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Optimal control of entanglement via quantum feedback

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It has recently been shown that finding the optimal measurement on the environment for stationary linear quadratic Gaussian control problems is a semidefinite program. We apply this technique to the control of the Einstein-Podolsky-Rosen correlations between two bosonic modes interacting via a parametric Hamiltonian at steady state. The optimal measurement turns out to be nonlocal homodyne measurement—the outputs of the two modes must be combined before measurement. We also find the optimal local measurement and control technique. This gives the same degree of entanglement but a higher degree of purity than the local technique previously considered [S. Mancini, Phys. Rev. A **73**, 010304(R) (2006)].

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I. INTRODUCTION

Quantum feedback control is a well-established theoretical technique for stabilizing an open quantum system in a state with certain desired properties [1–3]. The basic idea is to use the information that leaks from the system into a bath to undo the undesirable effects of coupling to this, or other, baths. Notable examples include protecting a "Schrödinger cat" superposition [4], correcting errors in encoded quantum information [5,6], maintaining a two-level atom in an arbitrary state [7], deterministically producing entangled states of spins [8,9] (which has been experimentally demonstrated [10]), and cooling various systems to (close to) their ground states [11–14].

Recently, one of us started to consider the application of quantum feedback to control entanglement. Preliminary studies have been carried out for two interacting qubits [15] and two interacting bosonic modes [16], damped to independent baths. Physically, two damped and interacting bosonic modes could be realized by optical cavity modes coupled by a $\chi^{(2)}$ nonlinearity. Such a nonlinearity induces only a finite amount of entanglement between the modes in steady state. By contrast, it was shown in Ref. [16] that performing homodyne detection on the two outputs, and using these currents to modulate the (linear) driving of the two modes, could, under ideal conditions, increase this entanglement without limit.

The quantum control problem of Ref. [16] has, like many which have been considered [1–3,8,14], an analogy in the class of classical linear quadratic Gaussian (LQG) problems [17]—that is, systems with linear dynamics and a linear map from inputs to outputs, an aim that can be expressed in terms of minimizing a quadratic function, and Gaussian noise in the dynamics and the outputs. The quantum LQG problem has recently been analyzed in detail by one of us and Doherty [18]. In particular, it was shown in Ref. [18] that in the quantum case, there is an extra level of optimization that naturally arises: choosing the optimal *unraveling* (a way to extract information from the bath) given a fixed system-bath coupling.

In this paper we reconsider the problem of Ref. [16] from the perspective of Ref. [18]. We formulate the problem as an LQG control problem and find the optimal unraveling. This is different from the unraveling used in Ref. [16] (it requires interfering the two output beams prior to homodyne detection) and leads to greater entanglement for any strength of the nonlinearity. This shows the usefulness of the general techniques of Ref. [18].

The paper is organized as follows. In Sec. II we summarize the general theory of quantum LQG control problems from Ref. [18] as needed for the current problem. In Sec. III the problem of maximizing the steady-state entanglement between two bosonic modes interacting via parametric Hamiltonian is addressed and the optimal unraveling found. As stated above, this involves interfering the output beams from the two modes prior to detection, which could be regarded as a nonlocal measurement. In Sec. IV we consider the constraint of local measurements (that is, independent measurements of the two outputs). We consider a variety of measurement and feedback schemes, including homodyne and heterodyne detection, and find that two schemes, considered in Ref. [16], and another (more symmetric) scheme are the best. However, when it comes to the purity of the stationary entangled state, considered in Sec. V, the more symmetric scheme is superior. Finally, Sec. VI is for conclusions.

II. QUANTUM LQG CONTROL PROBLEMS

A. Continuous Markovian unravelings of open systems

We consider open quantum systems whose average (that is, unconditional) evolution can be described by an autonomous differential equation for the state matrix ρ. The most general such equation is the Lindblad master equation [19]

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$$\dot{\rho} = -i[\hat{H},\rho] + \mathcal{D}[\hat{\mathbf{c}}]\rho \equiv \mathcal{L}_0\rho. \quad (1)$$

Here $\hat{H}=\hat{H}^\dagger$ is the system Hamiltonian (we use $\hbar=1$ throughout the paper), while $\hat{\mathbf{c}}$ is a vector of operators $\hat{\mathbf{c}}=(\hat{c}_1,\ldots,\hat{c}_K)^\top$, which need not be Hermitian, with K indicating the number of channels through which the system interacts with the environment. The action of $\mathcal{D}[\hat{\mathbf{c}}]$ on an arbitrary operator ρ is defined by

$$\mathcal{D}[\hat{\mathbf{c}}]\rho \equiv \sum_{k=1}^{K}\left[\hat{c}_k\rho\hat{c}_k^\dagger - \frac{1}{2}(\hat{c}_k^\dagger\hat{c}_k\rho + \rho\hat{c}_k^\dagger\hat{c}_k)\right]. \quad (2)$$

Master equations of this form can typically be derived if the system is coupled weakly to an environment that is large (i.e., with dense energy levels). Under these conditions, it is possible to measure the environment continually on a time scale much shorter than any system time of interest. This *monitoring* yields information about the system, producing a stochastic *conditional* system state ρ_c, which *on average* reproduces the unconditional state ρ. That is, the master equation is *unraveled* into stochastic quantum trajectories [20], with different measurements on the environment leading to different unravelings.

