Semiconductor model for quantum-dot-based microcavity lasers

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When it comes to laser phenomena in quantum-dot-based systems, usually atomic models are employed to analyze the characteristic behavior. We introduce a semiconductor theory, originating from a microscopic Hamiltonian, to describe lasing from quantum dots embedded in microcavities. The theory goes beyond two-level atomic models and includes modified contributions of spontaneous and stimulated emission as well as many-body effects. An extended version, which incorporates carrier-photon correlations, provides direct access to the photon autocorrelation function and thereby on the statistical properties of the laser emission. In comparison to atomic models, we find deviations in the dependence of the input/output curve on the spontaneous emission coupling β. Modifications of the photon statistics are discussed for high-quality microcavities with a small number of emitters.

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I. INTRODUCTION

Recently it became possible to combine high-Q optical microcavities with quantum-dot (QD) emitters as the active material. The cavity design has been realized in the form of microdisks [1–3], micropillars [4,5], or photonic crystals [6,7]. All of them have been used to demonstrate high Q-factors together with small mode volumes, resulting in highly selective emission into few cavity modes, or even a single mode. Correspondingly, large β-values, which describe the fraction of the spontaneous emission into the laser mode, are realized, leading to low threshold currents required to achieve lasing with such devices. In the limit of a β-factor of unity the so-termed "thresholdless" laser is obtained as the jump in the input-output curve vanishes [8–11]. Thus for the case that β becomes close to unity, one must look at the intensity correlations of the outcoupled light to obtain information on the photon statistics and, based on this, to identify the transition from dominating spontaneous to dominating stimulated emission. Experimentally this is possible in a Hanbury-Brown and Twiss (HBT)-like coincidence measurement setup [12]. At the transition from spontaneous to stimulated emission, a kink or peak is visible in the measured intensity correlation function. The list of recent efforts and publications where the threshold behavior of QD or quantum-well-based laser devices is investigated, partly by using HBT-like setups, show the current strong topical interest not only in the quantum optics, but also the nanotechnology community [2–7,13,14].

In the literature, semiconductor QD-based laser devices are modeled almost without exception by considering atomic two- or multi-level systems, resulting either in a set of rate equations [9,10], or a master equation for the reduced density matrix [11,15,16]. To describe the statistical properties of the emission from the microcavity, such as the intensity correlation function, the latter approach has been used. Alternatively, for two-level systems the quantum regression theorem can be applied [17].

On the basis of underlying atomic models, it is, however, not possible to consider intrinsic semiconductor effects, such as a modified source term of spontaneous emission and Coulomb effects [18,19]. Also, unlike conventional four-level gas lasers, QD-based microcavity lasers usually do not operate at full inversion, which leads to considerable differences in the input-output curve of these devices. Especially if characteristic values are derived from measured data, such as the β-factor, one must be aware of the differences between an atomic and a semiconductor laser model. Therefore a semiconductor approach is desirable if QD-based devices are studied. A general semiconductor laser model based on a microscopic Hamiltonian has been previously used to study the influence of the carrier dynamics and many-body effects [20,21], lasing without inversion [22], and noise spectra [23], but did not include correlations required to determine the photon statistics.

In this paper we introduce a microscopic theory to calculate both the light output and the intensity correlation function of microcavity lasers with QDs as the active material. Our semiconductor approach naturally includes a modified source term of spontaneous emission, Pauli-blocking effects of the occupied states, as well as many-body Coulomb effects.

The paper is structured as follows: In the next section we show how the equations of motion follow from the Hamiltonian for the coupled carrier-photon system and explain how operator averages are classified within the scheme of the cluster expansion technique. In Sec. III the laser equations are derived and the atomic rate equation limit is discussed. Section IV is concerned with higher order correlations needed to obtain the photon statistics in terms of the autocorrelation function. A direct comparison of a reduced two-level version of our equations to the master equation is used to verify our approach in Sec. IV B. Analytic results for the autocorrelation function are considered in Sec. IV C. Finally, results from the coupled equations for laser dynamics and photon correlations are presented in Sec. V.

II. THEORETICAL MODEL

Starting from the semiconductor Hamiltonian for the interacting carrier-photon system, we derive coupled equations of motion for the relevant expectation values that describe the carrier and photon population dynamics and—in an extended version—also the photon statistics. We employ a quantized light field together with a treatment of the carrier system in second quantization.

A. Hamiltonian

The carrier part of the Hamiltonian contains the single-particle contributions for conduction and valence band carriers with the energies $\varepsilon_\nu^{c,v}$ and the two-particle Coulomb interaction,

$$H_{\mathrm{carr}}^0 = \sum_\nu \varepsilon_\nu^c c_\nu^\dagger c_\nu + \sum_\nu \varepsilon_\nu^v v_\nu^\dagger v_\nu, \tag{1}$$

$$H_{\mathrm{Coul}} = \frac{1}{2}\sum_{\alpha'\nu\nu'\alpha} [V^{cc}_{\alpha'\nu,\nu'\alpha} c^\dagger_{\alpha'} c^\dagger_\nu c_{\nu'} c_\alpha + V^{vv}_{\alpha'\nu,\nu'\alpha} v^\dagger_{\alpha'} v^\dagger_\nu v_{\nu'} v_\alpha] + \sum_{\alpha'\nu\nu'\alpha} V^{cv}_{\alpha'\nu,\nu'\alpha} c^\dagger_{\alpha'} v^\dagger_\nu v_{\nu'} c_\alpha. \tag{2}$$

Here, the Fermi operators c_ν ($c_\nu^\dagger$) annihilate (create) a conduction-band carrier in the state $|\nu\rangle$, the operators v_ν ($v_\nu^\dagger$) are the equivalent for valence band carriers. The explicit nature of the single particle states will be specified later, and the corresponding form of the Coulomb matrix elements $V^{\lambda\lambda'}_{\alpha'\nu,\nu'\alpha}$ is discussed in detail in Ref. [18].

The Hamiltonian for the free part of the electromagnetic field has the form

$$H_{\mathrm{ph}} = \sum_q \hbar\omega_q \left(b_q^\dagger b_q + \frac{1}{2}\right), \tag{3}$$

where the Bose operators b_q ($b_q^\dagger$) annihilate (create) a photon in the mode q. The index q stands for the fundamental cavity mode q_l with energy $\hbar\omega_{q_l}$, used for the laser emission, or other nonlasing modes with $q \neq q_l$.

The two-particle Hamiltonian for the light-matter interaction in dipole approximation is given by

$$H_{\mathrm{D}} = -i\sum_{q,\alpha\nu} (g_{q\alpha\nu} c_\alpha^\dagger v_\nu b_q + g_{q\alpha\nu} v_\alpha^\dagger c_\nu b_q) + \mathrm{H.c.} \tag{4}$$

The light-matter coupling strength $g_{q\alpha\nu}$ is determined by the overlap of the mode function of the electromagnetic field with index q and the single-particle wave functions belonging to the states $|\alpha\rangle$ and $|\nu\rangle$, see Ref. [18]. We use the approximation of equal wave-function envelopes for conduction- and valence-band carriers, resulting in diagonal transitions between the corresponding conduction- and valence-band states, i.e., $q_{q\alpha\nu} = g_{q\nu}\delta_{\alpha\nu}$. The total Hamiltonian is the sum of all discussed contributions:

$$H = H_{\mathrm{carr}}^0 + H_{\mathrm{Coul}} + H_{\mathrm{ph}} + H_{\mathrm{D}}. \tag{5}$$

B. Factorization and truncation scheme

Using Heisenberg's equations of motion together with the Hamiltonian of the interacting system, we obtain the time evolution of the carrier and photon operators. From this coupled equations for operator averages, like the carrier population or photon number in the cavity, are derived. Occurring operator averages are classified into singlets, doublets, triplets, quadruplets, etc., according to the number of particles they involve. Considering interband transitions, it must be borne in mind that the excitation of one electron is described as the destruction of a valence band carrier and the creation of a conduction band carrier. For the corresponding interaction processes, a photon operator is connected to two carrier operators [18,24]. This fact is used to classify mixed expectation values with photon and carrier operators. For example, the electron population $f_\nu^e = \langle c_\nu^\dagger c_\nu\rangle$ is a singlet contribution, the source term of spontaneous emission $\langle c_\alpha^\dagger v_\alpha v_\nu^\dagger c_\nu\rangle$ and the photon-assisted polarization $\langle b_q^\dagger v_\nu^\dagger c_\nu\rangle$ are doublet terms.

