Lowest-Landau-level description of a Bose-Einstein condensate in a rapidly rotating anisotropic trap

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(Received 19 September 2006; published 18 January 2007)

A rapidly rotating Bose-Einstein condensate in a symmetric two-dimensional trap can be described with the lowest Landau-level set of states. In this case, the condensate wave function $\psi(x,y)$ is a Gaussian function of $r^2=x^2+y^2$, multiplied by an analytic function $P(z)$ of the single complex variable $z=x+iy$; the zeros of $P(z)$ denote the positions of the vortices. Here, a similar description is used for a rapidly rotating anisotropic two-dimensional trap with arbitrary anisotropy $(\omega_x/\omega_y \leqslant 1)$. The corresponding condensate wave function $\psi(x,y)$ has the form of a complex anisotropic Gaussian with a phase proportional to xy, multiplied by an analytic function $P(\zeta)$, where $\zeta \propto x+i\beta_- y$ and $0 \leqslant \beta_- \leqslant 1$ is a real parameter that depends on the trap anisotropy and the rotation frequency. The zeros of $P(\zeta)$ again fix the locations of the vortices. Within the set of lowest Landau-level states at zero temperature, an anisotropic parabolic density profile provides an absolute minimum for the energy, with the vortex density decreasing slowly and anisotropically away from the trap center.

DOI: 10.1103/PhysRevA.75.013620 PACS number(s): 03.75.Hh, 05.30.Jp, 67.40.Db

I. INTRODUCTION

The experimental creation of rapidly rotating Bose-Einstein condensates (BEC) generally involves anisotropic rotating trap potentials [1–3], yet most theoretical analyses of such systems have relied on an isotropic trap [4–9]. As emphasized by Ho [4], the low-lying states in a symmetric two-dimensional trap are closely analogous to those in the lowest Landau level for a charged particle in a uniform magnetic field. This analogy allows a simplified description in the limit that the rotation frequency Ω approaches the frequency ω_0 of the symmetric confining trap. If the typical interaction energy is small compared to the spacing $\approx 2\hbar\omega_0$ between adjacent Landau levels, then the condensate wave function ψ of the interacting rotating BEC can be constructed as a linear combination of the lowest Landau-level (LLL) states. The nth such state is simply proportional to z^n, where $z=x+iy$, multiplied by the ground-state Gaussian. It follows that such a linear combination involves an analytic function of z that is usually approximated by a polynomial $P(z) \propto \Pi_j(z-z_j)$, where the zeros z_j of the polynomial represent the positions of the vortices in the two-dimensional condensate. The identification of the vortices with the nodes in the wave function goes back at least to Feynman and Abrikosov [10,11]; subsequently this connection has reappeared in many different contexts [12–14].

If the vortex density in the axisymmetric rapidly rotating BEC is strictly uniform, then the overall density profile is also Gaussian with an effective condensate radius that grows and ultimately diverges as $\Omega \to \omega_0$ [4]. In fact, this system can lower its energy by slightly reducing the vortex density near the outer edge of the condensate, and the actual density profile has a quadratic shape (an inverted parabola) [5–8], as shown by both analytical and numerical studies.

The quantum-mechanical problem of a particle in a rotating two-dimensional anisotropic harmonic trap is exactly soluble [15–17], although the corresponding eigenstates have not been discussed previously in full detail. We here amplify Valatin's description [15] to construct the anisotropic analogs of the LLL states. Each such state $\varphi_{n0}(x,y)$ involves the anisotropic complex Gaussian ground-state eigenfunction $\varphi_{00}(x,y)$, multiplied by a polynomial $p_n(\zeta)$, where $\zeta \propto x + i\beta_- y$ is a single "stretched" complex variable, and $0 \leqslant \beta_- \leqslant 1$ is a real parameter that depends on the trap anisotropy and the rotation frequency. Thus a linear combination of these LLL states for an anisotropic trap again involves an analytic function of the single complex variable ζ (apart from the common overall factor φ_{00}). The corresponding zeros again represent the positions of the vortices, now in the rotating anisotropic BEC.

Section II focuses on the eigenstates of the single-particle Hamiltonian H_0 for a rotating anisotropic harmonic potential, starting with the classical trajectories and then obtaining the explicit form of the low-lying quantum mechanical states φ_{n0} that are the analogs of the lowest Landau-level states for a charged particle in a magnetic field. As in that case, the expectation value of both H_0 and the angular momentum L_z in the lowest Landau level can be reduced to corresponding expectation values of x^2 and y^2. The interacting dilute Bose–Einstein gas in this rapidly rotating anisotropic trap is treated in Sec. III. For a trial state constructed as a linear combination of $\varphi_{n0}(x,y)$, an anisotropic parabolic density profile provides the absolute minimum of the energy. The density of vortices is constant near the center but decreases slowly toward the edge of the condensate. Section IV contains a discussion and suggestions for additional study.

II. SINGLE-PARTICLE EIGENSTATES

Consider a particle of mass m in an anisotropic two-dimensional harmonic potential (for definiteness, I assume oscillator frequencies $\omega_x \leqslant \omega_y$) that rotates uniformly at an angular velocity $\mathbf{\Omega} = \Omega\hat{z}$ perpendicular to the plane of the motion. In the rotating frame, this potential is time independent, with the Hamiltonian

$$H_0 = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2) - \Omega(xp_y - yp_x), \quad (1)$$

where the last factor involves the angular momentum $L_z = xp_y - yp_x$. If $\omega_x < \omega_y$, the centrifugal force preferentially expands the condensate along the x axis. Although this case ($\omega_x < \omega_y$) is of principal interest here, it will also be valuable to see how the more familiar symmetric case emerges in the limit $\omega_x = \omega_y = \omega_0$.

A. Classical dynamical trajectories

The normal modes of Eq. (1) are readily determined to have the frequencies [15–19]

$$\omega_\pm^2 = \omega_\perp^2 + \Omega^2 \mp \sqrt{\frac{1}{4}(\omega_y^2 - \omega_x^2)^2 + 4\omega_\perp^2 \Omega^2}, \quad (2)$$

where $\omega_\perp^2 = \frac{1}{2}(\omega_x^2 + \omega_y^2)$ is the mean-squared oscillator frequency. This general result contains several important limits.

(1) In the symmetric case $\omega_x = \omega_y = \omega_0$, the plus (minus) modes have frequencies $\omega_\pm = \omega_0 \mp \Omega$. Specifically, the plus mode with frequency $\omega_+ = \omega_0 - \Omega$ has a reduced frequency when viewed from the rotating frame (as is evident physically) and a positive angular momentum (which explains the notation). Correspondingly, the minus mode has an increased frequency $\omega_- = \omega_0 + \Omega$ and a negative angular momentum.

(2) If $\omega_x < \omega_y$, then the modes are nondegenerate even for $\Omega = 0$, when they reduce to $\omega_+ = \omega_x$ and $\omega_- = \omega_y$.