For the purposes of this paper we can restrict ourselves to unravelings that yield an evolution for ρ_c that is continuous and Markovian. In that case, it must be of the form [21]

$$d\rho_c = \mathcal{L}_0\rho_c dt + d\mathbf{z}^\dagger(t)\Delta_c\hat{\mathbf{c}}\rho_c + \rho_c\Delta_c\hat{\mathbf{c}}^\dagger d\mathbf{z}(t). \quad (3)$$

Note that here the $\dagger$ indicates transpose ($\top$) of the vector and Hermitian adjoint of its components. We are also using the notation $\Delta_c\hat{o}\equiv\hat{o}-\langle\hat{o}\rangle_c$, where $\langle\hat{o}\rangle_c\equiv\mathrm{Tr}[\rho_c\hat{o}]$. Finally, we have introduced a vector $d\mathbf{z}=(dz_1,\ldots,dz_K)^\top$ of infinitesimal complex Wiener increments [22]. It satisfies $\mathrm{E}[d\mathbf{z}]=0$, where E denotes expectation value, and for efficient detection has the correlations [21]

$$d\mathbf{z}d\mathbf{z}^\dagger = I dt, \quad d\mathbf{z}d\mathbf{z}^\top = \Upsilon dt. \quad (4)$$

Here Υ is a symmetric complex matrix, which is constrained only by the condition $U\geq 0$, where

$$U \equiv \frac{1}{2}\begin{pmatrix} I+\mathrm{Re}[\Upsilon] & \mathrm{Im}[\Upsilon] \\ \mathrm{Im}[\Upsilon] & I-\mathrm{Re}[\Upsilon]\end{pmatrix}. \quad (5)$$

We call this the unraveling matrix because it parametrizes the possible diffusive unravelings of the master equation (1). That is, it parametrizes the possible sorts of stochasticity in Eq. (3) for the conditional state ρ_c. The stochasticity is provided by the randomness in the measurement record upon which the state ρ_c is conditioned. The record can be represented by a complex time-dependent vector containing the same noise vector as in quantum trajectory [21]:

$$\mathbf{J}^\top dt = \langle\hat{\mathbf{c}}^\top I + \hat{\mathbf{c}}^\dagger\Upsilon\rangle_c dt + d\mathbf{z}^\top. \quad (6)$$

Following the terminology from quantum optics [20], we will call $\mathbf{J}$ a current.

B. Linear systems

We now specialize to systems of N degrees of freedoms, with the nth described by the canonically conjugate pair obeying the commutation relations $[\hat{q}_n,\hat{p}_n]=i$. Defining a vector of operators

$$\hat{\mathbf{x}} = (\hat{q}_1,\hat{p}_1,\ldots,\hat{q}_N,\hat{p}_N)^\top, \quad (7)$$

we can write $[\hat{x}_n,\hat{x}_m]=i\Sigma_{nm}$, where Σ is the $(2N)\times(2N)$ symplectic matrix

$$\Sigma = \bigoplus_{n=1}^{N}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} = \Sigma^* = -\Sigma^\top = -\Sigma^{-1}. \quad (8)$$

For a system with such a phase-space structure we can define a Gaussian state as one with a Gaussian Wigner function [23]. We write the mean vector as $\langle\hat{\mathbf{x}}\rangle$ and its covariance matrix as V:

$$V_{nm} = (\langle\Delta\hat{x}_n\Delta\hat{x}_m\rangle + \langle\Delta\hat{x}_m\Delta\hat{x}_n\rangle)/2. \quad (9)$$

For these to define a quantum states, the necessary and sufficient condition is that [24]

$$V + i\Sigma/2 \geq 0. \quad (10)$$

To obtain linear dynamics for our system in phase space we require that $\hat{H}$ be quadratic and $\hat{\mathbf{c}}$ linear in $\hat{\mathbf{x}}$:

$$\hat{H} = (1/2)\hat{\mathbf{x}}^\top G\hat{\mathbf{x}} - \hat{\mathbf{x}}^\top\Sigma B\mathbf{u}(t), \quad \hat{\mathbf{c}} = \tilde{C}\hat{\mathbf{x}}, \quad (11)$$

where G is real and symmetric and B is real. The second term in $\hat{H}$ is linear in $\hat{\mathbf{x}}$ to ensure a linear map between the time-dependent classical input $\mathbf{u}(t)$ to the system and the output current $\mathbf{J}(t)$. For such a system, the unconditional master equation (1) has a Gaussian state as its solution, with the following moment equations:

$$d\langle\hat{\mathbf{x}}\rangle/dt = A\langle\hat{\mathbf{x}}\rangle + B\mathbf{u}(t), \quad (12)$$

$$dV/dt = AV + VA^\top + D. \quad (13)$$

Here $A\equiv\Sigma(G+\mathrm{Im}[\tilde{C}^\dagger\tilde{C}])$ and $D\equiv\Sigma\mathrm{Re}[\tilde{C}^\dagger\tilde{C}]\Sigma^\top$ are the drift and diffusion matrices, respectively.

For *conditional* evolution of linear quantum systems it is convenient to recast the complex current $\mathbf{J}$ of Eq. (6) as a real current with *uncorrelated* noises, as opposed to the complex current $\mathbf{J}$ with (in general) correlated noises:

$$\mathbf{y} \equiv (U)^{-1/2}\begin{pmatrix}\mathrm{Re}[\mathbf{J}] \\ \mathrm{Im}[\mathbf{J}]\end{pmatrix} = C\langle\hat{\mathbf{x}}\rangle + \frac{d\mathbf{w}}{dt}. \quad (14)$$