In the following, N-particle averages, schematically denoted as $\langle N\rangle$ and containing $2N$ carrier operators or an equivalent replacement of photon operators, are factorized into all possible combinations of averages involving one up to $(N-1)$-particle averages. For the difference between the full operator average and this factorization, we introduce a correlation function of order N, denoted as $\delta\langle N\rangle$. Schematically the factorization of singlets, doublets, triplets, and quadruplets is given by

$$\langle 1\rangle = \delta\langle 1\rangle, \tag{6a}$$

$$\langle 2\rangle = \langle 1\rangle\langle 1\rangle + \delta\langle 2\rangle, \tag{6b}$$

$$\langle 3\rangle = \langle 1\rangle\langle 1\rangle\langle 1\rangle + \langle 1\rangle\delta\langle 2\rangle + \delta\langle 3\rangle, \tag{6c}$$

$$\langle 4\rangle = \langle 1\rangle\langle 1\rangle\langle 1\rangle\langle 1\rangle + \langle 1\rangle\langle 1\rangle\delta\langle 2\rangle + \langle 1\rangle\delta\langle 3\rangle + \delta\langle 2\rangle\delta\langle 2\rangle + \delta\langle 4\rangle. \tag{6d}$$

Looking at the last equation, the first four terms on the right-hand side represent all possible combinations of singlets, singlets and doublets, singlets and triplets, and doublets, respectively. The last term is the remaining quadruplet correlation function. Continuing the series (6a)–(6d) leads to quintuplet terms and so on. Note that singlets cannot be factorized any further.

From Heisenberg's equations of motion an infinite hierarchy arises due to the two-particle parts of the Hamiltonian, here the Coulomb and the light-matter interaction. The essential idea of what has become known as the *cluster expansion method* [25] is to replace all occurring operator expectation values $\langle N\rangle$ according to Eqs. (6) so that equations of motion for the correlation functions $\delta\langle N\rangle$ are obtained. Then the hierarchy of correlation functions is truncated rather than the hierarchy of expectation values itself. This allows the consistent inclusion of correlations up to a certain order in all of the appearing operator expectation values. This truncation procedure has previously been used to describe the luminescence dynamics of quantum wells [26,27] and QDs [18,28,29]. If the hierarchy is truncated at the level of two-particle correlation functions, the so-called semiconductor luminescence equations (SLE) for the coupled carrier and photon populations emerge, which consistently include correlations up to the doublet level.

The purpose of this paper is the application of the cluster expansion method to the coupled carrier-photon system in order to describe the threshold behavior of semiconductor-QD lasers. The extensions of the SLE presented in this paper are twofold. By discriminating between the emission channels into the laser mode and the nonlasing modes and by explicitly considering the pump process, a semiconductor laser theory can be formulated, which contains the familiar rate equations as a limiting case. Furthermore, the hierarchy of coupled equations is extended to photon correlations (corresponding to the quadruplet-level) in order to access the photon statistics of the light emission. In this context it is important to consider the correct terms for the spontaneous and stimulated emission in order to relate the β-factor to the threshold properties and especially to the height of the "jump" in the input-output curve. The correct source term already appears on the doublet level and is, therefore, part of a laser theory obtained from extending the SLE.

An important motivation for our work is the observation that for large β values, typical for state-of-the-art microcavity systems, the "jump" in the input-output curve broadens and cannot be used for a clear identification of the onset of coherent light emission. The statistical properties of the light emission can be described in terms of the autocorrelation function at zero delay time, $g^{(2)}(\tau=0)=(\langle n^2\rangle-\langle n\rangle)/\langle n\rangle^2$. Here, $n=b^\dagger b$ is the photon number operator for the laser mode. By means of Eq. (6d), we can introduce $\delta\langle b^\dagger b^\dagger bb\rangle=\langle b^\dagger b^\dagger bb\rangle-2\langle b^\dagger b\rangle^2$. Since $\langle b\rangle=\langle b^\dagger\rangle=0$ for a system without coherent excitation, only a factorization into doublets is possible. The factor of 2 arises from the two realizations for this factorization. Then the autocorrelation function can be written in terms of a quadruplet correlation function:

$$g^{(2)}(\tau=0)=2+\frac{\delta\langle b^\dagger b^\dagger bb\rangle}{\langle b^\dagger b\rangle^2}. \tag{7}$$

C. Equations of motion

For the dynamical evolution of the photon number $\langle b_q^\dagger b_q\rangle$ in the mode q and the carrier populations $f_\nu^e=\langle c_\nu^\dagger c_\nu\rangle$, $f_\nu^h=1-\langle v_\nu^\dagger v_\nu\rangle$, the contribution of the light matter interaction H_{D} in the Heisenberg equations of motion leads to

$$\left(\hbar\frac{d}{dt}+2\kappa_q\right)\langle b_q^\dagger b_q\rangle=2\,\mathrm{Re}\sum_{\nu'}|g_{q\nu'}|^2\langle b_q^\dagger v_{\nu'}^\dagger c_{\nu'}\rangle, \tag{8}$$

$$\hbar\frac{d}{dt}f_\nu^{e,h}|_{\mathrm{opt}}=-2\,\mathrm{Re}\sum_q|g_{q\nu}|^2\langle b_q^\dagger v_\nu^\dagger c_\nu\rangle. \tag{9}$$

Note that we have scaled $\langle b_q^\dagger v_\nu^\dagger c_\nu\rangle\to g_{q\nu}\langle b_q^\dagger v_\nu^\dagger c_\nu\rangle$ to have the modulus of the coupling matrix elements appear. In Eq. (8) we have introduced the cavity loss rate $2\kappa_q$. For the laser mode, this is directly connected to the Q-factor of the fundamental cavity mode, $Q=\hbar\omega/2\kappa$. The dynamics of the photon number in a given mode is determined by the photon-assisted polarization $\langle b_q^\dagger v_\nu^\dagger c_\nu\rangle$ that describes the expectation value for a correlated event, where a photon in the mode q is created in connection with an interband transition of an electron from the conduction to the valence band. The sum over ν involves all possible interband transitions from various QDs. The dynamics of the carrier population in Eq. (9) is governed by contributions of photon-assisted polarizations from all possible modes q. The influence of carrier-carrier interaction on the carrier dynamics is discussed below.

The dynamical equation for the photon-assisted polarization is given by

$$\begin{aligned}\left[\hbar\frac{d}{dt}+\kappa_q+\Gamma+i(\tilde{\varepsilon}_\nu^e+\tilde{\varepsilon}_\nu^h-\hbar\omega_q)\right]\langle b_q^\dagger v_\nu^\dagger c_\nu\rangle\\ =f_\nu^e f_\nu^h-(1-f_\nu^e-f_\nu^h)\langle b_q^\dagger b_q\rangle\\ +i(1-f_\nu^e-f_\nu^h)\sum_\alpha V_{\nu\alpha\nu\alpha}\langle b_q^\dagger v_\alpha^\dagger c_\alpha\rangle\\ +\frac{1}{g_{q\nu}}\sum_\alpha g_{q\alpha}C^{\mathrm{x}}_{\alpha\nu\nu\alpha}+\delta\langle b_q^\dagger b_q c_\nu^\dagger c_\nu\rangle-\delta\langle b_q^\dagger b_q v_\nu^\dagger v_\nu\rangle.\end{aligned} \tag{10}$$

The free evolution of $\langle b_q^\dagger v_\nu^\dagger c_\nu\rangle$ is determined by the detuning of the QD transitions from the cavity resonances. In a semiconductor, the source term of spontaneous emission is described by an expectation value of four carrier operators $\langle c_\alpha^\dagger v_\alpha v_\nu^\dagger c_\nu\rangle$, see Ref. [18]. For uncorrelated carriers, the Hartree-Fock factorization of this source term leads to $f_\nu^e f_\nu^h$, which appears as the first term on the right-hand side of Eq. (10). Corrections to this factorization are provided by the Coulomb and light-matter interaction between the carriers and are included in $C^{\mathrm{x}}_{\alpha'\nu\nu'\alpha}=\delta\langle c_{\alpha'}^\dagger v_\nu^\dagger c_{\nu'} v_\alpha\rangle$.