(3) For an anisotropic trap ($\omega_x < \omega_y$) and rapid rotation with $\delta = 1 - \Omega/\omega_x \to 0^+$, the plus normal-mode frequency vanishes, with

$$\omega_+^2 \approx \frac{2\omega_x^2(\omega_y^2 - \omega_x^2)}{3\omega_x^2 + \omega_y^2}\delta. \quad (3)$$

Thus $-\partial\omega_+/\partial\Omega$ diverges for small δ like $\delta^{-1/2}$. In contrast, the minus normal-mode frequency remains finite at $\delta = 0$, with

$$\omega_-^2 \approx 3\omega_x^2 + \omega_y^2. \quad (4)$$

(4) In the case of a rapidly rotating nearly symmetric trap with two small parameters [17] $\delta = 1 - \Omega/\omega_x$ and $\eta = \omega_y/\omega_x - 1$, these eigenfrequencies simplify to

$$\frac{\omega_+}{\omega_x} \approx \sqrt{\delta(\eta + \delta)}, \quad \frac{\omega_-}{\omega_x} \approx 2 + \frac{1}{2}\eta - \delta. \quad (5)$$

Note the sensitivity of ω_+ in Eq. (5) to the order of limiting procedures: (i) first $\eta = 0$ (namely, $\omega_x = \omega_y = \omega_0$) and then $\delta \to 0$ or (ii) $\delta \to 0$ (namely, $\Omega \to \omega_x$) at fixed $\omega_x < \omega_y$. For a symmetric trap, the plus frequency ω_+ vanishes linearly with the small parameter δ; in contrast, for the anisotropic trap with $\omega_x < \omega_y$, the plus frequency vanishes like $\sqrt{\delta}$, with a coefficient proportional to $\sqrt{\eta}$.

Figure 1(a) shows the two different normal-mode frequencies $\omega_\pm$ (normalized to ω_x) as functions of Ω/ω_x for the typical anisotropy $\omega_y/\omega_x = 1.2$. For a symmetric trap with $\omega_x = \omega_y = \omega_0$, the plus (minus) normal modes with frequencies $\omega_\pm = \omega_0 \mp \Omega$ have counterclockwise (clockwise) circular orbits with positive (negative) helicity. To understand the physics of the two normal modes in the more general anisotropic case $\omega_x < \omega_y$, consider first the motion in the plus mode [15]. It has the form

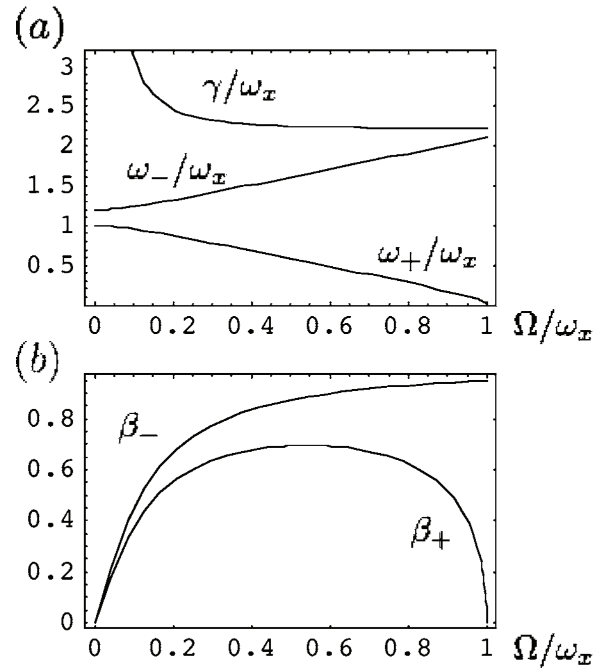

FIG. 1. Behavior of relevant dimensionless quantities as a function of dimensionless rotation speed Ω/ω_x for the typical anisotropy $\omega_y/\omega_x = 1.2$. (a) Dimensionless normal mode frequencies ω_+/ω_x and ω_-/ω_x and dimensionless auxiliary frequency γ/ω_x; (b) dimensionless polarization parameters β_+ and β_-.

$$x_+(t) = x_0 e^{-i\omega_+ t}, \quad y_+(t) = i\beta_+ x_0 e^{-i\omega_+ t}, \quad (6)$$

where $0 \leq \beta_+ \leq 1$ is a real non-negative dimensionless parameter and x_0 is an arbitrary amplitude. A detailed analysis shows that the parameter β_+ has two alternative representations

$$\beta_+ = \frac{\omega_x^2 - \omega_+^2 - \Omega^2}{2\Omega\omega_+} = \frac{2\Omega\omega_+}{\omega_y^2 - \omega_+^2 - \Omega^2}. \quad (7)$$

In the rotating frame, the plus orbit is an ellipse with major axis oriented along $\hat{x}$; the polarization parameter β_+ gives the ratio of the minor to major axes. The second form of Eq. (7) shows that $\omega_+/\beta_+ \approx (\omega_y^2 - \omega_x^2)/(2\omega_x)$ for $\delta \to 0$, so that ω_+ and β_+ both vanish in this limit like $\sqrt{\delta}$. Physically, this behavior reflects the rotation-induced cancellation of the harmonic confinement in the $\hat{x}$ direction; the orbit then becomes linearly polarized as $\Omega \to \omega_x$. The plus motion is counterclockwise, with positive helicity and positive angular momentum.

Similarly, the orbit for the minus mode has the parametric representation

$$x_-(t) = i\beta_- y_0 e^{-i\omega_- t}, \quad y_-(t) = y_0 e^{-i\omega_- t}, \quad (8)$$

where y_0 is an arbitrary amplitude and $0 \leq \beta_- \leq 1$ is a real non-negative parameter with two alternative representations [15]

$$\beta_- = \frac{\omega_-^2 - \omega_y^2 + \Omega^2}{2\Omega\omega_-} = \frac{2\Omega\omega_-}{\omega_-^2 - \omega_x^2 + \Omega^2}. \tag{9}$$

Unlike β_+, these relations show that β_- has a nonzero limit $\beta_- \approx 2\omega_x/(3\omega_x^2+\omega_y^2)^{1/2} \leqslant 1$ as $\delta \to 0$. In the rotating frame, the minus orbit is an ellipse with major axis oriented along $\hat{\boldsymbol{y}}$; the polarization parameter β_- gives the ratio of the minor to major axes. The minus motion is clockwise, with negative helicity and negative angular momentum.

It is again instructive to specialize these results to the case of a rapidly rotating nearly symmetric trap [17], making use of the small parameters $\delta = 1-\Omega/\omega_x$ and $\eta = \omega_y/\omega_x - 1$. A straightforward expansion yields the approximate polarization parameters

$$\beta_+ \approx \sqrt{\frac{\delta}{\eta+\delta}}, \quad \beta_- \approx 1 - \frac{1}{4}\eta. \tag{10}$$

Although β_- varies smoothly in this limit, the corresponding polarization β_+ for the positive mode has a singular behavior that depends on the relative magnitude of the two small parameters. More generally, β_+ and β_- both have the value 1 in the limit of a symmetric trap, independent of Ω, since the resulting motion is circularly polarized. Figure 1(b) shows the dependence of the two parameters $\beta_\pm$ on Ω/ω_x for an anisotropy $\omega_y/\omega_x = 1.2$. Note the singular slope of ω_+ and β_+ near the upper limit.