Here

$$C = 2(U)^{1/2}\bar{C}, \quad \bar{C}^\top \equiv (\mathrm{Re}[\tilde{C}^\top],\mathrm{Im}[\tilde{C}^\top]), \quad (15)$$

while $d\mathbf{w}$ is a vector of real Wiener increments satisfying $d\mathbf{w}d\mathbf{w}^\top=Idt$. For linear systems this conditional state ρ_c from Eq. (3) is Gaussian, with the conditional moment equations [18]

$$d\langle\hat{\mathbf{x}}\rangle_c = [A\langle\hat{\mathbf{x}}\rangle_c + B\mathbf{u}(t)]dt + (V_cC^\top + \Gamma^\top)d\mathbf{w}, \quad (16)$$

$$\dot{V}_c = AV_c + V_cA^\top + D - (V_cC^\top + \Gamma^\top)(CV_c + \Gamma). \quad (17)$$

Here

$$\Gamma = -(U)^{1/2} S\bar{C}\Sigma^{\top}, \quad S = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{18}$$

Note that the equation for V_c is deterministic. In many situations (including those considered later in this paper), has a so-called *stabilizing solution* [25]. This means that the long-time solution is independent of the initial state and also makes the long-time solution of Eq. (16) independent of state [18]. We will notate a stabilizing solution of Eq. (17) as W_U to emphasize that it depends upon the unraveling U.

Practically, the set of W_U's, for all possible unravelings U, is the set of real symmetric matrices satisfying the two linear matrix inequalities

$$W_U + i\Sigma/2 \geq 0, \tag{19}$$

$$D + AW_U + W_U A^{\top} \geq 0. \tag{20}$$

Moreover, given a W_U that satisfies these inequalities, an unraveling U (not necessarily unique) that will generate it can be found through the relation [18]

$$R^{\top} U R = D + AW_U + W_U A^{\top}, \tag{21}$$

with $R \equiv 2\bar{C}W_U + S\bar{C}\Sigma$.

C. Optimal quantum control

In feedback control, $\mathbf{u}(t)$ depends on the history of the measurement record $\mathbf{y}(s)$ for $s<t$. The typical aim of control over some interval $[t_0, t_1]$ is to minimize the expected value of a *cost function* [17]. This represents a penalty for the deviation of the state of the system from the desired state, plus the cost of the controls $\mathbf{u}$. It can be expressed as the integral of the sum of positive functions of $\hat{\mathbf{x}}(t)$ and $\mathbf{u}(t)$ for $t_0<t<t_1$. We are interested in the special case of LQG control [17]: a *linear* system with a *quadratic* cost function and having *Gaussian* noise. For an LQG control problem it can be shown that the optimal $\mathbf{u}$ is linear in the phase-space mean:

$$\mathbf{u}(t) = -\Xi(t)\langle\hat{\mathbf{x}}\rangle_c(t), \tag{22}$$

where the matrix $\Xi(t)$ can be determined from A, B, and the cost functions. It is independent of D, C, and Γ.

In this paper we are concerned only with the properties of the system at steady state, so our aim is to minimize $m = \mathrm{E}[h]$ in the limit $t_1 \to \infty$, where

$$h = \langle\hat{\mathbf{x}}^{\top} P \hat{\mathbf{x}}\rangle_c, \tag{23}$$

with $P \geq 0$. Note that in steady state

$$\mathrm{E}_{ss}[\langle\hat{\mathbf{x}}^{\top} P\hat{\mathbf{x}}\rangle_c] = \mathrm{tr}[W_U P] + \mathrm{E}_{ss}[\langle\hat{\mathbf{x}}\rangle_c^{\top} P \langle\hat{\mathbf{x}}\rangle_c]. \tag{24}$$

Assuming (as is the case in our system) a stabilizing solution W_U plus control over all relevant degrees of freedom of the system (as will be the case if B is invertible), the control can always be chosen to set $\langle\hat{\mathbf{x}}\rangle_c \to 0$, so that

$$m_{\mathrm{opt}} = \mathrm{tr}[PW_U]. \tag{25}$$

For such systems it turns out [18] that the same result (that is, $\langle\hat{\mathbf{x}}\rangle_c \to 0$) can always be achieved with Markovian feedback as introduced by Wiseman and Milburn [26]. This is a much simpler form of feedback; for a general linear system, it means

$$\mathbf{u}(t) = F(t)\mathbf{y}(t). \tag{26}$$

If F is time independent, the average evolution of the system is described simply by modifying the drift and diffusion matrices to

$$A' = A + BFC, \tag{27}$$

$$D' = D + BFF^{\top}B^{\top} + BF\Gamma + \Gamma^{\top}F^{\top}B^{\top}. \tag{28}$$

With B invertible it can be shown from Eq. (16) that the optimal choice (which makes $\langle\hat{\mathbf{x}}\rangle_c \to 0$) is $BF = -W_U C^{\top} - \Gamma^{\top}$.

III. CONTROLLING ENTANGLEMENT

We now specialize to the system examined in Ref. [16]: a nondegenerate parametric oscillator [27] where two damped bosonic modes c_1 and c_2 interact through a $\chi^{(2)}$ optical nonlinearity. Treating the pump mode classically, this results in a quadratic Hamiltonian for the two modes

$$\hat{H} = i\chi(\hat{c}_1^{\dagger}\hat{c}_2^{\dagger} - \hat{c}_1\hat{c}_2) = \chi(\hat{q}_1\hat{p}_2 + \hat{q}_2\hat{p}_1). \tag{29}$$

Here χ is the coupling constant, proportional to the $\chi^{(2)}$ coefficient and the amplitude of the pump. We have also defined quadratures for the two modes via $\hat{c}_j = (\hat{q}_j + i\hat{p}_j)/\sqrt{2}$.