A restriction of the source term of spontaneous emission to the factorization approximation is justified in certain situations, such as the laser applications considered here. High carrier densities efficiently screen the Coulomb interaction between the carriers and lead to strong dephasing that directly suppresses correlations [18]. The feedback of the laser cavity can support strong carrier-photon correlations that dominate over carrier-carrier correlations. The calculation of carrier-correlation contributions C^{x} to the source term of spontaneous emission is a central issue of Ref. [18], and a discussion about the sensitivity of C^{x} to dephasing can be found there. Note that in atomic systems, the spontaneous emission is always linear in the excited-state population. This difference to semiconductor systems is the origin of interesting new effects in QDs, which are unknown in atomic systems [19,28].

The stimulated emission-absorption term in Eq. (10), which is proportional to the photon number $\langle b_q^\dagger b_q\rangle$ in the mode q, provides feedback due to the photon population in the cavity. Hartree-Fock (singlet) contributions of the Coulomb interaction lead to the appearance of renormalized energies $\tilde{\varepsilon}$ and to the interband exchange contribution in Eq. (10) that couples the photon-assisted polarizations from different states α. The last two terms in Eq. (10) are carrier-photon correlations that are discussed in Sec. IV. Carrier-carrier and carrier-phonon interaction lead to dephasing, which corresponds to a damping of the photon-assisted transition amplitude. While the used formalism allows for a microscopic evaluation of these effects [27], this is not the purpose of this paper and dephasing is included via a phenomenological damping constant Γ.

Furthermore, in the above equations we have used the fact that in the incoherent regime polarizationlike averages of the form $\langle v_\nu^\dagger c_\nu \rangle$ vanish [26].

III. LASER EQUATIONS

To formulate the laser theory for QDs in optical microcavities, we have to specify the electronic structure of the system as well as the mode structure of the resonator. We consider QDs with two confined shells, referred to as *s*- and *p*-shell according to the in-plane symmetry of the corresponding single-particle eigenstates, which appear energetically below a continuum of delocalized wetting layer (WL) states. The QDs are embedded in a microcavity, which provides one (potentially degenerate) fundamental mode with a large quality (Q-) factor that is in resonance with the QD *s*-shell emission. Higher cavity modes are assumed to be energetically well-separated from the fundamental mode, and a continuum of leaky modes is used to define the spontaneous emission coupling β, i.e., only a fraction of the spontaneous emission at the laser transition energy involves the laser mode.

A. Dynamical equations

In the following scheme, several assumptions are included, which are justified by possible experimental conditions and which lead to a convenient formulation of the theory. They provide no principle limitations and their use can be circumvented at the cost of more complicated analytical and numerical formulations. (i) We assume that optical processes involving the laser mode (stimulated and spontaneous emission as well as photon reabsorption) are exclusively connected to the *s*-shell transitions. In this case, higher shells and WL states contribute only to the carrier dynamics. (ii) Ultrafast carrier scattering processes in QDs have been predicted in recent studies of carrier-carrier [30] and carrier-phonon [31] interaction. Based on these grounds, we assume that the carrier system is close to equilibrium, so that scattering processes can be described in relaxation-time approximation [30]. (iii) To include the simplest possible pump process, we consider carrier generation in the *p*-shell at a given rate P. This can be traced back either to resonant optical pumping in connection with rapid dephasing, or to carrier injection into the delocalized WL or bulk states and fast successive carrier capture and relaxation processes. (iv) For the nonlasing modes, stimulated emission and reabsorption of photons is neglected, which corresponds to a situation where photons spontaneously emitted into nonlasing modes rapidly leave the cavity. In the case of strong dephasing (provided by efficient carrier scattering) it is then possible to analytically solve the equation for the corresponding photon-assisted polarization and to introduce a rate of spontaneous emission into the nonlasing modes. (v) It has been shown in Ref. [28] that the major emission into the fundamental mode is due to those QDs, which are on resonance, whereas slightly detuned dots hardly contribute. Therefore in the following we can consider to a good approximation only those emitters in resonance with the fundamental mode, rather than using an inhomogeneously broadened sample of QDs. For a system of identical dots, which are on resonance with the nondegenerate fundamental mode of the cavity, the occurring energy differences with the laser mode, $\tilde{\varepsilon}_\nu^e + \tilde{\varepsilon}_\nu^h - \hbar\omega_{q_l}$, drop out in the equations of motion in the following.

So far we have derived the fundamental equations for the carrier and photon dynamics, which, on the singlet-doublet level, are known as semiconductor luminescence equations. In order to describe a pumped laser system, we must incorporate carrier generation and the β-factor into the theory, as well as deal with the correlations appearing in Eq. (10).

Regarding the treatment of many-body Coulomb effects, one can distinguish between two limiting cases. In the high-carrier density and high-temperature regime, the WL states accommodate a substantial carrier density that screens the Coulomb interaction between the QD carriers. At the same time, the Coulomb interaction between QD and WL carriers leads to broadening and energy shifts of the QD transitions. Calculations of QD gain spectra in this regime are the subject of Ref. [32]. In the low-temperature regime that was recently studied in several experiments [5,28], the population of the WL states is expected to be marginal. The remaining Coulomb interaction between the QD carriers leads to intra- and interband interaction effects and will be summarized in an effective transition energy and oscillator strength for the coupling to the laser mode.

While the main focus of this paper is on carrier-photon and photon-photon correlations in QD lasers, also the explicit inclusion of carrier-carrier Coulomb correlations, in terms of the cluster expansion or with alternative methods, is desirable. While this is an ongoing research subject, our approximations are supported by several arguments: (i) Since dominantly QDs with transitions in resonance with the high-Q laser mode contribute to the emission, possible line shifts are not explicitly included subsequently. (ii) In calculations of the source term of spontaneous emission for QDs *in microcavities* [28] it turned out that the role of C^{x} is small, when typical material parameters and high Q-values are considered. In this case the strong feedback of the cavity dominates over the correlations and the singlet factorization $f_\nu^e f_\nu^h$ provides a good approximation for the source term. Thus, in the following calculations, C^{x} is not included.

Under the discussed conditions, the equation of motion for the photon-assisted polarization of the laser mode takes the form

$$\left(\hbar\frac{d}{dt} + \kappa + \Gamma\right)\langle b^\dagger v_s^\dagger c_s\rangle = f_s^e f_s^h - (1 - f_s^e - f_s^h)\langle b^\dagger b\rangle + \delta\langle b^\dagger b c_s^\dagger c_s\rangle - \delta\langle b^\dagger b v_s^\dagger v_s\rangle, \tag{11}$$

where, from now on, the index $q = q_l$ is omitted for the laser mode. In the equation of motion for the photon-assisted polarization of the nonlasing modes, the negligible photon population allows the omission of the feedback term and carrier-photon correlations,

$$\left(\hbar\frac{d}{dt}+\kappa_q+\Gamma+i(\tilde{\varepsilon}_s^e+\tilde{\varepsilon}_s^h-\hbar\omega_q)\right)\langle b_q^\dagger v_s^\dagger c_s\rangle|_{q\neq q_l}=f_\nu^e f_\nu^h. \tag{12}$$