B. Bogoliubov canonical transformation to diagonal Hamiltonian

The structure of H_0 in Eq. (1) is unusual because the term $-\Omega L_z = \Omega(yp_x - xp_y)$ couples the otherwise independent x and y motions. This situation can be clarified by introducing the conventional ladder operators [20]

$$a_x = \frac{1}{\sqrt{2}}\left(\frac{x}{d_x} + i\frac{d_x p_x}{\hbar}\right), \quad a_x^\dagger = \frac{1}{\sqrt{2}}\left(\frac{x}{d_x} - i\frac{d_x p_x}{\hbar}\right), \tag{11}$$

where $d_x = \sqrt{\hbar/(m\omega_x)}$, and similarly for a_y and $a_y^\dagger$. With these operators, it is straightforward to see that the term ΩL_z is proportional to [16]

$$i\Omega[(\omega_y+\omega_x)(a_x^\dagger a_y - a_y^\dagger a_x) + (\omega_y - \omega_x)(a_x a_y - a_y^\dagger a_x^\dagger)]. \tag{12}$$

The first term is "diagonal" in the creation and annihilation operators, but the second is "off diagonal," similar to Bogoliubov's approximate Hamiltonian for a dilute Bose-Einstein gas [21]. Unfortunately, a direct diagonalization based on these "particle" operators involves considerable algebraic complexity [16,17].

Thus, it is preferable to return to the original single-particle Hamiltonian in Eq. (1). Since H_0 is quadratic in the coordinates and momenta, it can be diagonalized with a canonical transformation to new variables that obey the same Poisson brackets (in the classical case) or the same commutators (in the quantum case). Specifically, I follow Valatin [15] and introduce the generating function

$$S(x,y;Q_+,Q_-) = -m\gamma\left[\lambda_+\lambda_- Q_+Q_- + \frac{1}{2}(\lambda_+^2+\lambda_-^2)xy - \lambda_+Q_+y - \lambda_-Q_-x\right]. \tag{13}$$

Here, $Q_\pm$ are new canonical coordinates, $\lambda_\pm$ are dimensionless constants given by

$$\lambda_\pm^2 = \frac{\omega_\pm}{\mu_\pm}, \quad \text{with } \mu_\pm = \omega_\pm + \beta_+\beta_-\omega_\mp, \tag{14}$$

and $\gamma = \mu_+/\beta_+ = \mu_-/\beta_-$ has the dimensions of a frequency. It follows from Eqs. (2), (7), (9), and (14) that γ has various equivalent representations

$$\gamma = \frac{\omega_+}{\beta_+} + \omega_-\beta_- = \frac{\omega_-}{\beta_-} + \omega_+\beta_+ = \frac{\omega_-^2 - \omega_+^2}{2\Omega} = \frac{\sqrt{\frac{1}{4}(\omega_y^2-\omega_x^2)^2 + 4\omega_\perp^2\Omega^2}}{\Omega}. \tag{15}$$

For a symmetric trap with $\omega_x = \omega_y = \omega_0$, this frequency reduces to $\gamma = 2\omega_0$ for all Ω. In contrast, for an anisotropic trap, γ diverges as Ω becomes small and approaches the value $\approx (3\omega_x^2+\omega_y^2)/(2\omega_x) > 2\omega_x$ for $\Omega \to \omega_x$. Figure 1(a) shows the normalized parameter γ/ω_x as a function of Ω/ω_x for $\omega_y/\omega_x = 1.2$.

According to the general theory of classical Hamiltonian dynamics [22], *any* function such like $S(x,y;Q_+,Q_-)$ that depends on both the old and new coordinates will automatically generate a canonical transformation from old canonical variables to new canonical variables, with the corresponding momentum variables given by

$$p_x = \frac{\partial S}{\partial x}, \quad p_y = \frac{\partial S}{\partial y},$$

$$P_+ = -\frac{\partial S}{\partial Q_+}, \quad P_- = -\frac{\partial S}{\partial Q_-}. \tag{16}$$

The first set of equations immediately yields the relations

$$Q_+ = \left(\frac{\lambda_+^2+\lambda_-^2}{2\lambda_+}\right)x + \frac{p_y}{m\gamma\lambda_+},$$

$$Q_- = \left(\frac{\lambda_+^2+\lambda_-^2}{2\lambda_-}\right)y + \frac{p_x}{m\gamma\lambda_-} \tag{17}$$

that express the new coordinates $Q_\pm$ as linear combinations of the original coordinates and momenta. Similarly, the second set of equations can be used to find the corresponding relations for the new momenta $P_\pm$

$$P_+ = m\gamma\lambda_+\left(\frac{\lambda_+^2+\lambda_-^2}{2}-1\right)y+\lambda_+p_x,$$

$$P_- = m\gamma\lambda_-\left(\frac{\lambda_+^2+\lambda_-^2}{2}-1\right)x+\lambda_-p_y. \tag{18}$$

For future reference, note the following alternative relations

$$x=\lambda_+Q_+-\frac{P_-}{m\gamma\lambda_-},$$

$$y=\lambda_-Q_--\frac{P_+}{m\gamma\lambda_+}; \tag{19}$$

they express the original coordinates in terms of the new canonical variables and will be valuable in the subsequent analysis.

It is now straightforward to verify that the Hamiltonian has the following simple diagonal form when expressed in the new canonical variables

$$H_0=\frac{P_+^2}{2m}+\frac{1}{2}m\omega_+^2Q_+^2+\frac{P_-^2}{2m}+\frac{1}{2}m\omega_-^2Q_-^2. \tag{20}$$

One strategy is to substitute Eqs. (17) and (18) directly into Eq. (20), which eventually reproduces the original Eq. (1). This new Hamiltonian (20) has the great advantage of immediately providing a quantum description of two independent harmonic oscillators with mass m and frequencies $\omega_\pm$. The corresponding quantum-mechanical annihilation operators $\alpha_\pm$ and creation operators $\alpha_\pm^\dagger$ follow from general quantum theory [20]

$$\alpha_\pm=\frac{1}{\sqrt{2}}\left(\frac{Q_\pm}{d_\pm}+i\frac{d_\pm P_\pm}{\hbar}\right),\quad \alpha_\pm^\dagger=\frac{1}{\sqrt{2}}\left(\frac{Q_\pm}{d_\pm}-i\frac{d_\pm P_\pm}{\hbar}\right), \tag{21}$$

where $d_\pm=\sqrt{\hbar/(m\omega_\pm)}$ are the oscillator lengths for the two separate modes. These operators obey the usual commutation relations $[\alpha_\pm,\alpha_\pm^\dagger]=1$ (all other commutators vanish). Note that d_+ diverges as $\Omega\to\omega_x$, whereas d_- remains finite in the same limit. In terms of these operators, the Hamiltonian takes the form [16]

$$H_0=\frac{1}{2}\hbar\omega_+(\alpha_+^\dagger\alpha_++\alpha_+\alpha_+^\dagger)+\frac{1}{2}\hbar\omega_-(\alpha_-^\dagger\alpha_-+\alpha_-\alpha_-^\dagger). \tag{22}$$

C. Lowest Landau-level single-particle states for rotating anisotropic trap

The single-particle ground state φ_{00} is given by the prescription $\alpha_\pm\varphi_{00}=0$, which leads to the explicit representation

$$\varphi_{00}\propto\exp\left(-\frac{Q_+^2}{2d_+^2}-\frac{Q_-^2}{2d_-^2}\right) \tag{23}$$

as a Gaussian function of the two new coordinates $Q_\pm$. Correspondingly, the complete set of normalized single-particle states $\varphi_{n_+n_-}$ is specified by two non-negative integers $n_\pm$

$$\varphi_{n_+n_-}=\frac{(\alpha_+^\dagger)^{n_+}}{\sqrt{n_+!}}\frac{(\alpha_-^\dagger)^{n_-}}{\sqrt{n_-!}}\varphi_{00}. \tag{24}$$

Here, the eigenstate $\varphi_{n_+n_-}$ has $n_+(n_-)$ quanta with frequency $\omega_+(\omega_-)$; the energy eigenvalue is

$$\epsilon_{n_+n_-}=\hbar\omega_+\left(n_++\frac{1}{2}\right)+\hbar\omega_-\left(n_-+\frac{1}{2}\right). \tag{25}$$

This eigenstate $\varphi_{n_+n_-}$ has an angular momentum [16]

$$L_{n_+n_-}=-\frac{\partial\epsilon_{n_+n_-}}{\partial\Omega}=-\hbar\frac{\partial\omega_+}{\partial\Omega}\left(n_++\frac{1}{2}\right)-\hbar\frac{\partial\omega_-}{\partial\Omega}\left(n_-+\frac{1}{2}\right). \tag{26}$$

Since $\partial\omega_+/\partial\Omega$ is negative ($\partial\omega_-/\partial\Omega$ is positive), this result confirms that the plus (minus) mode has positive (negative) angular momentum. For the special case of a symmetric trap, the detailed form of these eigenstates is well known [23,24].