In this system, each mode (subsystem) interacts with its own environment through a channel so that the model fits within the general one described above, with $N=K=2$. The master equation is

$$\dot{\rho} = -i[\hat{H}, \rho] + \mathcal{D}[\hat{c}_1]\rho + \mathcal{D}[\hat{c}_2]\rho. \tag{30}$$

Defining

$$\hat{\mathbf{x}}^T \equiv (\hat{q}_1, \hat{p}_1, \hat{q}_2, \hat{p}_2)^T, \tag{31}$$

we have

$$G = \begin{pmatrix} 0 & 0 & 0 & \chi \\ 0 & 0 & \chi & 0 \\ 0 & \chi & 0 & 0 \\ \chi & 0 & 0 & 0 \end{pmatrix} \tag{32}$$

and

$$\tilde{C} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i & 0 & 0 \\ 0 & 0 & 1 & i \end{pmatrix}. \tag{33}$$

From the theory of the previous section we obtain

$$\bar{C} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{34}$$

$$A=\begin{pmatrix} -\frac{1}{2} & 0 & \chi & 0 \\ 0 & -\frac{1}{2} & 0 & -\chi \\ \chi & 0 & -\frac{1}{2} & 0 \\ 0 & -\chi & 0 & -\frac{1}{2} \end{pmatrix}, \tag{35}$$

$$D=\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{36}$$

For $\chi<1/2$, the system has a stationary state with mean zero and covariance matrix V which can be found by setting dV/dt to zero in Eq. (12). The result can be written in terms of 2×2 submatrices as

$$V=\begin{pmatrix} \gamma & \sigma \\ \sigma^T & \gamma \end{pmatrix}, \tag{37}$$

where the matrix elements of γ and σ are

$$\gamma_{qq}=\gamma_{pp}=\frac{1}{2}\left(\frac{1}{1-4\chi^2}\right), \tag{38}$$

$$\gamma_{qp}=\gamma_{pq}=0, \tag{39}$$

$$\sigma_{qq}=-\sigma_{pp}=\frac{\chi}{1-4\chi^2}, \tag{40}$$

$$\sigma_{qp}=\sigma_{pq}=0. \tag{41}$$

Since the steady state is Gaussian, it is completely characterized by the correlation matrix (37). Then, its degree of entanglement can be quantified by means of the logarithmic negativity [28]

$$L\equiv\begin{cases} -\log_2(2\tilde{\zeta}_-) & \text{if } \tilde{\zeta}<1, \\ 0 & \text{otherwise,} \end{cases} \tag{42}$$

where

$$\tilde{\zeta}_-\equiv\sqrt{(\det\gamma-\det\sigma)-\sqrt{(\det\gamma-\det\sigma)^2-\det V}} \tag{43}$$

is the lowest symplectic eigenvalue of the partial transposed Gaussian state characterized by V. As a first result we have the following.

(i) The quantity L is represented by curve (i) in Fig. 1. It is nonzero for all $\chi>0$ and is finite even as $\chi\to 1/2$. That is, the damping channels degrade the system state, preventing it from becoming maximally entangled like that of Ref. [31].

A Gaussian with a covariance matrix of the form of Eq. (37) is entangled if and only if the variance of the mixed quadratures

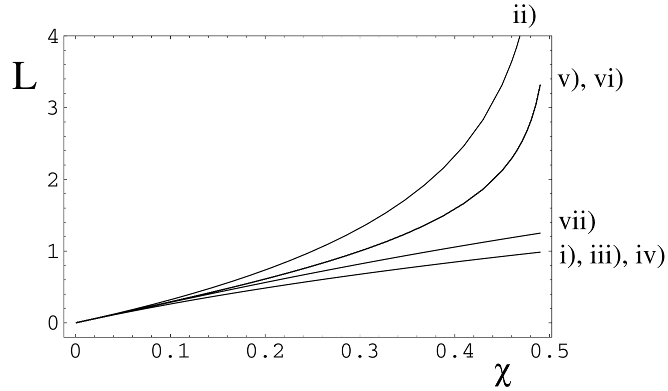

FIG. 1. The logarithmic negativity L of the steady quantum state of the nondegenerate OPO vs the optical nonlinearity strength χ. Different curves correspond to cases discussed in the text: (i) no feedback, (ii) the optimal feedback using nonlocal measurements (Fig. 2), (iii) the optimal feedback using local measurements without classical communication (Fig. 3); (iv)–(vi) various cases (including the optimal) of feedback using single-quadrature local measurements with classical communication (Fig. 4); (vii) the optimal feedback using local measurements of both quadratures (Fig. 4).

$$\hat{x}_j(\theta)=\cos\theta\hat{q}_j+\sin\theta\hat{p}_j \tag{44}$$

is less than the vacuum fluctuation level of unity [29,30]. It is easy to calculate that

$$\langle[\hat{x}_1(\theta)+\hat{x}_2(\pi-\theta)]^2\rangle=\frac{1}{1+2\chi}. \tag{45}$$

We see that the variances (45) go below 1 as soon as $\chi>0$, from which we infer entanglement. Since we must have $\chi<1/2$, the variances (45) are limited from below by 1/2. This is even though the variance in $\hat{x}_j(\theta)$, for either j and for all θ, is unbounded as $\chi\to 1/2$. This again shows that the stationary state has only a finite amount of entanglement and is not pure. For states of this form, the logarithmic negativity is in fact a simple function of the above variance:

$$L=-\log_2\left[\frac{1}{(1+2\chi)}\right]. \tag{46}$$

A. Optimal measurement and control

As we have seen, entanglement is manifest in the squeezing of the quadratures in Eq. (45) for all θ. Thus, as an aim for the feedback, we can choose the minimization of

$$\int\frac{d\theta}{2\pi}\langle[\hat{x}_1(\theta)+\hat{x}_2(\pi-\theta)]^2\rangle \tag{47}$$

in steady state. This evaluates to

$$\langle(\hat{q}_1-\hat{q}_2)^2\rangle/2+\langle(\hat{p}_1+\hat{p}_2)^2\rangle/2. \tag{48}$$

