta\phi_t)^2\overline{n(n-1)}, \tag{24}$$

where $\bar{n}$ and $\overline{n(n-1)}$ are the factorial moments of the initial density operator.

Another measurable quantity we consider here is the *waiting time distribution*. It describes the probability density for registering two consecutive clicks separated by the time interval τ, under the condition that the first one occurred at time t. Its non-normalized form is

$$\mathrm{W}_t(\tau) = \mathrm{Tr}[\hat{J}\hat{S}_\tau\hat{J}\hat{T}_t\rho], \tag{25}$$

and the mean waiting time is

$$\bar{\tau} = \mathcal{N}^{-1}\int_0^T d\tau \mathrm{W}_t(\tau)\tau, \quad \mathcal{N} = \int_0^T d\tau \mathrm{W}_t(\tau), \tag{26}$$

where T is the time interval during which one evaluates the averaging in experiments. As will be shown in Sec. III, T is an important parameter due to the presence of dark counts. After straightforward manipulations, using the "commutation relations" (12), we obtain

$$\mathrm{W}_t(\tau) = e^{-d\lambda\tau}[\eta^2 e^{-\lambda(2t+\tau)}\Phi_2^W + \eta d e^{-\lambda t}(1+e^{-\lambda\tau})\Phi_1^W + d^2\Phi_0^W],$$

where

$$\Phi_k^W = \Phi_k\{t+\tau, 1-e^{-\lambda t}[\eta+(1-\eta)e^{-\lambda\tau}]\}. \tag{27}$$

In Appendix A we consider the dead-time effect and show that it cannot be consistently incorporated into the SD model, because the QJS (4) is an unbounded superoperator and the resulting counting probability is non-normalizable. This is just one more mathematical inconsistency [39] of the SD model. In Appendix B we evaluate the expression (21) for three kinds of states: coherent, number, and thermal.

B. E model

We now repeat the same procedures for the E model in which the QJS is

$$\hat{J}\rho = \lambda(\eta\hat{\varepsilon}+d)\rho, \tag{28}$$

where $\hat{\varepsilon}\rho \equiv \hat{E}_-\rho\hat{E}_+$. The probability per unit time for detecting a photon is $\eta\lambda(1-p_0)$, where $p_0 = \langle 0|\rho|0\rangle$, so the detector "sees" whether there is any photon in the cavity. In principle, the parameter λ is different from the one in the SD model, but in the context of this paper it will be always clear which one we are dealing with. The field UTE is described by an equation similar to Eq. (3), obtained by doing the substitution $\{\hat{a},\hat{a}^\dagger\} \to \{\hat{E}_-,\hat{E}_+\}$ in the nonunitary evolution (second term on the RHS). So the no-count state ρ_S obeys the equation

$$\dot{\rho}_S = -\frac{\lambda}{2}(\hat{\Lambda}\rho_S + \rho_S\hat{\Lambda}) + \lambda q\hat{\varepsilon}\rho_S - d\lambda\rho_S \tag{29}$$

[similar to Eq. (5)], where $\hat{\Lambda} \equiv \hat{E}_+\hat{E}_- = 1 - \hat{\Lambda}_0$, $\hat{\Lambda}_0 \equiv |0\rangle\langle 0|$. Setting the transformation

$$\rho_S = e^{-d\lambda t}e^{-\lambda t\hat{\Lambda}/2}\rho_1 e^{-\lambda t\hat{\Lambda}/2} \tag{30}$$

in Eq. (29) and using the property $\exp(\alpha\hat{\Lambda}) = \hat{\Lambda}_0 + e^{\alpha}\hat{\Lambda}$, we obtain the differential equation for ρ_1:

$$\dot{\rho}_1 = \lambda q e^{-\lambda t}(\hat{\Lambda}_0 + e^{\lambda t/2}\hat{\Lambda})\hat{E}_-(\hat{\Lambda}\rho_1\hat{\Lambda})\hat{E}_+(\hat{\Lambda}_0 + e^{\lambda t/2}\hat{\Lambda}). \tag{31}$$

We solve this equation by projecting it onto orthogonal subspaces spanned by projectors $\{\hat{\Lambda}, \hat{\Lambda}_0\}$. Moreover, since at the end we shall be interested only in calculating probabilities, we consider only the diagonal part in the Fock basis for the quantities of interest, thus disregarding terms whose trace is null, such as $\hat{\Lambda}\rho\hat{\Lambda}_0$.