It is important to re-express the ground-state wave function φ_{00} in terms of the original canonical coordinates x and y. Valatin [15] uses the generating function $S(x,y;Q_+,Q_-)$ in Eq. (13) to obtain the explicit (factorized) expression

$$\varphi_{00}(x,y)\propto\exp\left[-\frac{m\gamma(\beta_+x^2+\beta_-y^2)}{2\hbar(1+\beta_+\beta_-)}\right]\times\exp\left\{i\frac{mxy}{\hbar}\left[\frac{\gamma}{1+\beta_+\beta_-}-\frac{1}{2}\left(\frac{\omega_+}{\beta_+}+\frac{\omega_-}{\beta_-}\right)\right]\right\}. \tag{27}$$

Each of these two factors has an interesting structure.

(1) The first factor is a real anisotropic Gaussian with characteristic lengths a_x and a_y given by

$$a_x^2=\frac{1+\beta_+\beta_-}{\beta_+}\frac{\hbar}{m\gamma},\quad a_y^2=\frac{1+\beta_+\beta_-}{\beta_-}\frac{\hbar}{m\gamma}. \tag{28}$$

In the limit of a symmetric trap, this Gaussian ground state becomes $\varphi_{00}(x,y)\propto\exp[-m\omega_0(x^2+y^2)/(2\hbar)]$, with the expected oscillator length $d_0=\sqrt{\hbar/(m\omega_0)}$. More generally, the ground-state density for a rotating anisotropic trap is an anisotropic Gaussian, given by the corresponding normalized wave function

$$|\varphi_{00}(x,y)|^2=\frac{1}{\pi a_xa_y}\exp\left(-\frac{x^2}{a_x^2}-\frac{y^2}{a_y^2}\right). \tag{29}$$

For a rapidly rotating anisotropic trap ($\omega_x<\omega_y$ and $\delta=1-\Omega/\omega_x\to0$), the length a_x diverges because $\beta_+\to0$, but a_y remains finite. In this limit, the ground-state density becomes an essentially one-dimensional strip with Gaussian transverse profile and finite width a_y [25,26].

(2) The second factor of φ_{00} involves a complex phase proportional to xy, which reflects the irrotational flow induced by the rotating anisotropic trap [27–29]. The factor in square brackets (the coefficient of $imxy/\hbar$) has a rather intricate structure. It vanishes for a symmetric trap, because $\beta_\pm=1$ and $\omega_++\omega_-=\gamma=2\omega_0$. It also vanishes for a stationary anisotropic trap, but this limit requires a detailed analysis because each term separately diverges as $\Omega\to0$. This phase

will be seen to play an essential role in the following construction of the lowest Landau-level states.

With Eqs. (17) and (18), the operators $\alpha_\pm$ and $\alpha_\pm^\dagger$ defined in Eqs. (21) are readily expressed in terms of the original coordinates x, y and momenta p_x, p_y. It is not hard to verify explicitly that $\alpha_\pm\varphi_{00}(x,y)=0$. The more interesting question is the form of the low-lying states $\varphi_{n0}(x,y)$, which are the analogs of the lowest Landau-level states but now for an anisotropic rotating trap. In this case, the state φ_{n0} has n quanta of the plus mode (whose frequency ω_+ becomes small $\propto\sqrt{\delta}$ for $\delta=1-\Omega/\omega_x\to 0$), and zero quanta of the minus mode (which has a finite frequency ω_- in the same limit).

The basic relation $\alpha_+\varphi_{00}=0$ implies that

$$(Q_+/\sqrt{2}d_+)\varphi_{00} = -i(d_+P_+/\sqrt{2}\hbar)\varphi_{00}.$$

Thus

$$\alpha_+^\dagger\varphi_{00} = (\sqrt{2}Q_+/d_+)\varphi_{00} = -i(\sqrt{2}d_+P_+/\hbar)\varphi_{00},$$

and a straightforward calculation yields

$$\varphi_{10}(x,y) = \alpha_+^\dagger\varphi_{00}(x,y) = \zeta\varphi_{00}(x,y), \tag{30}$$

where φ_{00} is the normalized ground state, and

$$\zeta = \frac{\sqrt{2}(x+i\beta_- y)}{d_+\lambda_+(1+\beta_+\beta_-)} = \sqrt{\frac{2m\gamma\beta_+}{\hbar}}\frac{x+i\beta_- y}{1+\beta_+\beta_-} = \sqrt{\frac{2}{1+\beta_+\beta_-}}\frac{x+i\beta_- y}{a_x} \tag{31}$$

is a dimensionless complex variable involving a “stretched” combination $x+i\beta_- y$. As shown in Fig. 1(b), the polarization parameter β_- is real and less that 1 in the limit of rapidly rotating anisotropic trap. This complex variable reduces to [4] $\zeta=(x+iy)/d_0$ for a rotating symmetric trap, where $d_0=\sqrt{\hbar/(m\omega_0)}$. In the more general case of a rotating anisotropic trap, the characteristic length that appears in Eq. (31) is essentially a_x from Eq. (28), apart from the common factor $\sqrt{\frac{1}{2}(1+\beta_+\beta_-)}$ and the additional factor β_- for y; both these factors remain finite as $\Omega\to\omega_x$. This quasi-isotropic behavior for ζ is very different from the anisotropy seen in the two lengths a_x and a_y that determine the x and y structure of the ground-state density $|\varphi_{00}|^2$.

The higher states within the lowest Landau level have a similar structure. For example, $\varphi_{20}=(\alpha_+^\dagger)^2\varphi_{00}/\sqrt{2}$ can be written as

$$\varphi_{20}(x,y) = \frac{\alpha_+^\dagger}{\sqrt{2}}\varphi_{10}(x,y) = \frac{\alpha_+^\dagger}{\sqrt{2}}\zeta\varphi_{00}(x,y) = \frac{1}{\sqrt{2}}(-[\zeta,\alpha_+^\dagger]+\zeta^2)\varphi_{00}(x,y). \tag{32}$$

The commutator is readily evaluated with Eqs. (21), (17), (18), and (31), yielding

$$[\zeta,\alpha_+^\dagger] = \frac{1-\beta_+\beta_-}{1+\beta_+\beta_-} \equiv c, \tag{33}$$

which defines the constant c (it depends on the trap frequencies ω_x, ω_y and the rotation speed Ω). In general, $0\leq c\leq 1$, but it vanishes identically for a symmetric trap since $\beta_\pm=1$ in this case. Thus $\varphi_{20}=(\zeta^2-c)\varphi_{00}/\sqrt{2}$, namely an even polynomial in ζ times the complex Gaussian ground state φ_{00}. As a check on this analysis, note that $[\alpha_+,\alpha_+^\dagger]\varphi_{00}=\varphi_{00}$, and direct calculation verifies that $[\alpha_+,\zeta]=1$.