This is exactly of the form of Eq. (23), with

$$P = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}. \tag{49}$$

From the symmetry of the problem, we can assume that the optimal conditional covariance matrix shares the same structure as the unconditional matrix: namely,

$$W_U = \begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & \alpha & 0 & -\beta \\ \beta & 0 & \alpha & 0 \\ 0 & -\beta & 0 & \alpha \end{pmatrix}. \tag{50}$$

Thus the quantity to be minimized is

$$m = \mathrm{tr}[PW_U] = 2(\alpha - \beta). \tag{51}$$

We have to find the minimum of m constrained by $W_U + (i/2)\Sigma \geq 0$ and $D + AW_U + W_U A^T \geq 0$. In terms of α and β this becomes

$$\min(\alpha - \beta), \tag{52}$$

$$\alpha - \frac{1}{2}\sqrt{1 + 4\beta^2} \geq 0, \tag{53}$$

$$\frac{1}{2} - (\alpha \pm \beta)(1 \mp 2\chi) \geq 0. \tag{54}$$

Taking $\alpha = \frac{1}{2}\sqrt{1+4\beta^2}$ we get $m = 2(\sqrt{1+4\beta^2} - 2\beta)$, which decreases monotonically with β. But from condition (54) we obtain

$$\frac{1}{2} - \frac{1}{2}(\sqrt{1 + 4\beta^4} \pm 2\beta)(1 \mp 2\chi) \geq 0. \tag{55}$$

That is,

$$\beta \leq \chi \frac{1-\chi}{1-2\chi} \quad \left(0 \leq \chi < \frac{1}{2}\right). \tag{56}$$

Thus, choosing $\beta = \chi\frac{1-\chi}{1-2\chi}$ and $\alpha = \frac{1}{2}\sqrt{1+4\beta^2}$, we obtain the minimum m_{opt}. In this case we have the following result.

(ii) The logarithmic negativity takes a simple analytical form

$$L_{\mathrm{opt}} = -\log_2[m_{\mathrm{opt}}] = -\log_2[1 - 2\chi]. \tag{57}$$

If now we wish to know how to achieve this optimal result, we can use Eq. (21) to get

$$U = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \tag{58}$$

Since $U = U^{1/2}$, we can easily derive the matrix C of Eq. (14):

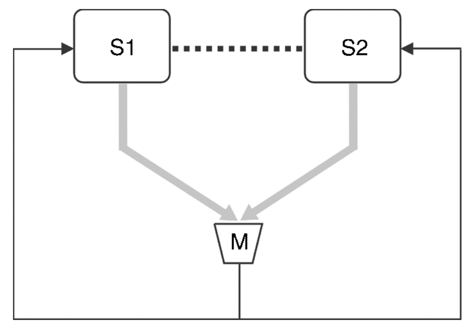

FIG. 2. Schematic representation of feedback action based on nonlocal measurements. $S1$ and $S2$ are the two interacting subsystems, and M is a common measurement box.

$$C = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}. \tag{59}$$

This tells us that the optimal unraveling is the measure of $\hat{q}_1 - \hat{q}_2$ (first and second rows) and $\hat{p}_1 + \hat{p}_2$ (third and fourth rows). Intuitively this makes sense, as it is the variances of these quantities that we wish to minimize, from Eq. (48). Note, however, that these are *nonlocal* measurements in the sense that they involve combinations of observables belonging to the different subsystems. That is, the output beam from mode 1 must be mixed at a beam splitter with the output beam from mode 2 and then the two beam-splitter outputs subject to homodyne detection. One of these detections can measure the output quadrature corresponding to $\hat{q}_1 - \hat{q}_2$, and the other can measure that corresponding to $\hat{p}_1 + \hat{p}_2$. This situation of nonlocal measurements is schematically represented in Fig. 2.

IV. CONTROL USING LOCAL MEASUREMENTS

Having shown that the optimal control protocol involves a nonlocal measurement, it is natural to ask how much improvement this offers over protocols involving only local measurements on the two subsystems [16]. Although the measurements are local, there is, of course, a Hamiltonian interaction (29) between the two subsystems, which produces the entanglement. The feedback Hamiltonians we consider are local in the sense that they do not contain a product of subsystems' operators. However, we use the term purely local feedback only for the cases where the current from the first (second) subsystem is used to control the first (second) subsystem (see Fig. 3). This contrasts with feedback action requiring classical communication so that the first current can control the second subsystem and vice versa (see Fig. 4).

A. Single-quadrature measurements

We begin by considering a homodyne measurement of the output beam of each mode. From Eq. (45) we see that there are no preferred quadratures to be measured provided that their angles sum up to π. Without loss of generality we can

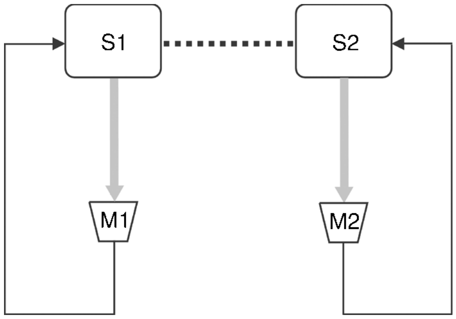

FIG. 3. Schematic representation of purely local feedback action, based on local measurements and requiring no classical communication. $S1$ and $S2$ are the two interacting subsystems, and $M1$ and $M2$ are local measurement boxes.

assume to measure $\hat{q}_1$ and $\hat{q}_2$. In terms of the parameters of Sec. II A we require

$$\Upsilon = \mathrm{diag}(1,1), \tag{60}$$

so that $J_1 \propto q_1$ and $J_2 \propto q_2$.