Multiplying Eq. (31) by $\hat{\Lambda}$ on both sides we obtain

$$\frac{d}{dt}(\hat{\Lambda}\rho_1\hat{\Lambda}) = \lambda q\hat{\Lambda}\hat{E}_-(\hat{\Lambda}\rho_1\hat{\Lambda})\hat{E}_+\hat{\Lambda}, \tag{32}$$

whose solution is

$$\hat{\Lambda}\rho_1\hat{\Lambda} = \hat{\Lambda}(e^{q\lambda t\hat{\varepsilon}}\rho_0)\hat{\Lambda} \tag{33}$$

and we note again that all the composite superoperators, such as $\exp(y\hat{\varepsilon})$, are understood as power expansions. Now, multiplying Eq. (31) by $\hat{\Lambda}_0$ on both sides and using the solution (33) we get an equation for $\hat{\Lambda}_0\rho_1\hat{\Lambda}_0$,

$$\frac{d}{dt}(\hat{\Lambda}_0\rho_1\hat{\Lambda}_0) = \lambda q\hat{\Lambda}_0(e^{-\lambda t(1-q\hat{\varepsilon})}\hat{\varepsilon}\rho_0)\hat{\Lambda}_0, \tag{34}$$

with solution

$$\hat{\Lambda}_0\rho_1\hat{\Lambda}_0 = \hat{\Lambda}_0\left[\frac{1-q\hat{\varepsilon}\hat{R}_t}{1-q\hat{\varepsilon}}\rho_0\right]\hat{\Lambda}_0, \quad \hat{R}_t \equiv e^{-\lambda t(1-q\hat{\varepsilon})}. \tag{35}$$

Thus the *diagonal* form of the no-count superoperator, which we write just in terms of the projector $\hat{\Lambda}_0$ and the unit operator, is

$$\hat{S}_t\rho_0 = e^{-d\lambda t}\left[\hat{R}_t + \hat{\Lambda}_0\frac{1-\hat{R}_t}{1-q\hat{\varepsilon}}\hat{\Lambda}_0\right]\rho_0, \tag{36}$$

where we use the notation $(\Lambda_0\hat{Q}\Lambda_0)\rho \equiv \Lambda_0(\hat{Q}\rho)\Lambda_0$.

Repeating steps (16)–(18), we first obtain the conditioned density operator

$$\hat{h}^{(0)}\rho = e^{-d\lambda t}\left[\hat{R}_t + \hat{\Lambda}_0\frac{\hat{R}_{t_m} - \hat{R}_t}{1-q\hat{\varepsilon}}\hat{\Lambda}_0\right](\lambda\eta\hat{\varepsilon})^m\rho. \tag{37}$$

After evaluating the time integrals as in (17) we get

$$\hat{n}_t(m) = e^{-d\lambda t}\left[\left(1 - \hat{\Lambda}_0\frac{1}{1-q\hat{\varepsilon}}\hat{\Lambda}_0\right)\hat{R}_t\frac{(\lambda t\eta\hat{\varepsilon})^m}{m!} + \hat{\Lambda}_0\frac{(\lambda\eta\hat{\varepsilon})^m}{1-q\hat{\varepsilon}}\int_0^t dx\hat{R}_x\frac{x^{m-1}}{(m-1)!}\hat{\Lambda}_0\right]$$