The general lowest Landau-level state follows from similar arguments [it is essential here that the commutator (33) is a pure number, independent of x and y]

$$\varphi_{n0}(x,y) = \frac{1}{\sqrt{n!}}p_n(\zeta)\varphi_{00}(x,y), \tag{34}$$

where $p_n(\zeta)$ is a polynomial of order n that obeys the symmetry condition $p_n(-\zeta)=(-1)^n p_n(\zeta)$. These polynomials are easily obtained recursively with the relation

$$p_{n+1}(\zeta) = \zeta p_n(\zeta) - c\frac{dp_n(\zeta)}{d\zeta}, \tag{35}$$

with the first few given explicitly as $p_0=1$, $p_1=\zeta$, $p_2=\zeta^2-c$, $p_3=\zeta^3-3c\zeta,\ldots$. The Hermite polynomials $H_n(x)$ obey a similar recursion relation [30] $H_{n+1}(x)=2xH_n(x)-H_n'(x)$. Direct comparison shows that

$$p_n(\zeta) = \left(\frac{c}{2}\right)^{n/2} H_n\left(\frac{\zeta}{\sqrt{2c}}\right), \tag{36}$$

which readily reproduces the explicit forms given above for small $n=0,\ldots,3$. Oktel [17] obtained an analogous but less general result in the special limit of small anisotropy and rapid rotation. For a symmetric trap (with $\beta_\pm=1$ and $c=0$), it follows directly that $p_n(\zeta)$ reduces to the nth power of $(x+iy)/d_0$.

D. Expectation value of single-particle H_0 for general lowest-Landau-level state

Let $\psi_{\text{LLL}}=\Sigma_n c_n\varphi_{n0}$ be a general linear combination of lowest-Landau-level states $\{\varphi_{n0}\}$, normalized with the condition $\int dxdy|\psi_{\text{LLL}}|^2=1$. The expectation value of H_0 in Eq. (1) is given by the equivalent operators in Eq. (22)

$$\langle H_0\rangle = \int dxdy\,\psi_{\text{LLL}}^*\left[\frac{1}{2}\hbar\omega_+(\alpha_+^\dagger\alpha_+ + \alpha_+\alpha_+^\dagger) + \frac{1}{2}\hbar\omega_-(\alpha_-^\dagger\alpha_- + \alpha_-\alpha_-^\dagger)\right]\psi_{\text{LLL}}, \tag{37}$$

where the angular brackets denote the expectation value with the state ψ_{LLL}. Since $\alpha_-\psi_{\text{LLL}}$ vanishes (by construction), this quantity reduces to

$$\langle H_0\rangle = \frac{1}{2}\hbar\omega_- + \frac{1}{2}\hbar\omega_+\int dxdy\,\psi_{\text{LLL}}^*(\alpha_+^\dagger\alpha_+ + \alpha_+\alpha_+^\dagger)\psi_{\text{LLL}}, \tag{38}$$

where the first term is just the zero-point energy of the unoccupied minus mode. For a symmetric trap, this expectation value is readily expressed in terms of the expectation value $\langle x^2+y^2\rangle$ [4,8]. As shown below, a similar but more intricate result holds for the rotating anisotropic trap.

It is convenient to start from Eqs. (19) that express x and y in terms of the new canonical variables $Q_\pm$ and $P_\pm$. In turn, these operators are simply linear combinations of the corresponding oscillator variables $\alpha_\pm^\dagger$ and $\alpha_\pm$, as follows from Eq. (21). For example,

$$x = \frac{d_+\lambda_+}{\sqrt{2}}(\alpha_+ + \alpha_+^\dagger) - \frac{\hbar}{\sqrt{2}id_-\lambda_- m\gamma}(\alpha_- - \alpha_-^\dagger). \quad (39)$$

The expectation value of x^2 is then given by

$$\langle x^2\rangle = \frac{\hbar}{2m\gamma\beta_+}\langle\alpha_+^\dagger\alpha_+ + \alpha_+\alpha_+^\dagger\rangle + \frac{\hbar}{2m\gamma\beta_+}\langle(\alpha_+^\dagger)^2 + (\alpha_+)^2\rangle + \frac{\hbar\beta_-}{2m\gamma}, \quad (40)$$

where the cross terms between plus and minus operators vanish because $\langle\alpha_-\rangle = \langle\alpha_-^\dagger\rangle = 0$, and I use the relations $d_\pm^2\lambda_\pm^2 = (\hbar/m\omega_\pm)(\omega_\pm/\mu_\pm) = \hbar/m\gamma\beta_\pm$. A similar calculation gives

$$\langle y^2\rangle = \frac{\hbar\beta_+}{2m\gamma}\langle\alpha_+^\dagger\alpha_+ + \alpha_+\alpha_+^\dagger\rangle - \frac{\hbar\beta_+}{2m\gamma}\langle(\alpha_+^\dagger)^2 + (\alpha_+)^2\rangle + \frac{\hbar}{2m\gamma\beta_-}, \quad (41)$$

and an appropriate linear combination leads to the quantity in Eq. (38). In this way, the desired LLL expectation value $\langle H_0\rangle$ has the simple form

$$\langle H_0\rangle = \frac{1}{2}\hbar\omega_- - \frac{1}{4}\hbar\omega_+\left(\beta_+\beta_- + \frac{1}{\beta_+\beta_-}\right) + \frac{1}{2}m\gamma\omega_+\left(\beta_+\langle x^2\rangle + \frac{1}{\beta_+}\langle y^2\rangle\right). \quad (42)$$

For a symmetric trap with $\omega_x = \omega_y = \omega_0$, this result reduces to the familiar LLL expression $\langle H_0\rangle = \hbar\Omega + m\omega_0(\omega_0 - \Omega)\langle r^2\rangle$ [4,8], where $r^2 = x^2 + y^2$.

In the case of a symmetric condensate, the special properties of the LLL states yield a simple well-known relation between the expectation value of the angular momentum and the mean-squared radius [4,8]

$$\frac{\langle L_z\rangle}{\hbar} = \frac{\langle r^2\rangle}{d_0^2} - 1. \quad (43)$$

For an anisotropic condensate, an analogous result follows from the expectation value $\langle L_z\rangle = \langle xp_y - yp_x\rangle$ in a LLL state. Equations (19) for x and y and the corresponding relations for p_x and p_y lead to an expression involving the combinations $\langle\alpha_+^\dagger\alpha_+ + \alpha_+\alpha_+^\dagger\rangle$ and $\langle(\alpha_+^\dagger)^2 + (\alpha_+)^2\rangle$. Comparison with Eqs. (40) and (41) and use of Eq. (15) then yields the following generalization of Eq. (43)

$$\frac{\langle L_z\rangle}{\hbar} = \frac{m\gamma}{2\hbar}\langle x^2 + y^2\rangle + \frac{m\omega_-}{2\hbar}\left(\beta_- - \frac{1}{\beta_-}\right)\langle x^2 - y^2\rangle - \frac{1}{2}\left(\beta_- + \frac{1}{\beta_-}\right). \quad (44)$$

For a symmetric trap ($\omega_x = \omega_y = \omega_0$), it follows by inspection that this result has the correct limit (43), because $\gamma \to 2\omega_0$, $\langle y^2\rangle = \langle x^2\rangle$, and $\beta_- \to 1$.