The Hamiltonian term $-\hat{\mathbf{x}}^\top \Sigma BF\mathbf{y}$ would represent the feedback Hamiltonian. Since we measure q_1 and q_2, it is natural to act on the quadrature to $\hat{q}_1 - \hat{q}_2$ in order to minimize its variance. That is, we choose the feedback to be proportional to the conjugate quadrature:

$$\hat{H}_{\mathrm{fb}} = \lambda_-[J_1(t) - J_2(t)] \times [\hat{p}_1 - \hat{p}_2] + \lambda_+[J_1(t) + J_2(t)] \times [\hat{p}_1 + \hat{p}_2]. \tag{61}$$

Here $\lambda_\pm$ represents possible feedback strengths. Equation (61) represents the most general feedback action that accounts for the symmetry between the two subsystems. Note that this feedback can be performed locally because the Hamiltonian (61) contains no products of operators for both subsystems. However, in general it requires classical communication, so that the controller for mode 1 can apply a Hamiltonian proportional to J_2 and vice versa. Equation (61)

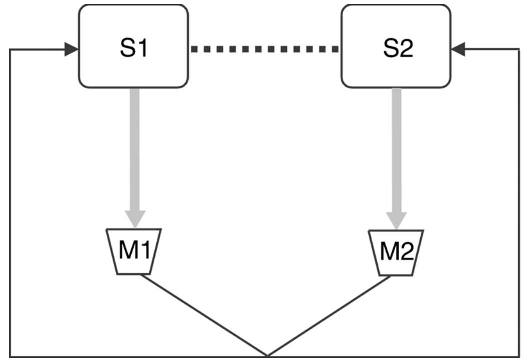

FIG. 4. Schematic representation of feedback action based on local measurements but requiring classical communication. $S1$ and $S2$ are the two interacting subsystems, and $M1$ and $M2$ are local measurement boxes.

is obtained by choosing the feedback driving like

$$BF = \frac{1}{\sqrt{2}}\begin{pmatrix} \lambda_+ + \lambda_- & \lambda_+ - \lambda_- & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \lambda_+ - \lambda_- & \lambda_+ + \lambda_- & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{62}$$

As a consequence of the feedback action the matrices A and D are modified according to Eqs. (27) and (28) to

$$A' = \begin{pmatrix} -\frac{1}{2} + \lambda_- + \lambda_+ & 0 & \chi - \lambda_- + \lambda_+ & 0 \\ 0 & -\frac{1}{2} & 0 & -\chi \\ \chi - \lambda_- + \lambda_+ & 0 & -\frac{1}{2} + \lambda_- + \lambda_+ & 0 \\ 0 & -\chi & 0 & -\frac{1}{2} \end{pmatrix}, \tag{63}$$

$$D' = \frac{1}{2}\begin{pmatrix} (1-\lambda_- - \lambda_+)^2 + (\lambda_- - \lambda_+)^2 & 0 & 2(1-\lambda_- - \lambda_+)(\lambda_- - \lambda_+) & 0 \\ 0 & 1 & 0 & 0 \\ 2(1-\lambda_- - \lambda_+)(\lambda_- - \lambda_+) & 0 & (1-\lambda_- - \lambda_+)^2 + (\lambda_- - \lambda_+)^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{64}$$

The stationary covariance matrix that results from these is of the form of Eq. (37) with

$$\gamma_{qq} = \frac{-1 + 4(1+\chi)\lambda_+ - 2(1+2\chi)\lambda_+^2 + \lambda_-^2(-2 + 4\chi + 8\lambda_+) - 4\lambda_-(-1 + \chi + 4\lambda_+ - 2\lambda_+^2)}{2(1 + 2\chi - 4\lambda_-)(-1 + 2\chi + 4\lambda_+)}, \tag{65}$$

$$\gamma_{qp} = \gamma_{pq} = 0, \tag{66}$$

$$\gamma_{pp} = \frac{1}{2}\left(\frac{1}{1-4\chi^2}\right), \tag{67}$$

$$\sigma_{qq} = \frac{\lambda_-^2(1-4\lambda_+) - \lambda_+^2 + 4\lambda_-\lambda_+^2 + \chi(-1+2\lambda_- - 2\lambda_-^2 + 2\lambda_+ - 2\lambda_+^2)}{(1+2\chi-4\lambda_-)(-1+2\chi+4\lambda_+)}, \quad (68)$$

$$\sigma_{qp} = \sigma_{pq} = 0, \quad (69)$$

$$\sigma_{pp} = -\frac{\chi}{1-4\chi^2}. \quad (70)$$

We have maximized the logarithmic negativity (42) over λ_+ and λ_- with the constraint that V be a stable solution to (13). In the range $0<\chi<1/2$, this constraint is

$$\lambda_\pm < \frac{1}{4} \mp \frac{\chi}{2}. \quad (71)$$

We summarize the results by distinguishing four limit cases for which L becomes dependent on a single parameter.

(iii) If we set $\lambda_\pm=\lambda$, we have a purely local feedback, without classical communication. This case does not show any improvement with respect to the no-feedback case—that is, the optimal value of parameter is $\lambda=0$ [it corresponds to curve (iii) of Fig. 1]. This is because, with local measurements and no communication, the correlations between the two subsystems cannot be increased.