for $m>0$ and $\hat{n}_t(0) = \hat{S}_t$. Finally, analogously to expression (18), we obtain the m-count superoperator

$$\hat{N}_t(m) = e^{-d\lambda t}\left\{\left(1 - \hat{\Lambda}_0\frac{1}{1-q\hat{\varepsilon}}\hat{\Lambda}_0\right)\hat{R}_t\frac{(\hat{J}t)^m}{m!} + \hat{\Lambda}_0\frac{1}{1-q\hat{\varepsilon}}\frac{(d\lambda t)^m}{m!}\hat{\Lambda}_0 + \hat{\Lambda}_0\frac{\lambda\eta\hat{\varepsilon}}{1-q\hat{\varepsilon}}\int_0^t dx\hat{R}_x\frac{[d\lambda t + \eta\hat{\varepsilon}\lambda x]^{m-1}}{(m-1)!}\hat{\Lambda}_0\right\}, \tag{38}$$

where the last term is zero for $m=0$. One can easily verify that the superoperator $\hat{N}_t(m)$, Eq. (38), satisfies relation (19).

After lengthy however straightforward calculations we obtain the following expressions for the initial factorial photocounts moments:

$$\bar{m}_t = d\lambda t + \eta\bar{n}(1-\Xi_1), \tag{39}$$

$$\overline{m(m-1)}_t = (d\lambda t)^2 + 2\eta\bar{n}d\lambda t(1-\Xi_1) + \eta^2[\overline{n(n-1)}(1-\Omega) - 2\bar{n}\lambda t\Xi_2], \tag{40}$$

where

$$\Xi_k \equiv \frac{1}{\bar{n}}\mathrm{Tr}\left[\frac{\hat{\varepsilon}^k}{1-\hat{\varepsilon}}\hat{R}_t^0\rho\right], \quad \hat{R}_t^0 \equiv \hat{R}_t(q=1), \tag{41}$$

$$\Omega \equiv \frac{2}{\overline{n(n-1)}}\mathrm{Tr}\left[\left(\frac{\hat{\varepsilon}}{1-\hat{\varepsilon}}\right)^2\hat{R}_t^0\rho\right]. \tag{42}$$

Using Eq. (25), the waiting time distribution density is found to be

$$\mathrm{W}_t(\tau) = e^{-d\lambda\tau}\left\{(\lambda d)^2[1 - \mathrm{Tr}(\hat{R}_t^0\rho)] + \mathrm{Tr}\left[\left(\hat{J}\hat{R}_\tau + \lambda d\hat{\Lambda}_0\frac{1-\hat{R}_\tau}{1-q\hat{\varepsilon}}\hat{\Lambda}_0\right)\hat{J}\hat{R}_t^0\rho\right]\right\}. \tag{43}$$

In Appendix A we show that the dead-time effect can be incorporated into the E model; however, its effect is quite small compared to the nonunit QE effect, so we disregard it in this paper. In Appendix B we obtain formulas for Eqs. (41)–(43) in terms of ρ_n and evaluate them for the coherent, number, and thermal states.

III. VERIFYING THE CPM

Basing ourselves on published experimental data [48] we chose the following numerical values for the model param-

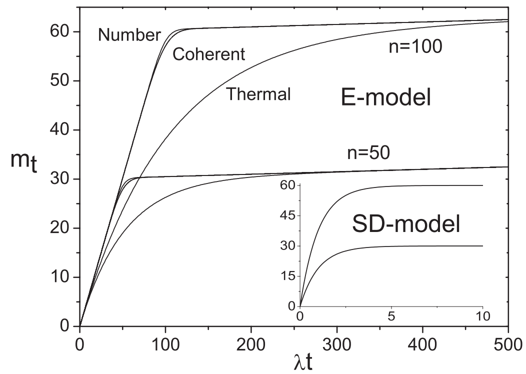

FIG. 1. Mean photocount number $\bar{m}_t$ in the E model for coherent, number, and thermal states (indicated in the figure; the lower curves are labeled analogously) as a function of time for two values of the initial photon number: the lower curves correspond to $\bar{n}=50$ and the upper to $\bar{n}=100$. In the inset we plot $\bar{m}_t$ for the SD model, which is independent from field state.