III. INTERACTING GAS IN A ROTATING ANISOTROPIC TRAP

A dilute interacting Bose-Einstein condensate with N particles in a trap is described by a condensate wave function ψ that is here normalized to unity, with $\int dxdy|\psi|^2 = 1$. The Gross-Pitaevskii (GP) energy functional for this system involves both the noninteracting Hamiltonian H_0 from Eq. (1) and the interaction energy

$$E[\psi] = \int dxdy\left(\psi^* H_0\psi + \frac{1}{2}g_{2D}N|\psi|^4\right), \quad (45)$$

where g_{2D} is a two-dimensional coupling constant with dimensions of energy times area. If the condensate is confined in a tight axial harmonic trap with oscillator length $d_z = \sqrt{\hbar/(m\omega_z)}$, then $g_{2D} = \sqrt{8\pi}\hbar^2 a_s/(md_z)$, where a_s is the s-wave scattering length [8,9,31]. In contrast, for a condensate that is uniform in the z direction with axial length Z, the analogous relation is $g_{2D} = 4\pi\hbar^2 a_s/(mZ)$. The Euler-Lagrange equation for the wave function is the stationary GP equation

$$H_0\psi + g_{2D}N|\psi|^2\psi = \mu\psi, \quad (46)$$

where the chemical potential μ is fixed by the normalization of ψ.

A. Lowest Landau-level limit for rapid rotation

If the trap rotates rapidly, the condensate wave function ψ can be approximated by a general LLL state $\psi_{LLL} = \Sigma_n c_n\varphi_{n0}$. In this case, the expectation value of H_0 simplifies considerably to Eq. (42). Correspondingly, the total energy functional takes the approximate form

$$E_{\rm LLL}[\psi] = \frac{1}{2}\hbar\omega_- - \frac{1}{4}\hbar\omega_+\left(\beta_+\beta_- + \frac{1}{\beta_+\beta_-}\right) + \int dxdy\left[\frac{1}{2}m\gamma\omega_+\left(\beta_+x^2 + \frac{1}{\beta_+}y^2\right)|\psi|^2 + \frac{1}{2}g_{2D}N|\psi|^4\right], \quad (47)$$

where I now follow Ref. [8] and omit the subscript LLL on the condensate wave function.

If the restriction to the LLL states is ignored, the absolute minimum of $E_{\rm LLL}[\psi]$ is found by varying $|\psi|^2$ subject solely to the normalization condition. The resulting approximate GP equation

$$\frac{1}{2}m\gamma\omega_+\beta_+x^2 + \frac{1}{2}m\gamma\omega_+\frac{1}{\beta_+}y^2 + g_{2D}N|\psi|^2 = \mu \quad (48)$$

implies an anisotropic density distribution

$$|\psi_{\rm min}(x,y)|^2 = \frac{\mu}{g_{2D}N}\left(1 - \frac{x^2}{R_x^2} - \frac{y^2}{R_y^2}\right), \quad (49)$$

with characteristic condensate radii given by

$$R_x^2 = \frac{2\mu}{m\gamma\omega_+\beta_+}, \quad R_y^2 = \frac{2\mu\beta_+}{m\gamma\omega_+}. \tag{50}$$

Note that the ratio $R_y/R_x = \beta_+ \propto \omega_+$ vanishes as $\Omega \to \omega_x$, but the behavior of the individual condensate radii requires a study of the chemical potential μ.

As emphasized by various authors [4–9], this density is very similar to the familiar Thomas-Fermi form for a nonrotating condensate in a stationary two-dimensional trap. In that case, the repulsive interactions expand the condensate and reduce the kinetic energy compared to the trap energy and the interaction energy. The situation here is very different, because the approximate LLL wave function explicitly incorporates the full single-particle Hamiltonian, including the kinetic energy; in this context, the appearance of the squared coordinates x^2 and y^2 arises from the special properties of the LLL states, specifically the result in Eq. (42).

The normalization condition for $|\psi|^2$ in Eq. (49) readily yields the condition

$$\mu = \sqrt{\frac{m\gamma\omega_+ g_{2D} N}{\pi}}. \tag{51}$$

Note that the chemical potential μ vanishes proportional to $\sqrt{\omega_+} \propto \delta^{1/4}$ for a rapidly rotating anisotropic trap, where $\delta = 1 - \Omega/\omega_x \to 0$. Equation (49) then shows that the central density $|\psi_{\rm min}(0,0)|^2$ also has the same behavior in this limit.

A combination of Eqs. (50) and (51) gives the condensate radii

$$R_x^2 = \frac{2}{\beta_+}\sqrt{\frac{g_{2D}N}{\pi m\gamma\omega_+}}, \quad R_y^2 = 2\beta_+\sqrt{\frac{g_{2D}N}{\pi m\gamma\omega_+}}. \tag{52}$$

Correspondingly, the normalized minimizing density (49) becomes

$$|\psi_{\rm min}(x,y)|^2 = \frac{2}{\pi R_x R_y}\left(1 - \frac{x^2}{R_x^2} - \frac{y^2}{R_y^2}\right). \tag{53}$$

Since ω_+ and β_+ are both proportional to $\sqrt{\delta}$ as $\delta \to 0$ for fixed trap anisotropy, it is clear that R_x^2 grows like $\delta^{-3/4}$ and R_y^2 shrinks like $\delta^{1/4}$ for small δ, which reflects the conservation of total number of particles. In particular, the total area $\pi R_x R_y$ diverges like $\omega_+^{-1/2} \propto \delta^{-1/4}$. This anisotropy of the minimizing N-body condensate in the limit $\delta \to 0$ is quite different from the anisotropy of the LLL single-particle ground state φ_{00}, where Eq. (28) shows that a_x grows but a_y approaches a constant as $\delta \to 0$. For the minimizing density (53), it is straightforward to evaluate the mean-squared displacements $\langle x^2 \rangle = (1/6)R_x^2$ and $\langle y^2 \rangle = (1/6)R_y^2$. The mean angular momentum in Eq. (44) then becomes

$$\frac{\langle L_z \rangle}{\hbar} = \frac{m}{12\hbar}(\omega_+\beta_+ + \omega_-\beta_-)R_x^2 + \frac{m}{12\hbar}\left(\frac{\omega_+}{\beta_+} + \frac{\omega_-}{\beta_-}\right)R_y^2 - \frac{1}{2}\left(\beta_- + \frac{1}{\beta_-}\right). \tag{54}$$

The condition $g_{2D}n(0) \ll \hbar\omega_-$ for the validity of the lowest Landau-level approximation can now be made explicit. Use of $g_{2D}n(0) \approx 2g_{2D}N/(\pi R_x R_y)$ from Eq. (53), Eq. (52) for the condensate radii and the relation $g_{2D} \approx 4\pi\hbar^2 a_s/(mZ)$ for a uniform condensate of thickness Z yields

$$2\sqrt{\frac{\gamma\omega_+}{\omega_-^2}\frac{Na_s}{Z}} \ll 1. \tag{55}$$

Since $\omega_+ \to 0$ and ω_- remains nonzero for sufficiently rapid rotation ($\Omega \to \omega_x$), this condition can always be satisfied.