(iv) If we set $\lambda_-=0$ and $\lambda_+=\lambda$, we do require classical communication (see Fig. 4). However, also this case does not show improvement with respect to the no-feedback case; the optimal value of parameter is $\lambda=0$ [it corresponds to curve (iv) of Fig. 1]. This is because the corresponding feedback Hamiltonian is not effective acting on the antisqueezed quadrature q_1+q_2.

(v) If we set $\lambda_-=\lambda$ and $\lambda_+=0$, we again require classical communication (see Fig. 4). In this case the feedback Hamiltonian coincides with that used in Ref. [16]. The optimal value of the feedback parameter is $\lambda=\chi$ and gives rise to a great improvement in the logarithmic negativity with respect to the no-feedback case [it corresponds to curve (v) of Fig. 1]. By approaching the instability point $\chi\to 1/2$ the logarithmic negativity increases indefinitely.

(vi) If we set $\lambda_\pm=\pm\lambda$, we once again require classical communication (see Fig. 4). The optimal value of the parameter is $\lambda=-\chi$ and gives rise exactly to the same values of the logarithmic negativity as for case (v) [thus corresponding to curve (vi) of Fig. 1]. However, in Sec. V it will become clear that case (vi) is superior to case (v) in other ways.

That case (vi) gives the best result is not surprising, since it gives rise to a feedback Hamiltonian that resembles that in Eq. (29), once it is remembered that $J_1 \propto q_1$ and $J_2 \propto q_2$. Note that although in cases (v) and (vi) the entanglement increases without bound as $\chi\to 1/2$, the logarithmic negativity is still less than that of the optimal control using nonlocal measurements for all values of χ as shown by Fig. 1.

B. Joint quadratures measurements

Since Eq. (48) contains both q and p, one might think that performing joint quadratures measurements in both subsystems would be an effective route to controlling entanglement. Of course it is not possible to measure both quadratures with perfect efficiency, but it is possible to measure each quadrature with an efficiency of 1/2. This can be achieved by heterodyne measurement—for example, [33]. In terms of the parameters of Sec. II A we require $\Upsilon=0$ so that $J_1 \propto c_1$ and $J_2 \propto c_2$.

Bearing in mind the results of the preceding subsection [that is, that scheme (vi) performed best] we restrict our consideration to feedback that gives rise to a Hamiltonian resembling the one in Eq. (29). Hence we choose the feedback driving as

$$BF = \begin{pmatrix} 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & -\mu \\ \mu & 0 & 0 & 0 \\ 0 & 0 & -\mu & 0 \end{pmatrix}, \quad (72)$$

corresponding to the feedback Hamiltonian

$$\hat{H}_{\text{fb}} = -i\frac{\mu}{\eta}\{[J_1(t)\hat{c}_2 - J_1^*(t)\hat{c}_2^\dagger] + [J_2(t)\hat{c}_1 - J_2^*(t)\hat{c}_1^\dagger]\}. \quad (73)$$

Here μ represents the feedback strength and $\eta\equiv 1/2$ accounts for the half unit efficiency. Also in this case feedback can be performed locally because the Hamiltonian (73) contains no products of operators for both subsystems. However, it requires classical communication, so that the controller for mode 1 can apply a Hamiltonian proportional to J_2 and vice versa.

As a consequence of feedback action, the matrices A and D are modified according to Eqs. (27) and (28) to

$$A' = \begin{pmatrix} -\frac{1}{2} & 0 & \chi+\mu & 0 \\ 0 & -\frac{1}{2} & 0 & -\chi-\mu \\ \chi+\mu & 0 & -\frac{1}{2} & 0 \\ 0 & -\chi-\mu & 0 & -\frac{1}{2} \end{pmatrix}, \quad (74)$$

$$D' = \frac{1}{2}\begin{pmatrix} 1+2\mu^2 & 0 & -2\mu & 0 \\ 0 & 1+2\mu^2 & 0 & 2\mu \\ -2\mu & 0 & 1+2\mu^2 & 0 \\ 0 & 2\mu & 0 & 1+2\mu^2 \end{pmatrix}. \quad (75)$$

Proceeding as above, the stationary covariance matrix elements resulting are given by

$$\gamma_{qq} = \frac{-1+4\chi\mu+2\mu^2}{2[-1+4(\chi+\mu)^2]} = \gamma_{pp}, \quad (76)$$

$$\gamma_{qp} = \gamma_{pq} = 0, \quad (77)$$

$$\sigma_{qq} = -\frac{\chi+2\chi\mu^2+2\mu^3}{-1+4(\chi+\mu)^2} = -\sigma_{pp}, \quad (78)$$

$$\sigma_{qp} = \sigma_{pq} = 0. \quad (79)$$

We have maximized the logarithmic negativity (42) over μ with the constraint that V be a stable solution to Eq. (13). In the range $0<\chi<1/2$ this is

$$-\frac{1}{2}-\chi<\mu<\frac{1}{2}-\chi. \quad (80)$$

We summarize the results hereafter.

(vii) The optimal value of the parameter μ is found to be $\mu=(-1-2\chi+\sqrt{1+4\chi^2})/2$. This gives rise to a small improvement in the logarithmic negativity with respect to the no-feedback case—it corresponds to curve (vii) of Fig. 1.

Although this case does improve entanglement, it is not as good as the best homodyne scheme (vi). This can be understood as follows. In controlling a quantum system, one has always to reach a trade-off between information gain and introduced disturbance. Heterodyne detection allows us to gain information about both system quadratures, in contrast to homodyne detection, at the expense of introducing more noise via the feedback. In our system, it is apparent that a high degree of entanglement can be produced by controlling only one pair of quadratures, so the noise introduced by heterodyne-based feedback produces inferior performance relative to homodyne-based feedback. In other contexts (with other Hamiltonians) heterodyne-based feedback may outperform homodyne-based feedback.