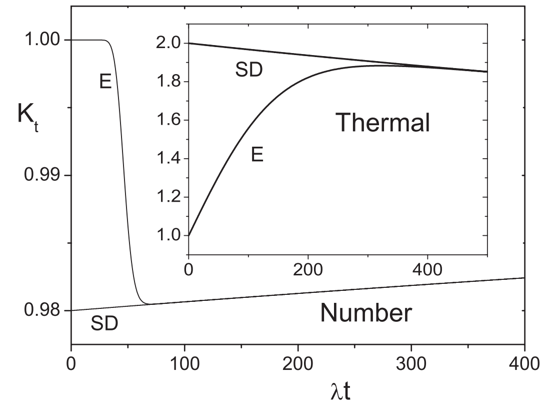

FIG. 2. Normalized second factorial moment K_t, Eq. (44), for the SD and E models (as indicated in the graph with abbreviations) for the number state (and the thermal state in the inset) for $\bar{n}=50$. For the coherent state one has $K_t=1$ at all times for both models.

eters: $\eta=0.6$ for the QE and $d=5\times10^{-3}$ for the dark count rate (normalized by the ideal counting rate). We do not attribute any fixed value to λ since our analysis will be given in terms of the dimensionless λt. Many photodetection quantities in different contexts were reported in, e.g., [1,3,17,32,34,35,38,39], so here we shall consider few of them that could check the validity of either, the SD or E model in photocounting experiments.

First we analyze the counting statistics. In Fig. 1 we plot $\bar{m}_t$ as function of λt for both models for two values of the initial mean photon number, $\bar{n}=50$ and 100. Initially, $\bar{m}_t$ increases steeply due to photon absorption, and after some time the growth turns linear with much smaller slope due to the dark counts. We call the time interval during which the photons are absorbed (representing the duration of the steep increase in the number of counts) the *effective counting time* t_E. In the E model t_E is proportional to the initial average photon number, contrary to the SD model [as seen from Fig. 1 and formulas (23) and (39)]. So the experimental analysis of the dependence of t_E on $\bar{n}$ seems to us a feasible way for verifying which model could hold in practice, because, according to the SD-model, t_E does not depend on $\bar{n}$. Moreover, one could also check the validity of each model by verifying whether $\bar{m}_t$ depends on the initial field state: in the SD model it is independent of the field state, while in the E model $\bar{m}_t$ is quite sensible to it: in Fig. 1 one sees a notable difference between thermal and coherent states, although not so much between number and coherent states. This can be explained by a great difference in the values of Mandel's Q factor [49] characterizing the statistics of photons in the initial state: it equals -1 and 0 for number and coherent states, respectively, whereas it is very large ($Q_{th}=\bar{n}$) for the thermal states with large mean numbers of photons.

Now we analyze the normalized second factorial moment

$$K_t \equiv \overline{m(m-1)}_t/\bar{m}_t^2 \tag{44}$$

for the same initial states with mean photon number $\bar{n}=50$. For the number and thermal states K_t as a function of λt is shown in Fig. 2, and for the coherent state we get $K_t=1$, so it is not plotted. In the asymptotic time limit and for the nonzero dark count rate, the same value $K_\infty\to1$ holds for both models; however, the transient is model dependent. In the SD model without considering dark counts K_t is time independent, written as $K=\overline{n(n-1)}/\bar{n}^2$ [$\bar{n}$ and $\overline{n(n-1)}$ correspond to the initial field state]; nevertheless, it depends on the initial field state: $K=2$ for the thermal state and $K=1-1/\bar{n}$ for the number state. By including the dark counts in the analysis this constant behavior is slowly modified as time goes on; see Fig. 2.