In the special case of a symmetric trap, the minimizing density has the isotropic form [8]

$$|\psi_{\rm min}(r)|^2 = \frac{2}{\pi R_0^2}\left(1 - \frac{r^2}{R_0^2}\right) \tag{56}$$

with

$$R_0^2 = \left[\frac{2g_{2D}N}{\pi m\omega_0(\omega_0 - \Omega)}\right]^{1/2}. \tag{57}$$

This squared condensate radius diverges as $\Omega \to \omega_0$. Similarly, the absolute minimum of the LLL energy functional (47) for a symmetric trap becomes [8] $E_{\rm LLL}|_{\rm min} = \hbar\Omega + \frac{2}{3}m\omega_0^2 R_0^2$. As a simple check, it is easy to verify that $-\partial E_{\rm LLL}|_{\rm min}/\partial\Omega = \hbar(\frac{1}{3}R_0^2/d_0^2 - 1)$; this result agrees with Eq. (43) because $\langle r^2 \rangle = \langle x^2 + y^2 \rangle = (1/3)R_0^2$ in the symmetric limit.

B. Density of vortices

The general LLL state $\psi_{\rm LLL}$ is a linear combination of the states φ_{n0}. Apart from a normalization factor $1/\sqrt{n!}$, each of these states is a polynomial $p_n(\zeta)$ multiplied by the ground state φ_{00}, where ζ from Eq. (31) is proportional to $x + i\beta_- y$. Thus the general LLL state also involves a polynomial in ζ that can formally be factorized to write

$$\psi_{\rm LLL} \propto \varphi_{00}\prod_j (\zeta - \zeta_j). \tag{58}$$

The corresponding LLL particle density $n_{\rm LLL} = |\psi_{\rm LLL}|^2$ becomes

$$n_{\rm LLL} \propto |\varphi_{00}|^2 \prod_j |\zeta - \zeta_j|^2. \tag{59}$$

Apart from an additive constant, the logarithm of this relation gives [4,8,32,33]

$$\sum_j \ln|(x - x_j)^2 + \beta_-^2(y - y_j)^2| = \frac{x^2}{a_x^2} + \frac{y^2}{a_y^2} + \ln n_{\rm LLL}(\boldsymbol{r}) = \frac{1}{a_x^2}\left(x^2 + \frac{\beta_- y^2}{\beta_+}\right) + \ln n_{\rm LLL}(\boldsymbol{r}). \tag{60}$$

Here I use Eqs. (28) and (29) for the anisotropic ground state and note that $a_y^2\beta_- = a_x^2\beta_+$.

To include the anisotropy of the complex variable $\zeta \propto x + i\beta_- y$, it is convenient to introduce the rescaled variables $x' = x$ and $y' = \beta_- y$. Application of the rescaled

Laplacian $\nabla'^2=\partial^2/\partial x'^2+\partial^2/\partial y'^2$ readily gives

$$\sum_j \nabla'^2 \ln|\boldsymbol{r}'-\boldsymbol{r}'_j|^2 = \frac{2}{a_x^2}\left(1+\frac{1}{\beta_+\beta_-}\right)+\nabla'^2 \ln n_{\text{LLL}}(\boldsymbol{r}). \tag{61}$$

Since $\nabla'^2 \ln|\boldsymbol{r}'|^2=4\pi\delta(x')\delta(y')=(4\pi/\beta_-)\delta(x)\delta(y)$, this relation implies

$$n_v(\boldsymbol{r}) = \frac{m\gamma}{2\pi\hbar}+\frac{\beta_-}{4\pi}\nabla'^2 \ln n_{\text{LLL}}(\boldsymbol{r}), \tag{62}$$

where $n_v(\boldsymbol{r})=\Sigma_j\delta^{(2)}(\boldsymbol{r}-\boldsymbol{r}_j)$ is the two-dimensional vortex density, and I have again used Eq. (28). For a rapid rotation speed $\Omega\lesssim\omega_x\leqslant\omega_y$ and any trap anisotropy, the frequency $\gamma=(\omega_-^2-\omega_+^2)/(2\Omega)$ is given in terms of elementary expressions from Eqs. (3) and (4), as shown in Fig. 1(a) for a typical anisotropy ratio $\omega_y/\omega_x=1.2$.

To estimate the vortex density n_v in Eq. (62), assume that the equilibrium particle density n_{LLL} is that given by the absolute minimum solution in Eq. (49), with $n_{\text{LLL}}(\boldsymbol{r})\propto 1-x^2/R_x^2-y^2/R_y^2$. A straightforward calculation then yields

$$n_v(x,y) \approx \frac{m\gamma}{2\pi\hbar} - \frac{\beta_-}{2\pi(1-x^2/R_x^2-y^2/R_y^2)^2} \times\left[\frac{1}{R_x^2}+\frac{1}{\beta_-^2R_y^2}+\left(\frac{y^2}{R_y^2}-\frac{x^2}{R_x^2}\right)\left(\frac{1}{\beta_-^2R_y^2}-\frac{1}{R_x^2}\right)\right]. \tag{63}$$

For an isotropic trap ($\omega_x=\omega_y=\omega_0$), this expression reduces to the well-known axisymmetric result [7,32,33]

$$n_v(r) \approx \frac{m\omega_0}{\pi\hbar} - \frac{1}{\pi R_0^2}\frac{1}{(1-r^2/R_0^2)^2}, \tag{64}$$

where R_0^2 is given in Eq. (57).

Comparison of these two expressions shows some interesting differences.

(1) For fixed angular velocity $\Omega\lesssim\omega_0$, the vortex density (64) in a rapidly rotating symmetric trap decreases gradually and isotropically away from the center of the condensate. Such behavior has been observed at lower angular velocities in the mean-field Thomas-Fermi regime [34]. In contrast, the general expression for the vortex density in Eq. (63) displays explicit anisotropy between x and y.

(2) To sharpen this analysis, it is convenient to focus on the central vortex density

$$n_v(0) \approx \frac{m\gamma}{2\pi\hbar} - \frac{1}{2\pi}\left(\frac{\beta_-}{R_x^2}+\frac{1}{\beta_-R_y^2}\right). \tag{65}$$

For a symmetric trap, Eq. (64) shows that $n_v(0)$ increases monotonically with increasing $\Omega\lesssim\omega_0$, because the mean condensate radius R_0 grows in the same limit. For an anisotropic trap, in contrast, the frequency γ and R_x^{-2} both decrease with increasing Ω, whereas R_y^{-2} increases. Thus the combined effect of anisotropy and rapid rotation can, in principle, yield a central vortex density $n_v(0)$ that varies nonmonotonically with increasing Ω, as can be seen for typical numerical examples.

It is not clear whether either of these behaviors would be observable in practice.

The present discussion has focused on the "macroscopic" parabolic density profile that provides an absolute minimum of the energy in the rotating frame, ignoring the local distortions associated with the vortex cores. In practice, these phenomena have very different length scales: in the rapidly rotating limit, the vortex core and the intervortex distance are both of order $\sqrt{\hbar/(m\Omega)}$, whereas the condensate radii R_x and R_y are much larger.