As a result, Eq. (12) can be solved in the adiabatic limit and the part $q\neq q_l$ of the sum in Eq. (9) can be evaluated, yielding a time constant $\tau_{\rm nl}$ for the spontaneous emission into nonlasing modes according to the Weisskopf-Wigner theory [17],

$$\frac{2}{\hbar}\,\mathrm{Re}\sum_{q\neq q_l}\frac{|g_{qs}|^2}{\kappa_q+\Gamma+i(\tilde{\varepsilon}_s^e+\tilde{\varepsilon}_s^h-\hbar\omega_q)}=\frac{1}{\tau_{\rm nl}}. \tag{13}$$

In a laser theory, one typically distinguishes between the rate of spontaneous emission into lasing and nonlasing modes, $1/\tau_l$ and $1/\tau_{\rm nl}$, respectively. Both rates add up to the total spontaneous emission rate $1/\tau_{\rm sp}$. Then the spontaneous emission factor is given by

$$\beta=\frac{\frac{1}{\tau_l}}{\frac{1}{\tau_{\rm sp}}}=\frac{\frac{1}{\tau_l}}{\frac{1}{\tau_l}+\frac{1}{\tau_{\rm nl}}} \tag{14}$$

and the rate of spontaneous emission into nonlasing modes can be expressed according to

$$\frac{1}{\tau_{\rm nl}}=\frac{1-\beta}{\tau_{\rm sp}}. \tag{15}$$

For a further discussion of the time constants, see the Appendix.

From Eq. (9) one can now determine the population dynamics in the s-shell. For the spontaneous emission into nonlasing modes, the adiabatic solution of Eq. (12) is used according to Eqs. (13) and (15). Furthermore, we include a transition rate of carriers from the p- to the s-shell in relaxation-time approximation, $R_{p\to s}^{e,h}=(1-f_s^{e,h})f_p^{e,h}/\tau_{\rm r}^{e,h}$, and $g\equiv g_{q_l,s}$ to obtain

$$\frac{d}{dt}f_s^{e,h}=-2|g|^2\,\mathrm{Re}\langle b^\dagger v_s^\dagger c_s\rangle-(1-\beta)\frac{f_s^e f_s^h}{\tau_{\rm sp}}+R_{p\to s}^{e,h}. \tag{16}$$

Here the first term describes the carrier dynamics due to the interaction with the laser mode, while the second term represents the loss of carriers into nonlasing modes. The blocking factor $1-f_s^{e,h}$ in $R_{p\to s}^{e,h}$ ensures that the populations cannot exceed unity.

The carrier dynamics for the p-shell can be written as

$$\frac{d}{dt}f_p^{e,h}=P(1-f_p^e-f_p^h)-\frac{f_p^e f_p^h}{\tau_{\rm sp}^p}-R_{p\to s}^{e,h}, \tag{17}$$

where a carrier generation rate P is included together with the Pauli-blocking factor $(1-f_p^e-f_p^h)$. The second term describes spontaneous recombination of p-shell carriers and the third contribution is the above-discussed carrier relaxation.

The resulting set of equations (16) and (17), together with Eqs. (8) and (11) allows one to calculate the coupled dynamics for the photon number and the carrier population. It turns out that the inclusion of the carrier-photon correlations, which are given by the last two terms in Eq. (11), does not change the results for the input-output characteristics, shown below in Fig. 3. Neglecting the carrier-photon correlations in Eq. (11), the resulting set of equations corresponds to a truncation of the hierarchy on the doublet level. Note, however, that the inclusion of these correlations in Eq. (11) is of critical importance if $\langle b^\dagger v_\nu^\dagger c_\nu\rangle$ is used for the calculation of higher-order correlation functions, cf. Sec. IV.

B. Rate equation limit

In the following we show how the frequently used atomic rate equation model [9,11] can be obtained from the above developed semiconductor theory as a limiting case. (i) The semiconductor specific source term of spontaneous emission $f_\nu^e f_\nu^h$ in Eqs. (11), (16), and (17) is replaced by the electron population f_ν^e. This happens because successive destruction of more than one carrier always yields zero in the case of a two-level system, where only one electron is present per independent emitter, in which case we use $c_\nu c_\nu=v_\nu v_\nu=c_\nu v_\nu=0$. Then the source of spontaneous emission $\langle c_\alpha^\dagger v_\alpha v_\nu^\dagger c_\nu\rangle$ arising in Eq. (10) can be found to reduce to $\langle c_\alpha^\dagger c_\alpha\rangle=f_\alpha^e$. (ii) Full inversion of the laser transition is assumed, $1-f_s^h=\langle v_s^\dagger v_s\rangle=0$, which is usually well-justified for atomic four-level laser systems (but not for QDs). (iii) The adiabatic solution of Eq. (11) is inserted into Eq. (8). Introducing the number of excited emitters $\bar{N}=f_s^e N$, where N is the total number of emitters that arises from the sum over all states in Eq. (8), we find

$$\frac{d}{dt}\langle b^\dagger b\rangle=-2\kappa\langle b^\dagger b\rangle+\frac{\beta}{\tau_{\rm sp}}(1+\langle b^\dagger b\rangle)\bar{N}. \tag{18}$$

The photon population is determined by the interplay of the cavity losses 2κ and the photon generation due to spontaneous processes $\propto\bar{N}$ and stimulated processes $\propto\langle b^\dagger b\rangle\bar{N}$. For the number of excited emitters we obtain

$$\frac{d}{dt}\bar{N}=-\frac{\beta}{\tau_{\rm sp}}\langle b^\dagger b\rangle\bar{N}-\frac{1}{\tau_{\rm sp}}\bar{N}+P, \tag{19}$$

where, for atomic laser systems quite common, a constant pumping $NR_{p\to s}^{e,h}=P$ has been used, which describes the carrier-generation rate in the laser-transition level. The carrier recombination is determined by the stimulated emission into the laser mode $\propto\beta/\tau_{\rm sp}=1/\tau_l$, and by the spontaneous emission $\propto 1/\tau_{\rm sp}$ into all available modes.

For a direct comparison with the semiconductor model, results of the rate equations (18) and (19) for the input-output curves and various values of the β-factor are shown in Fig. 1. We use a typical set of parameters: $\tau_{\rm sp}=50$ ps (spontaneous emission of QDs enhanced by the Purcell effect), $N=\tilde{N}/\beta$ with $\tilde{N}=20$ (the number of emitters is increased with decreasing β in order to have the thresholds occur at the same pump rate), and $\kappa=20$ μeV. The corresponding cavity lifetime is about 17 ps, yielding a Q-factor of roughly 30 000. The curves show the typical intensity jump $\propto\beta^{-1}$

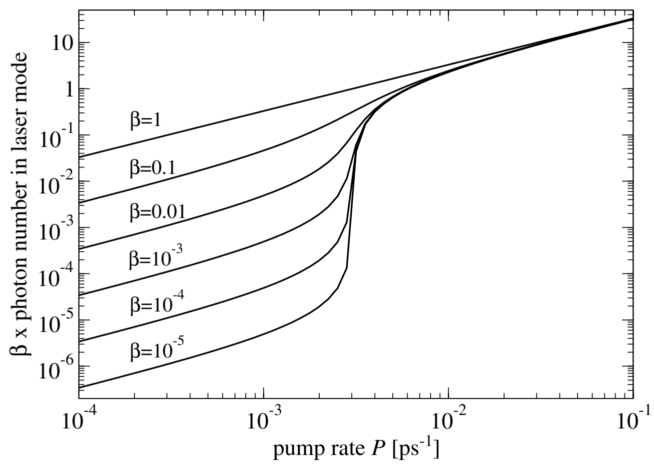

FIG. 1. Calculated output curves for the atomic limit and $\beta=1-10^{-5}$ from top to bottom. The photon number is scaled with β in order to have the thresholds occur at equal pump intensities for better comparison.

from below to above threshold. In the limit $\beta=1$ the kink in the input-output curve disappears.