In the E model in the absence of dark counts K_t starts at the value

$$\lim_{t\to0} K_t = \frac{\mathrm{Tr}(\hat{\varepsilon}^2\rho)}{[\mathrm{Tr}(\hat{\varepsilon}\rho)]^2} = \frac{1-\rho_0-\rho_1}{(1-\rho_0)^2}, \tag{45}$$

which is exactly 1 for the number state and very close to 1 for the thermal state with the chosen values of $\bar{n}$. With the course of time, K_t attains the same values as for the SD model (for respective initial field states) when all the photons have been counted. By taking in account the dark count effect such a behavior is slightly modified, yet it is quite different from the behavior in the SD model, as shown in Fig. 2. This is another possible manner for verifying the applicability of the SD or E model.

We now turn our attention to the waiting time analysis. It is important to define the time interval in which we do the average: if one has a non zero dark counting rate, then by performing the average over a very large time interval, we shall always get for the mean waiting time the value $\bar{\tau}\sim(\lambda d)^{-1}$, which is nothing but the mean time interval between consecutive dark counts. Since experimentally the average is done over finite time intervals, we shall proceed in the same way: the mean waiting time for initial times, when the photon number is significative, is roughly $(\eta\lambda)^{-1}$ (because $\eta\lambda$ is the effective counting rate), so we shall take the average over a time interval $\nu=10(\eta\lambda)^{-1}$. This means that if one does not detect consecutive counts within the time ν, such a measurement will not contribute to the average. In an

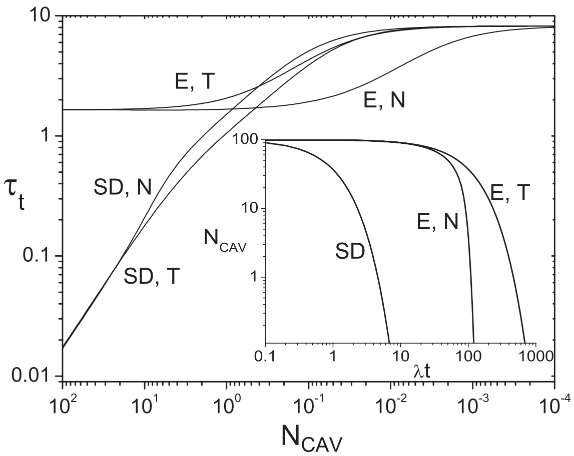

FIG. 3. Mean waiting time $\overline{\tau}_t$ as function of N_{CAV} for the number (N) and thermal (T) states for the SD and E models. While there are photons in the cavity $\overline{\tau}_t$ is constant for the E model, but increases with time for the SD model. In the inset we plot N_{CAV} as function of λt for these states (in the SD model N_{CAV} is state independent).

ideal case this procedure is not necessary because the probability for registering consecutive clicks separated by a large time interval is zero.

In Fig. 3 we plot the mean waiting time for the SD and E models for the number and thermal initial states (for the coherent state we obtain a curve almost identical to the one for a number state) with $\bar{n}=100$ as a function of the mean photon number in the cavity at the moment of the first click,

$$N_{CAV} = \mathrm{Tr}[\hat{n}\hat{T}_t\rho_0] = \begin{cases} \bar{n}e^{-\lambda t} & \text{for SD model} \\ \bar{n}\Xi_1 & \text{for E model.} \end{cases} \tag{46}$$

(For completeness, in the inset of Fig. 3 we plot N_{CAV} as a function of λt for both models.) For the E model, we see that when N_{CAV} becomes less than 1, the waiting time starts to increase dramatically due to the dominance of dark counts, which are much more rare events than absorption of photons. This is a drastic difference from the ideal case, in which no counts occur after all the photons having been absorbed, so the mean waiting time saturates at the inverse value of the counting rate, as shown in [39]. Moreover, from Fig. 3 one verifies that as long as there are photons in the cavity the mean waiting time is nearly time independent within the E model (and truly independent in the ideal case [39]) and does increase substantially in time for the SD model. This is another notable qualitative difference we suppose one could verify experimentally.