Thus it is possible to separate these two effects and write the number density in the form $|\psi(\boldsymbol{r})|^2\approx|\bar{\psi}(\boldsymbol{r})|^2|f(\boldsymbol{r})|^2$ as the product of a slowly varying envelope $|\bar{\psi}(\boldsymbol{r})|^2$ and a rapidly varying function $|f(\boldsymbol{r})|^2$ that vanishes at the center of each vortex [5,8,9,35,36]. For a large vortex lattice, the modulating function $|f(\boldsymbol{r})|^2$ is effectively periodic, and it is convenient to normalize it so that $\int_{\text{cell}}d^2r|f(\boldsymbol{r})|^2=1$ over each unit cell. Substitution of this approximate order parameter into the LLL energy functional (47) yields integrals of products of slowly varying functions involving powers of $|\bar{\psi}|^2$ and rapidly varying periodic modulation functions involving powers of $|f|^2$. As shown in detail in Ref. [8], these integrals can be factorized into averages of $|f|^2$ or $|f|^4$ over a unit cell and integrals of $|\bar{\psi}|^2$ or $|\bar{\psi}|^4$ over the whole condensate. If the vortex lattice is treated as triangular and unbounded, this analysis yields a simple renormalization [5,7–9,37] of the interaction constant $g_{2\text{D}}\to bg_{2\text{D}}$, where $b=\int_{\text{cell}}d^2r|f(\boldsymbol{r})|^4\approx 1.1596$ is the numerical value for a triangular Abrikosov vortex lattice [38]. Apart from this rescaling of the interaction parameter, the description remains essentially unchanged.

IV. CONCLUSIONS AND DISCUSSION

This work has examined the behavior of a two-dimensional Bose-Einstein condensate in an anisotropic harmonic trap (with general trap frequencies $\omega_x\leqslant\omega_y$) that rotates rapidly at an angular velocity $\Omega\lesssim\omega_x$. The single-particle Hamiltonian in Eq. (1) is exactly soluble [15–17], although the detailed form of the low-lying quantum-mechanical states in Eq. (34) for an arbitrary anisotropy has apparently not appeared previously.

The ground-state wave function $\varphi_{00}(x,y)$ in Eq. (27) is an anisotropic Gaussian with Ω-dependent characteristic lengths a_x and a_y given in (28). In addition, the ground state has a phase proportional to xy [28,29]. In the quantum problem, this behavior reflects the classical velocity potential for an irrotational (vortex-free) fluid confined in a rotating elliptical boundary [27].

Similar to the case of a rapidly rotating symmetric condensate, the two eigenfrequencies $\omega_\pm$ in Eq. (2) for a rapidly rotating anisotropic condensate have very different magnitudes: ω_+ vanishes as $\Omega\to\omega_x$ (the smaller of the two trap frequencies), but ω_- remains finite in this same limit. If the mean interaction energy $\sim g_{2\text{D}}n(0)$ is small compared to the gap $\hbar\omega_-$ between the ground state and the first excited Landau level, then the system can be described with the set of lowest Landau-level states $\varphi_{n0}(x,y)$, where n is a non-negative integer describing the number of plus quanta, each with the small energy $\hbar\omega_+$. Apart from the Gaussian factor $\varphi_{00}(x,y)$, these states involve a polynomial $p_n(\zeta)$ of order n, where ζ single complex variable proportional to $x+i\beta_- y$ and $\beta_-\leqslant 1$ is a positive constant for any $0<\Omega\leqslant\omega_x$.

Within the set of lowest Landau-level states $\psi_{\rm LLL}=\Sigma_n c_n\varphi_{n0}$, the expectation value $\langle H_0\rangle$ of the single-particle Hamiltonian can be reduced to an anisotropic linear combination of $\langle x^2\rangle$ and $\langle y^2\rangle$. An anisotropic Thomas-Fermi-like density $|\psi_{\rm min}(x,y)|^2\propto 1-x^2/R_x^2-y^2/R_y^2$ provides an absolute lower bound for the total energy of the interacting system. Here the condensate radii R_x and R_y are given in Eq. (52); R_x^2 grows and R_y^2 shrinks as $\Omega\to\omega_x$, while the area $\pi R_x R_y$ of the elliptical condensate grows slowly in the same limit.

As emphasized by Ho [4], the particle density $n_{\rm LLL}(x,y)=|\psi_{\rm LLL}(x,y)|^2$ in the lowest Landau-level limit also contains a description of the associated vortex density $n_v(x,y)$. This situation arises because the general linear combination of lowest Landau-level states is essentially a polynomial $P(\zeta)$ in the single complex variable $\zeta\propto x+i\beta_- y$, and the zeros $\{\zeta_j\}$ of $P(\zeta)$ represent the positions of the vortices in the xy plane. The actual particle density $n_{\rm LLL}$ has small-scale structure arising from the vortex cores, superposed on the parabolic global shape. If this fine-grain aspect is ignored, the resulting vortex density follows immediately in Eq. (63).

This work raises several interesting questions.

(1) Feynman's familiar expression for the mean vortex density $n_F=m\Omega/(\pi\hbar)$ in a large symmetric rotating condensate requires modification for an anisotropic rotating condensate because of the irrotational flow induced by the rotating walls. Specifically, this irrotational flow contributes to the total angular momentum and thus lowers the energy $E'=E-\Omega L_z$ in the rotating frame, delaying the first transition to a state containing a single quantized vortex [28]. Indeed, the measured critical angular velocity Ω_c for the appearance of the first vortex in a rigid elliptical or rectangular cylinder containing uniform superfluid ^{4}He exceeds that for a circular cylinder by a factor that increases with increasing anisotropy [39], confirming the theoretical prediction. A similar but more complicated situation occurs for slowly rotating weakly interacting anisotropic Bose-Einstein condensates [16]. Although the analogous situation with many vortices has not been studied in detail, the increased critical angular velocity for the appearance of the first vortex suggests that the mean vortex density at moderate rotation speeds in an anisotropic trap is likely to be smaller than in a corresponding symmetric trap.

In contrast, for a rapidly rotating anisotropic condensate, Eq. (65) gives the central density $n_v(0)\approx m\gamma/(2\pi\hbar)$, apart from finite-size corrections associated with the condensate radii. This value of $n_v(0)$ typically exceeds n_F, because $\gamma/(2\Omega)\to(3\omega_x^2+\omega_y^2)/(4\omega_x^2)>1$ in the limit $\Omega\to\omega_x$. The physical basis for such enhanced vortex density is not immediately obvious, and additional clarification would be desirable.

(2) The present LLL approach ignores the fine-grain structure of the vortex lattice (apart from the renormalization of the interaction parameter g_{2D} by the "Abrikosov parameter" $b\sim 1.16$) [5,7–9,37]. In the special case of a nearly symmetric trap rotating close to the limit of instability, Oktel [17] finds that the vortex lattice remains almost exactly triangular, with the principal lattice planes aligned with the direction of weak trap confinement. It is not clear whether this situation holds for arbitrary anisotropy and less extreme rotation speeds.

(3) As Ω approaches the weak confining frequency ω_x, the condensate becomes essentially a one-dimensional strip, and the vortices must then rearrange themselves to form parallel rows [25,26]. Ultimately, such behavior may be preempted by some sort of transition to a correlated (nonsuperfluid) state, as has been predicted for a rapidly rotating symmetric trap [40].

It will be interesting to investigate these various questions in detail.

ACKNOWLEDGMENTS

I thank G. Bertsch for early guidance on the literature about rotating deformed nuclei and rotating anisotropic oscillator potentials. A. Aftalion, T.L. Ho, and Z. Tešanović have made valuable comments and suggestions, for which I am most grateful.

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