IV. PHOTON STATISTICS

A. Extended laser equations

Now we turn to the extended set of laser equations including carrier-photon and photon-photon correlation functions. To access intensity correlations, we must calculate the correlation function in Eq. (7), which is a quadruplet contribution. This implies that the treatment within the cluster expansion has to be extended to the quadruplet level. Only photons from the laser mode are assumed to build up correlations, and because we consider only QDs in resonance with the cavity, the free evolution energy terms drop out and are therefore not explicitly given in the following.

The time evolution of the intensity correlation function is given by

$$\left(\hbar\frac{d}{dt}+4\kappa\right)\delta\langle b^\dagger b^\dagger bb\rangle = 4|g|^2\sum_{\nu'}\delta\langle b^\dagger b^\dagger b v^\dagger_{\nu'}c_{\nu'}\rangle, \quad (20)$$

where the sum involves all resonant laser transitions from various QDs. In this equation another quadruplet function enters, which represents a correlation between the photon-assisted polarization and the photon number. For the corresponding equation of motion we obtain

$$\begin{aligned}\left(\hbar\frac{d}{dt}+3\kappa+\Gamma\right)&\delta\langle b^\dagger b^\dagger b v^\dagger_\nu c_\nu\rangle\\ &= -2|g|^2\langle b^\dagger v^\dagger_\nu c_\nu\rangle^2-(1-f^e_\nu-f^h_\nu)\delta\langle b^\dagger b^\dagger bb\rangle\\ &\quad+2f^h_\nu\delta\langle b^\dagger bc^\dagger_\nu c_\nu\rangle-2f^e_\nu\delta\langle b^\dagger bv^\dagger_\nu v_\nu\rangle-2\delta\langle b^\dagger bc^\dagger_{\nu'}v^\dagger_\nu c_\nu v_{\nu'}\rangle\\ &\quad+\sum_{\nu'}\delta\langle b^\dagger b^\dagger v^\dagger_{\nu'}v^\dagger_\nu c_\nu c_{\nu'}\rangle. \end{aligned}\quad (21)$$

Here we have again scaled $\delta\langle b^\dagger b^\dagger b v^\dagger_\nu c_\nu\rangle\to g\,\delta\langle b^\dagger b^\dagger b v^\dagger_\nu c_\nu\rangle$ with the light-matter coupling g for the laser mode. The triplet photon-carrier correlations in the third line are the same as in Eq. (11), and their evolution is given by

$$\begin{aligned}\left(\hbar\frac{d}{dt}+2\kappa\right)&\delta\langle b^\dagger bc^\dagger_\nu c_\nu\rangle\\ &=-2|g|^2\,\mathrm{Re}\Bigg[\delta\langle b^\dagger b^\dagger b v^\dagger_\nu c_\nu\rangle+\sum_{\nu'}\delta\langle b^\dagger v^\dagger_{\nu'}c^\dagger_\nu c_{\nu'}c_\nu\rangle\\ &\quad+(\langle b^\dagger b\rangle+f^e_\nu)\langle b^\dagger v^\dagger_\nu c_\nu\rangle\Bigg], \end{aligned}\quad (22)$$

$$\begin{aligned}\left(\hbar\frac{d}{dt}+2\kappa\right)&\delta\langle b^\dagger bv^\dagger_\nu v_\nu\rangle\\ &=2|g|^2\,\mathrm{Re}\Bigg[\delta\langle b^\dagger b^\dagger b v^\dagger_\nu c_\nu\rangle-\sum_{\nu'}\delta\langle bc^\dagger_{\nu'}v^\dagger_\nu v_{\nu'}v_\nu\rangle\\ &\quad+(\langle b^\dagger b\rangle+f^h_\nu)\langle b^\dagger v^\dagger_\nu c_\nu\rangle\Bigg]. \end{aligned}\quad (23)$$

The correlation functions in the sum, which have been scaled as $\delta\langle b^\dagger v^\dagger_{\nu'}c^\dagger_\nu c_{\nu'}c_\nu\rangle\to g\,\delta\langle b^\dagger v^\dagger_{\nu'}c^\dagger_\nu c_{\nu'}c_\nu\rangle$, $\delta\langle bc^\dagger_{\nu'}v^\dagger_\nu v_{\nu'}v_\nu\rangle \to g\,\delta\langle bc^\dagger_{\nu'}v^\dagger_\nu v_{\nu'}v_\nu\rangle$, obey equations of motion

$$\begin{aligned}\left(\hbar\frac{d}{dt}+\kappa+\Gamma\right)&\delta\langle bc^\dagger_{\nu'}v^\dagger_\nu v_{\nu'}v_\nu\rangle\\ &=(1-\delta_{\nu\nu'})[(1-f^e_{\nu'}-f^h_{\nu'})\delta\langle b^\dagger bv^\dagger_\nu v_\nu\rangle\\ &\quad-|g|^2\langle b^\dagger v^\dagger_\nu c_\nu\rangle^*\langle b^\dagger v^\dagger_{\nu'}c_{\nu'}\rangle^*], \end{aligned}\quad (24)$$

$$\begin{aligned}\left(\hbar\frac{d}{dt}+\kappa+\Gamma\right)&\delta\langle b^\dagger v^\dagger_{\nu'}c^\dagger_\nu c_{\nu'}c_\nu\rangle\\ &=(1-\delta_{\nu\nu'})[(1-f^e_{\nu'}-f^h_{\nu'})\delta\langle b^\dagger bc^\dagger_\nu c_\nu\rangle\\ &\quad+|g|^2\langle b^\dagger v^\dagger_\nu c_\nu\rangle\langle b^\dagger v^\dagger_{\nu'}c_{\nu'}\rangle]. \end{aligned}\quad (25)$$

In the following, we give arguments why the correlation functions, which are determined by Eqs. (24) and (25), and the last term of Eq. (21) only contribute if correlations between *different* QDs exist, i.e., superradient coupling plays a role in the system. The effect of superradiance is known to rely on weak dephasing, which is difficult to realize under the considered high-excitation conditions. We refer to the dipole selection rules in cubic crystals, where optical transitions with a given circular light polarization are coupled to a particular electron spin and the corresponding hole total angular momentum. Specifically, the s-shell states for electrons are spin degenerate and the two spin states are coupled to different light polarizations. If we consider correlations between photons with the same circular polarization, we find that they are linked to states for which only one electron or hole per s-shell and QD are available. In other words, annihilating two valence-band electrons in the case of $\langle bc^\dagger_{\nu'}v^\dagger_\nu v_{\nu'}v_\nu\rangle$ and two conduction-band electrons in the case of $\langle b^\dagger v^\dagger_{\nu'}c^\dagger_\nu c_{\nu'}c_\nu\rangle$ is only possible if these carriers belong to different QDs. Hence for $\nu=\nu'$ these expectation values, and according to their definition

$$\langle bc^{\dagger}_{\nu'}v^{\dagger}_{\nu}v_{\nu'}v_{\nu}\rangle = -\langle bc^{\dagger}_{\nu'}v_{\nu'}\rangle f^{v}_{\nu}(1-\delta_{\nu\nu'}) + \delta\langle bc^{\dagger}_{\nu'}v^{\dagger}_{\nu}v_{\nu'}v_{\nu}\rangle, \tag{26}$$

$$\langle b^{\dagger}v^{\dagger}_{\nu'}c^{\dagger}_{\nu}c_{\nu'}c_{\nu}\rangle = -\langle b^{\dagger}v^{\dagger}_{\nu'}c_{\nu'}\rangle f^{c}_{\nu}(1-\delta_{\nu\nu'}) + \delta\langle b^{\dagger}v^{\dagger}_{\nu'}c^{\dagger}_{\nu}c_{\nu'}c_{\nu}\rangle, \tag{27}$$

also the corresponding correlation functions $\delta\langle bc^{\dagger}_{\nu'}v^{\dagger}_{\nu}v_{\nu'}v_{\nu}\rangle$ and $\delta\langle b^{\dagger}v^{\dagger}_{\nu'}c^{\dagger}_{\nu}c_{\nu'}c_{\nu}\rangle$ vanish exactly. The correlation functions referring to different QDs $\nu\neq\nu'$ are related to superradiant coupling. The same applies to the expectation value