IV. SUMMARY AND CONCLUSIONS

In this paper we have generalized the continuous photodetection model through a careful quantum treatment of nonideal effects that are ubiquitous in experiments. We derived general expressions for the fundamental operations in the presence of nonunit quantum efficiency and dark counts, and calculated explicitly the photocounts and the waiting time probability distributions for initial coherent, number, and thermal field states. By calculating the first and second factorial moments of the photocounts and the mean waiting time, we showed that in standard photodetection experiments one could check the applicability of the QJS of the SD or E model. Namely, we indicated three different ways for revealing the actual QJS: (i) quantitatively, one should study the time dependence of the normalized second factorial photocounts moment. Qualitatively, we showed that the models can be also distinguished by measuring: (ii) whether the effective detection time depends on the initial average photon number in the cavity and (iii) whether the mean waiting time is modified as time goes on. To that end we have considered three different kinds of field in the cavity: the number, coherent, and thermal states; each one on its own permitted one to do comparisons between the two studied QJS's. Results with other kinds of fields could also be presented here, such as, for instance, the binomial state or the so-called squeezed state; however, no new physics related to the goals of the paper appears. A last remark, if the experimental data would depart significantly from the theoretical prediction, one should reconsider both models and try to look for alternative mechanisms to reproduce the outcomes.

In conclusion, we believe that our theoretical treatment could provide clues for an experimental verification of the CPM, contributing valuable insights into the quantum nature of the photodetection in cavities, as well as giving rise to the possibility of field-state manipulation through detector post-action on the field.

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APPENDIX A: CAVITY DAMPING AND DEAD TIME

First we include the effect of cavity damping in our treatment. In quantum optics experiments the background photon number is negligible, so we can model the cavity as a thermal reservoir with zero mean excitation number, described by the standard master equation [7]. Then the UTE equation in the SD model should be

$$\dot{\rho} = -i\omega(\hat{n}\rho - \rho\hat{n}) - \frac{\lambda}{2}(\hat{n}\rho + \rho\hat{n} - 2\hat{A}\rho) - \frac{\lambda c}{2}(\hat{n}\rho + \rho\hat{n} - 2\hat{A}\rho), \tag{A1}$$

where λc is the cavity damping rate. From it, following the steps of Sec. II we obtain the no-count superoperator

$$\hat{S}_t\rho_0 = e^{-d\lambda t}\hat{U}_t(e^{\tilde{q}\tilde{\phi}_t\hat{A}}\rho_0), \quad \tilde{\phi}_t = \frac{1}{p}(1 - e^{-\lambda p t}), \tag{A2}$$

$$p \equiv 1 + c, \quad \tilde{q} \equiv 1 - \eta + c = p - \eta. \tag{A3}$$

The value of c should be at the most of order of 10^{-1} in order to make viable the CPM. In this case we see that if one takes into account the QE drawback, the cavity damping does not modify substantially the resulting expressions. Therefore we disregard its effect in this paper.

The dead-time effect means that immediately after a click the detector is unable to register another count within a quite small time interval x, $\lambda x \ll 1$. In our framework we can describe this effect as the occurrence of the UTE during the time x immediately after the count, so the conditioned density operator $\hat{h}$, Eq. (15), becomes

$$\hat{h}\rho \equiv \hat{S}_{t-t_m-x}\hat{T}_x\hat{J}\hat{S}_{t_m-t_{m-1}-x}\hat{T}_x\hat{J}\cdots\hat{J}\hat{S}_{t_1}\rho = \hat{S}_{t-t_m}\hat{\Theta}\hat{J}\hat{S}_{t_m-t_{m-1}}\hat{\Theta}\hat{J}\cdots\hat{\Theta}\hat{J}\hat{S}_{t_1}\rho, \tag{A4}$$

where the *dead-time superoperator*, under the condition $\lambda x \ll 1$, is found to be

$$\hat{\Theta} \equiv \hat{S}_{-x}\hat{T}_x = \exp(x\hat{J}) \tag{A5}$$

for both SD and E models with respective QJS's.