V. PURITY

The fact that for optimal nonlocal measurement (ii) and local measurement of cases (v) and (vi) the controlled entanglement can increase without bound means that feedback is able to recycle the information lost by the system into the environment through the amplitude damping. However the Einstein-Podoslsky-Rosen (EPR) correlations [31] imply not only an arbitrarily entangled state, but also a pure state. We now check what the purity of our stationary state is under the various feedback control schemes.

The measure of purity is provided by the negentropy; that is, the negative of the von Neumann entropy. For a Gaussian it can be written as [32]

$$S_- = -g(\zeta_+) - g(\zeta_-), \quad (81)$$

where

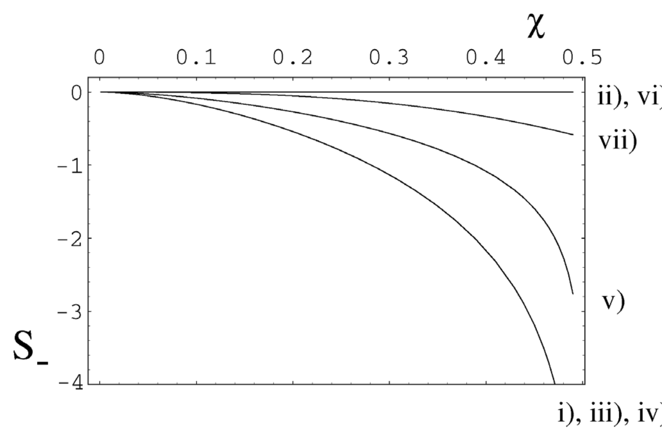

FIG. 5. The negentropy S_- of the steady quantum state of the nondegenerate OPO vs the optical nonlinearity strength χ. Different curves correspond to cases (i)–(vii) discussed in the text.

$$g(x) \equiv \left(x+\frac{1}{2}\right)\log_2\left(x+\frac{1}{2}\right) - \left(x-\frac{1}{2}\right)\log_2\left(x-\frac{1}{2}\right) \quad (82)$$

and

$$\zeta_\pm \equiv \sqrt{(\det\gamma+\det\sigma) \pm \sqrt{(\det\gamma+\det\sigma)^2-\det V}} \quad (83)$$

are the symplectic eigenvalues of the Gaussian state characterized by V.

We have numerically evaluated the quantity (81) for nonlocal and local measurements with feedback, and the results are shown in Fig. 5. The lower curve corresponds to the worst cases (i), (iii), and (iv). Above is the curve corresponding to case (v) and showing that the state does not remain pure. In this case we have the entropy, as well as the amount of entanglement, increasing without limit as χ increases. The fact that they both increase indefinitely may sound strange. However, it must be remembered that the limit $\chi\to 1/2$ allows infinite energy to come into the state. Next, there is the curve corresponding to case (vii), for which the negentropy is bounded as $\chi\to 1/2$. Finally, at the top are the curves corresponding to optimal nonlocal measurement with feedback (ii) and case (vi). In these cases the entropy is always zero and the purity of the state is restored by the feedback action.

By comparing the results of Fig. 5 with those of Fig. 1, we see that the purity is not simply related to the degree of entanglement. For instance, the order of the schemes in the two graphs is not the same (in both cases a higher curve corresponds to better performance). However, the optimal (maximum entanglement and maximum purity) schemes are easy to identify: for the feedback schemes using local measurement, case (vi) is best, while the global optimum is scheme (ii) using nonlocal measurement.

VI. CONCLUSIONS

Summarizing, we have found the optimal nonlocal measurement plus feedback, as well as the optimal local measurement plus feedback, that allows one to control steady state EPR correlations for two bosonic modes interacting via parametric Hamiltonian $\propto\chi$. Both these actions allow one to produce arbitrary amounts of entanglement as $\chi\to 1/2$, although more in the former case. Moreover, they both do this while producing a pure state—that is, they permit us to recover the coherence of our open quantum system. (Incidentally the possibility of coherence recovery by means of feedback was forecast in Ref. [34] for finite-dimensional systems by an information-theoretic approach.)

The schemes presented could be implemented using an experimental setup similar to that of Ref. [35]. The two damped and interacting bosonic modes correspond to optical cavity modes coupled by a $\chi^{(2)}$ nonlinearity. When emerging from the optical cavity they are combined by a beam splitter (or not) and subjected to homodyne detection to realize a nonlocal (or local) measurement. Then, the currents are used to realize the feedback action through amplitude modulation of classical driving fields (lasers). Any delays in classical communication (including the feedback loop) must be much smaller than the typical time scale of the system and much smaller than the inverse of relevant bandwidth, which seems within reach of present day technology. Moreover, it was pointed out in Ref. [16] that quantum feedback control is quite robust against non unit overall efficiency.

Finally, our local measurement plus feedback protocol requires only classical communication and Gaussian operations (linear displacements). This may appear to contradict the impossibility to enhance (distill) entanglement by means of Gaussian local operations and classical communication (LOCC) stated in Ref. [36]. The key point is that, in contrast with Ref. [36], here the LOCC operations continuously happen while the entangling interaction is "on." Thus, the approach presented may shed further light on the subject of entanglement distillation.

While we have used a semidefinite program to find the optimal measurement and feedback action for general LQG systems, to find the optimum local measurement scheme we used simple optimization informed by the symmetries of the system. The question of defining an efficient program to find the optimal *local* measurement for feedback control of general LQG systems remains open.

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