$$\langle b^{\dagger}b^{\dagger}v^{\dagger}_{\nu'}v^{\dagger}_{\nu}c_{\nu}c_{\nu'}\rangle = 2\langle b^{\dagger}v^{\dagger}_{\nu}c_{\nu}\rangle\langle b^{\dagger}v^{\dagger}_{\nu'}c_{\nu'}\rangle(1-\delta_{\nu\nu'}) + \delta\langle b^{\dagger}b^{\dagger}v^{\dagger}_{\nu'}v^{\dagger}_{\nu}c_{\nu}c_{\nu'}\rangle, \tag{28}$$

which also vanishes together with the corresponding correlation function for $\nu=\nu'$. Under the assumption that superradiance is weak in the system, the discussed correlation functions are neglected. If, however, the phenomenon of superradiant coupling itself is to be studied, the correlation functions must be included via their own equations of motion. Finally, the term $\delta\langle b^{\dagger}bc^{\dagger}_{\nu'}v^{\dagger}_{\nu}c_{\nu}v_{\nu'}\rangle$ in Eq. (21) is a generalization of the correlations to the source term of spontaneous emission $C^{x}_{\alpha'\nu\nu'\alpha}=\delta\langle c^{\dagger}_{\alpha'}v^{\dagger}_{\nu}c_{\nu'}v_{\alpha}\rangle$. For consistency reasons, this contribution is neglected in accordance with the above discussed omission of C^{x}.

Effects due to the Coulomb interaction of carriers can be included along the same lines as discussed in Secs. II and III. The contributions to Eq. (21), that remain on the quadruplet level, are given by

$$i\hbar\frac{d}{dt}\,\delta\langle b^{\dagger}b^{\dagger}bv^{\dagger}_{\nu}c_{\nu}\rangle|_{\text{Coul}} = -2\sum_{\alpha}(1-f^{e}_{\nu}-f^{h}_{\nu})V_{\nu\alpha\nu\alpha}\delta\langle b^{\dagger}b^{\dagger}bv^{\dagger}_{\nu}c_{\nu}\rangle - 2(f^{e}_{\alpha}+f^{h}_{\alpha})V_{\nu\alpha\nu\alpha}\delta\langle b^{\dagger}b^{\dagger}bv^{\dagger}_{\nu}c_{\nu}\rangle. \tag{29}$$

The result shows an analogous structure like the Hartree-Fock Coulomb terms for $\langle b^{\dagger}v^{\dagger}_{\nu}c_{\nu}\rangle$ in Eq. (10) and can be interpreted accordingly as a renormalization of the single-particle energies and as interband exchange interaction causing additional renormalizations of the transition energies as well as a redistribution of oscillator strength between different QD transitions.

Coulomb interaction contributions to $\delta\langle b^{\dagger}bc^{\dagger}_{\nu}c_{\nu}\rangle$ and $\delta\langle b^{\dagger}bv^{\dagger}_{\nu}v_{\nu}\rangle$ are analogous to those contributing to the carrier dynamics of f^{e}_{ν} and f^{h}_{ν} discussed in detail in Ref. [18]. Their inclusion is, however, beyond the scope of this paper and will be the subject of future investigations.

B. Verification of the treatment of correlations: Comparison with the master equation model

In Sec. III B we have shown that the developed laser model can be reduced to the well-known rate equations, if the semiconductor is replaced by two-level systems. In the extended laser model derived in Sec. IV A, the calculation of photon correlations is based on the cluster expansion that provides a truncation of the hierarchy of correlation functions. The aim of this section is to verify that such a truncation on the quadruplet level provides correct results for the photon-intensity correlations, described by the autocorrelation function $g^{(2)}$.

In quantum optics, access to photon-intensity fluctuations can either be obtained by invoking the quantum regression theorem [17], or by using a master equation to calculate the diagonal density matrix for the coupled atom-photon system [11,15,16]. Both methods are not directly applicable in semiconductors due to the presence of many-body effects and the modified source term of spontaneous emission. However, our semiconductor model can be reduced to a description of two-level systems. This provides a verification method for our approach and the possibility to study how well the truncation of correlations within the cluster expansion scheme works, as carrier-photon correlations are treated on an exact level in the master equation within the two-level approach.

We have shown in Sec. III B that the source term of spontaneous emission reduces to f^{e}_{ν} under the assumption that only one electron is present in each two-level atomic system. Additionally, the equation of motion (21) changes to

$$\left(\hbar\frac{d}{dt}+3\kappa+\Gamma\right)\delta\langle b^{\dagger}b^{\dagger}bv^{\dagger}_{\nu}c_{\nu}\rangle = -2g^{2}\langle b^{\dagger}v^{\dagger}_{\nu}c_{\nu}\rangle^{2} - 4|g|^{2}\langle b^{\dagger}v^{\dagger}_{\nu}c_{\nu}\rangle\text{Re}\langle b^{\dagger}v^{\dagger}_{\nu}c_{\nu}\rangle - (1-f^{e}_{\nu}-f^{h}_{\nu})\delta\langle b^{\dagger}b^{\dagger}bb\rangle + 2\delta\langle b^{\dagger}bc^{\dagger}_{\nu}c_{\nu}\rangle, \tag{30}$$

again neglecting the quadruplet-level correlation functions appearing on the right-hand side and scaling with the light-matter coupling strength. All other equations of motion for the correlations remain unmodified under the two-level assumptions. In order to quantitatively compare to the master equation given in Ref. [11], we must once more assume a fully inverted system, which is done by setting $1-f^{h}_{\nu}=0$ in Eqs. (11) and (30). Due to the coupling to the correlation functions in Eq. (11), an adiabatic solution in the spirit of the rate equations (18) and (19) is no longer possible. Nevertheless, the numerical steady-state solution can be directly compared to the results of the master equation.

To remain as close as possible to the semiconductor model, we solve the atomic two-level version of Eqs. (8) and (11) together with Eqs. (16) and (17) for the population dynamics of the laser and pump level (with $f^{e}_{\nu}f^{h}_{\nu}$ replaced by f^{e}_{ν} for the spontaneous emission). This allows us to avoid the introduction of a number of excited two-level systems. For the direct comparison with the master equations, $R_{p\to s}$ is used as a measure for the carrier generation rate at the laser transition level.

Figure 2 shows numerical results from our truncated cluster expansion model applied to two-level systems, in comparison to results obtained from the master equation in the formulation of Rice and Carmichael [11]. The values for the parameters κ, $\tilde{N}$, and τ_{sp} were taken from Sec. III and are the same as for Fig. 1. Additionally relaxation rates entering $R^{e,h}_{p\to s}$ for both electrons and holes of 1 ps, and a dephasing $\Gamma=1.36$ meV, corresponding to a time of approximately

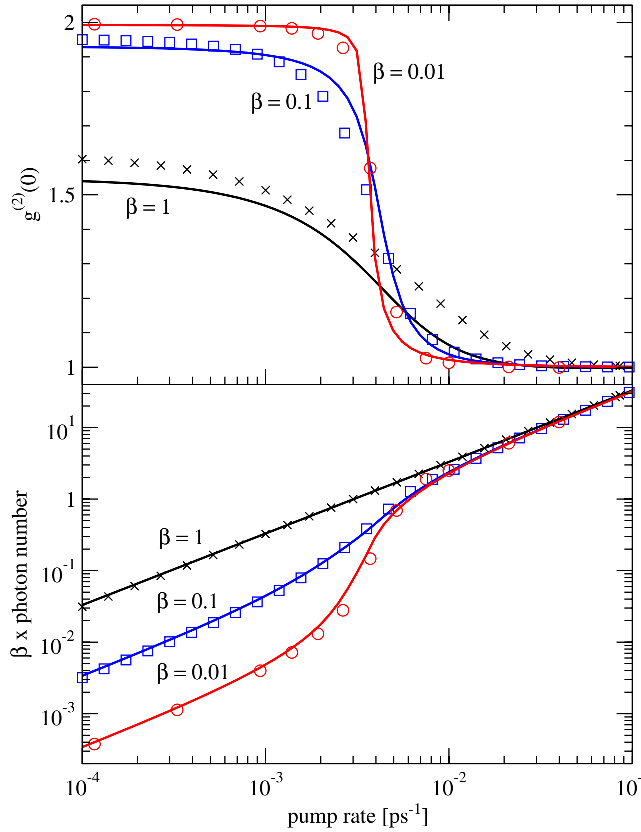

FIG. 2. (Color online) Autocorrelation function (top) and input-output curve (bottom) for a fully inverted two-level system. Comparison between the master equation (symbols) and the two-level version of the semiconductor theory (solid lines) for $\beta=1$, 0.1, and 0.01. On the x-axis the pump rate into the laser level is given. For the modified semiconductor theory, this corresponds to an effective carrier generation rate in the s-shell.