In the SD model the resulting dead-time superoperator is unbounded, as well as $\hat{J}$, so it can bring some mathematical inconsistencies. For example, the m-count superoperator with dead-time effect is found to be

$$\hat{N}_t(m) = \hat{S}_t\frac{[d\lambda t + e^{d\lambda x}\hat{z}/(p\phi_z)]^m}{m!}, \tag{A6}$$

where

$$\hat{z} \equiv e^{\eta\phi_x\hat{A}} - e^{\eta\phi_x\hat{A}\exp(-p\lambda t)}. \tag{A7}$$

If one evaluates, for instance, $\mathrm{Tr}[\Sigma_m m^k\hat{N}_t(m)\rho]$, one will find a divergent result because $\hat{z}$ increases much faster than the decreasing terms.

In the E model $\hat{J}$ is a bounded superoperator, so the dead-time corrections will be of order $\eta\lambda x \ll 1$, much less relevant than the nonunit QE drawback.

APPENDIX B: EVALUATION OF TRACES

In this appendix we derive general expressions for both SD and E models and evaluate them for a general initial density operator $\rho = \Sigma\rho_n|n\rangle\langle n|$ (nondiagonal elements do not contribute to the trace in the expressions below). We shall analyze three particular field states: the coherent state

$$\rho_n = e^{-\bar{n}}\bar{n}^n/n!, \quad \overline{n(n-1)} = \bar{n}^2,$$

number state

$$\rho_n = \delta_{n,\bar{n}} \text{ with integer } \bar{n},$$

and thermal state

$$\rho_n = (1-\alpha)\alpha^n, \quad \alpha = \bar{n}/(\bar{n}+1), \quad \overline{n(n-1)} = 2\bar{n}^2.$$

In the SD model, formula (27) results in

(i) coherent state

$$\Phi_k^W = \bar{n}^k\exp[-\eta\bar{n}\phi_\tau e^{-\lambda t}], \tag{B1}$$

(ii) number state

$$\Phi_k^W = \frac{\bar{n}!}{(\bar{n}-k)!}(1-\eta\phi_\tau e^{-\lambda t})^{\bar{n}-k}, \tag{B2}$$

(iii) thermal state

$$\Phi_k^W = \frac{k!(1-\alpha)\alpha^k}{[1-\alpha(1-\eta\phi_\tau e^{-\lambda t})]^{k+1}}. \tag{B3}$$

Formula (46) yields $N_{CAV} = \bar{n}e^{-\lambda t}$ for all states.

In the E model we need to expand the superoperators as series of $\hat{\varepsilon}$ and evaluate the sums. For Eqs. (41) and (42) we obtain

$$\Xi_k = \frac{e^{-\lambda t}}{\bar{n}}\sum_{n,l,m=0}^{\infty}\frac{(\lambda t)^m}{m!}\rho_{n+l+m+k} = \frac{e^{-\lambda t}}{\bar{n}}\sum_{n,m=0}^{\infty}(n+1)\frac{(\lambda t)^m}{m!}\rho_{n+m+k}, \tag{B4}$$

$$\Omega = \frac{2e^{-\lambda t}}{\overline{n(n-1)}}\sum_{n,l,m=0}^{\infty}n\frac{(\lambda t)^m}{m!}\rho_{n+l+m+1} = \frac{e^{-\lambda t}}{\overline{n(n-1)}}\sum_{n,m=0}^{\infty}n(n-1)\frac{(\lambda t)^m}{m!}\rho_{n+m}. \tag{B5}$$