500 fs, were used. The upper part of the figure shows the second order correlation function atop the input/output curve for various values of the β-factor. Looking at the input-output curves, we see that the equation of motion approach agrees convincingly well with the results from the master equation for all values of β. Regarding the autocorrelation function in the top panel, there is excellent agreement for small values of the β-factor. A deviation of roughly 5% becomes apparent as β is increased to unity, and the results are in good agreement regarding the onset and the end of the transition from thermal to coherent light emission.

We point out that the deviation between lines and symbols in Fig. 2 is a measure for the applicability of the cluster expansion method leading to a truncation of carrier-photon and photon-photon correlations beyond the quadruplet level. Clearly one has to consider the tradeoff between deviations due to this truncation and the possibility to include semiconductor effects. The influence of the latter will be discussed below. The agreement between the truncated (cluster expansion) and nontruncated (master equation) description of atomic two-level systems depends on parameters like the cavity lifetime and the spontaneous emission rate. (For the above comparison, typical values of current microcavities have been used.) As long as the semiconductor theory is used for parameters where its two-level version is in agreement with the master equation, we are reassured that the truncation of the cluster expansion can be applied with respect to the photonic correlations. The semiconductor theory contains additional carrier-carrier correlation effects, which are well-described by means of the cluster expansion method. In fact, the cluster expansion was developed to treat many-body effects of carriers [25]. Successful applications include the photoluminescence of QDs [18,29] and exciton formation in quantum wells [27].

C. Analytical results for $g^{(2)}(0)$

Before the numerical results of the semiconductor model are presented, it is instructive to study analytical solutions for $g^2(0)$ in the two limiting cases of strong and weak pumping. For this purpose we use the stationary limit of Eqs. (8), (11), and (20)–(23). Considering the resonant s-shell contributions from identical QDs, we replace $\Sigma_{\nu'}$ by the number of QDs N. Inserting in Eq. (8) the photon-assisted polarization from Eq. (11), ignoring spontaneous emission for the above-threshold solution, and expressing the higher-order correlations with the help of Eqs. (22) and (23), we obtain from Eqs. (7) and (20)

$$g^{(2)}(0)-1=-\frac{\kappa(\kappa+\Gamma)}{2|g|^2\langle b^\dagger b\rangle}\left(1+\frac{|g|^2N}{\kappa(\kappa+\Gamma)}(1-f_s^e-f_s^h)\right). \tag{31}$$

In the limit $\langle b^\dagger b\rangle/N\gg 1$, the right-hand side vanishes. Hence we obtain $g^{(2)}(0)=1$, i.e., well above threshold the light is coherent.

For the limiting case of weak pumping, we seek again the stationary solution of our coupled system of equations, now under the assumption that in Eq. (11) the stimulated emission term and the higher-order correlations $\delta\langle b^\dagger b c_\nu^\dagger c_\nu\rangle$, $\delta\langle b^\dagger b v_\nu^\dagger v_\nu\rangle$ can be neglected. A convenient way to solve for the intensity correlation function $\delta\langle b^\dagger b^\dagger bb\rangle$ is to insert Eq. (21) into Eq. (20). The higher-order correlations in Eq. (21) are replaced by the static solution of Eqs. (22) and (23), while in the latter $\langle b^\dagger v_\nu^\dagger c_\nu\rangle$ is replaced by Eq. (8), and $\delta\langle b^\dagger b^\dagger b v_\nu^\dagger c_\nu\rangle$ is traced back to $\delta\langle b^\dagger b^\dagger bb\rangle$ with the stationary solution of Eq. (20). As explained above, we ignore the quadruplet correlations occurring in Eq. (21). Together with Eqs. (7) and (8) we finally obtain

$$\left(\frac{\kappa(3\kappa+\Gamma)}{|g|^2N}+(1-f_s^e-f_s^h)\right)[g^{(2)}(0)-2] = -\frac{2(f_s^e+f_s^h)}{N}[g^{(2)}(0)-1]-\frac{2\kappa(3\kappa+2\Gamma)}{|g|^2N^2}. \tag{32}$$

To evaluate this formula further, we restrict ourselves to the case

$$\frac{\kappa^2}{|g|^2N}\gg 1, \tag{33}$$

or $2\kappa/\hbar \gg N/\tau_l$, i.e., the cavity loss rate is much larger than the total rate of spontaneous emission into the laser mode. In

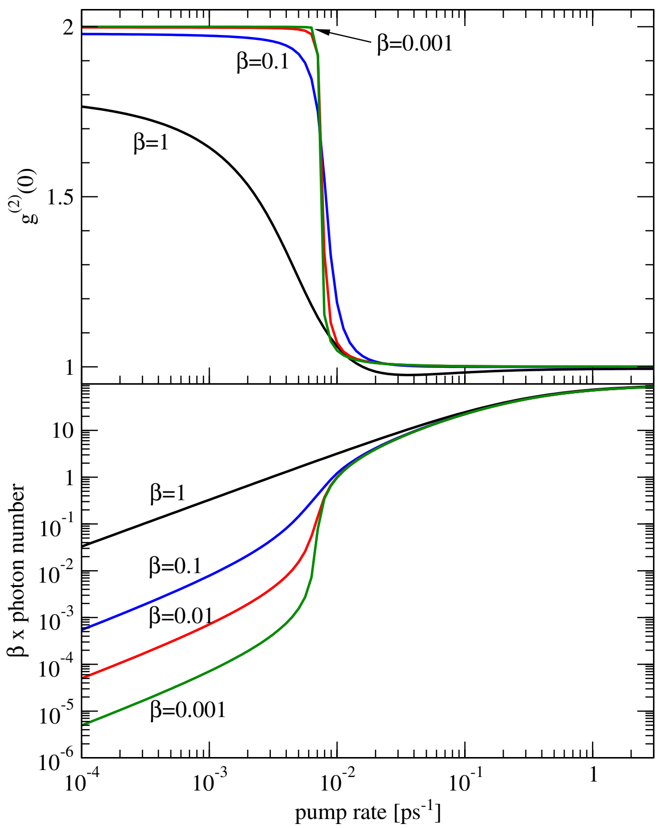

FIG. 3. (Color online) Calculated output curve (lower panel) and autocorrelation function $g^{(2)}(\tau=0)$ (upper panel) for $\beta=1$, 0.1, 0.01, and 0.001. The main parameters are the same as in Fig. 2 and also correspond to those in Fig. 1.

this "bad cavity limit" [11], in which typical semiconductor lasers operate, we obtain as an analytical result of our theory

$$g^{(2)}(0) = 2 - \frac{2}{N}. \tag{34}$$

This is an important finding because it provides the statistics of thermal light in the limit of many QDs, $g^{(2)}(0)=2$, and in the opposite limit of a single QD it gives the statistics of a single-photon emitter, $g^{(2)}(0)=0$.

V. NUMERICAL RESULTS

We now present numerical solutions of the extended semiconductor laser theory including carrier-photon correlations based on Eqs. (8), (11), (16), (17), and (20)–(23) using an adaptive time integration algorithm. Again, we use the same parameters as in Secs. III B and IV B, but different relaxation times for electrons and holes are taken: $\tau_r^e=1$ ps, $\tau_r^h=500$ fs.

In Fig. 3 the autocorrelation function is shown atop the input-output curve for various values of β. There are several striking features: (i) The jump of the intensity curve from below to above threshold is no longer determined by $1/\beta$, as in Figs. 1 and 2, obtained from a laser theory for two-level systems. This is of particular importance since measurements of the input-output characteristics are often used to experimentally deduce the β-factor according to the predictions of the two-level models. If the atomic $1/\beta$-behavior would be used to extract the β-factors from the curves in Fig. 3, one would obtain 0.017 instead of 0.1, 0.0017 instead of 0.01, and 0.00017 instead of 0.001. (ii) For small β values, the *s*-shaped intensity jump is accompanied by a decrease of the second-order coherence from the Poisson value $g^{(2)}(0)=2$ for thermal light to $g^{(2)}(0)=1$ for coherent laser light. Using larger β values, the abrupt drop of the autocorrelation function becomes softer, and below threshold $g^{(2)}$ remains smaller than two. This decrease in the autocorrelation function is already a result of the relatively high cavity quality (long cavity lifetime). For a shorter cavity lifetime and a large number of emitters, also at $\beta=1$ a value of $g^{(2)}(0)=2$ is obtained in the sub-threshold regime, see below. (iii) At high pump intensities saturation effects due to Pauli blocking become visible in the input-output curve, effectively limiting the maximum output that can be achieved. Additionally, effects of quenching were observed in master equation treatments [15].

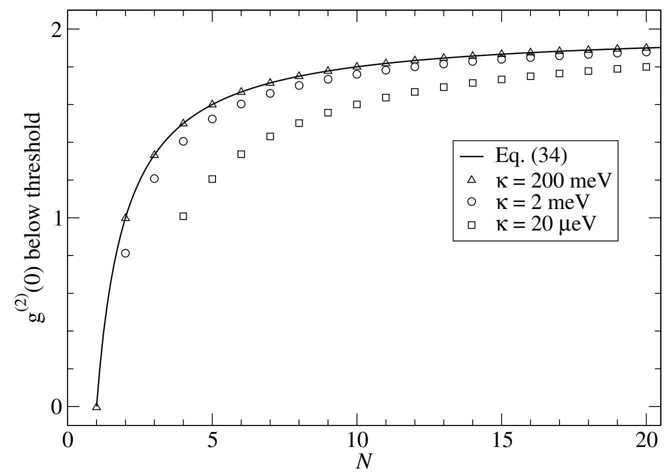

FIG. 4. Sub-threshold values of $g^{(2)}(0)$ for $\beta=1$ obtained from the extended semiconductor laser model. The analytical approximation in the "bad cavity" limit, Eq. (34), is compared to numerical results obtained with different values for the inverse cavity lifetime κ.

In Sec. IV C we have discussed the analytical solution of the semiconductor model for the autocorrelation function $g^{(2)}(0)$ below threshold in the limiting case that $\kappa^2 \gg |g|^2 N$. In Fig. 4 we show the subthreshold value of the autocorrelation function versus the number of emitters N. The analytical solution (solid line), which was derived for the limit of large κ, is compared to numerical solutions of the extended semiconductor laser model (open symbols) for $\beta=1$ and various values of κ. All other parameters are the same as those used in Fig. 3. If $\kappa=200$ meV, the condition for the analytical solution is fulfilled and perfect agreement between analytical and numerical results is obtained. In this case, the thermal emission $g^{(2)}(0)=2$ below the laser threshold is approached for a large number of emitters N. In the limit of one single QD, the antibunching signature $g^{(2)}(0)=0$ is numerically obtained. On the other hand, in the theoretical limit of an infinitely good cavity $\kappa \to 0$, a constant value of $g^{(2)}(0)=1$ is expected for atomic models [11]. The case of larger cavity lifetimes is displayed (circles and squares) and the trend of a decrease of the subthreshold value is observed. For the case of a small number of emitters in a very good cavity, photon correlations become so strong that the truncation on quadruplet order becomes insufficient.

VI. DISCUSSION AND OUTLOOK

In conclusion, we have developed a semiconductor laser theory that includes carrier-photon correlations and allows one to determine the photon statistics of the light emission. The theory has been applied to describe microcavity lasers with QDs as active material. It has been demonstrated how (i) the model can be reduced to obtain the commonly used rate equations, and (ii) how the incorporation of two-level assumptions makes it possible to compare the photon correlations to those obtained from a master equation. By these means, we have verified that the truncation method of the arising hierarchy of equations of motion can be applied in the considered parameter regime, which is typical for current state-of-the-art microcavity lasers.

Using a numerical evaluation of the theory, we have demonstrated modifications of the characteristic emission properties due to semiconductor effects. Especially the jump in the input-output curve from below to above threshold is found not to scale with $1/\beta$, as it does in the two-level case.

Most importantly, our approach opens up the possibility to include the full spectrum of semiconductor effects in a consistent and well-defined manner. Besides a more complete inclusion of Coulomb correlations beyond the singlet level, relaxation and dephasing processes can also be treated on a microscopic level. Furthermore, with respect to Coulomb and light-matter interaction-induced correlations between different QDs, effects of superradiant coupling can be studied. While this is not the focus of this paper, it outlines the direction of future work.

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APPENDIX: INTERPLAY OF TIME CONSTANTS

For the evaluation of our theory, we treat the rate of spontaneous emission into the nonlasing modes $1/\tau_{\mathrm{nl}}$ as an extrinsic parameter, which is determined by the properties of the laser resonator. Depending on the particular cavity design, other high-Q resonator modes as well as a quasicontinuum of low-Q leaky modes can contribute.

The spontaneous emission into the laser mode can be calculated from the light-matter coupling strength $|g|^2=|g_{q_l s}|^2$ for this mode, κ, and Γ. Restricting the adiabatic solution of Eq. (11) to the spontaneous emission into the laser mode and using Eq. (9) to define the corresponding rate $1/\tau_l$ according to

$$\frac{d}{dt} f_s^{e,h}\Big|_{l,\mathrm{spont}} = -\frac{f_s^e f_s^h}{\tau_l}, \tag{A1}$$

we find

$$\frac{1}{\tau_l} = \frac{2}{\hbar}\frac{|g|^2}{\kappa+\Gamma}. \tag{A2}$$

With τ_l and τ_{nl} the β-factor follows from Eq. (14).

In this paper we present the figures in the common style where the β-factor is varied, as it is the most important parameter characterizing the cavity efficiency, while the total rate of spontaneous emission $1/\tau_{\mathrm{sp}}=1/\tau_l+1/\tau_{\mathrm{nl}}$ is held constant. To achieve such a situation, for various β-values both τ_{nl} and τ_l need to be changed. Note that the latter requires a change of the light-matter coupling strength according to Eq. (A2), which is possible for a given dipole coupling by a modification of the mode functions, and/or by a change of the lifetime of the cavity mode.

In Ref. [5] the presented theory is applied to pillar microcavities with various resonator diameters. In such a situation, the spontaneous emission into nonlasing modes is practically constant due to the unchanging contributions of leaky modes, while the spontaneous emission into the laser mode is modified by the Purcell effect.

The Purcell factor F_P is defined as the ratio of the rate of spontaneous emission into the cavity mode, $1/\tau_l$, to the rate of spontaneous emission into free space, $1/\tau_{\mathrm{free}}$. We can express the β-factor in terms of F_P as

$$\beta = \frac{F_P}{dF_P + \frac{\tau_{\mathrm{free}}}{\tau_{\mathrm{nl}}}} = \frac{\frac{1}{\tau_l}}{\frac{d}{\tau_l} + \frac{1}{\tau_{\mathrm{nl}}}}, \tag{A3}$$

where additionally a possible degeneracy d of the fundamental mode has been included, see the article by J.-M. Gérard in Ref. [33].

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