Regarding the evaluation of the mean waiting time (43), one needs to evaluate the expressions

$$\mathrm{Tr}\left[\Lambda_0\left(\frac{\hat{\varepsilon}^k}{1-q\hat{\varepsilon}}e^{\lambda\hat{\varepsilon}\beta}\rho\right)\Lambda_0\right] = \Psi_k(q,\beta), \tag{B6}$$

$$\mathrm{Tr}[\hat{\varepsilon}^k e^{\lambda\hat{\varepsilon}\beta}\rho] = \Psi_k(q=1,\beta), \tag{B7}$$

where

$$\Psi_k(q,\beta) \equiv \sum_{n,l=0}^{\infty}q^n\frac{(\lambda\beta)^l}{l!}\rho_{n+l+k}. \tag{B8}$$

For the thermal state we can evaluate the expressions obtained in Sec. II B directly using the "eigenstate" relation $\hat{\varepsilon}\rho = \alpha\rho$ and $\mathrm{Tr}[\Lambda_0\rho\Lambda_0] = \rho_0$.

For the coherent state we use the formula

$$\sum_{k=0}^{\infty}\frac{x^k}{[k!(k+n)!]} = \frac{I_n(2\sqrt{x})}{x^{n/2}},$$

where $I_k(x)$ is the modified Bessel function [50], to obtain

$$\Xi_k = \frac{e^{-\lambda t-\bar{n}}}{\bar{n}}\sum_{n=0}^{\infty}(n+1)\left(\frac{\bar{n}}{\lambda t}\right)^{(n+k)/2}I_{n+k}(2\sqrt{\bar{n}\lambda t}), \tag{B9}$$

$$\Omega = \frac{e^{-\lambda t-\bar{n}}}{\overline{n(n-1)}}\sum_{n=2}^{\infty}n(n-1)\left(\frac{\bar{n}}{\lambda t}\right)^{n/2}I_n(2\sqrt{\bar{n}\lambda t}), \tag{B10}$$

$$\Psi_k(q,\beta) = e^{-\bar{n}}\left(\frac{\bar{n}}{\lambda t}\right)^{k/2}\sum_{n=0}^{\infty}\left(\frac{\bar{n}q^2}{\lambda t}\right)^{n/2}I_{n+k}(2\sqrt{\bar{n}\lambda t}). \tag{B11}$$

The above series can be transformed in a finite integral using

$$\sum_{k=0}^{\infty}t^kI_{k+\nu}(z) = \frac{e^{tz/2}}{z^\nu}\int_0^z\tau^\nu e^{-t\tau^2/(2z)}I_{\nu-1}(\tau)d\tau,$$

valid for $\mathrm{Re}(\nu) > 0$.

For the number state, using $\sum_{k=0}^{n} x^k/k! = e^x \Gamma(n+1,x)/n!$, where $\Gamma(\alpha,x) = \int_x^\infty t^{\alpha-1} e^{-t} dt$ is the incomplete complementary gamma function [50], we obtain

$$\Xi_k = \frac{\Gamma(\bar{n}-k+2,\lambda t) - \lambda t \Gamma(\bar{n}-k+1,\lambda t)}{\bar{n}(\bar{n}-k)!}, \tag{B12}$$

$$\Omega = \frac{\Gamma(\bar{n}+1,\lambda t) - 2\lambda t \Gamma(\bar{n},\lambda t) + (\lambda t)^2 \Gamma(\bar{n}-1,\lambda t)}{\overline{n(n-1)}(\bar{n}-2)!}, \tag{B13}$$

$$\Psi_k(q,\beta) = q^{\bar{n}-k} e^{\lambda\beta/q} \frac{\Gamma(\bar{n}-k+1,\lambda\beta/q)}{(\bar{n}-k)!}. \tag{B14}$$

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