Preparation of pure and mixed polarization qubits and the direct measurement of figures of merit

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Nonclassical joint measurements can hugely improve the efficiency with which certain figures of merit of quantum systems are measured. We use such a measurement to determine a particular figure of merit, the purity, for a polarization qubit. In the process we highlight some of subtleties involved in common methods for generating decoherence in quantum optics.

Quantum information science has the potential to dramatically increase the speed of certain information processing tasks such as factoring large numbers [1]. The superiority of quantum information processors over classical ones arises in part because the number of internal states of the processor increases exponentially with the number of inputs and output bits, rather than polynomially. This very property that makes quantum systems such powerful information processors also makes them notoriously difficult to characterize, since describing the state of a system at a particular stage of a calculation requires exponentially more measurements than there are input and output bits. Even with the small quantum information processors of ten or so qubits that have been demonstrated so far [2], this characterization—known technically as quantum state tomography [3]—would require up to a million independent measurements and rapidly becomes impractical as systems become larger.

Given this problem, it is often desirable to describe the system in terms of a few figures of merit that encapsulate the relevant properties of the state for some particular application. The simplest such figure of merit is the fidelity to the expected state of the system [4]. If the expected state is $|\phi\rangle$, then one will measure the expectation value of the projector $\mathbf{P}_\phi = |\phi\rangle\langle\phi|$, and the fidelity is given by $F = \mathrm{Tr}\{\mathbf{P}_\phi \rho\}$, where ρ is the state of the system. The fidelity measurement will yield one if the system is in the expected state and less than one if it is not. Other figures of merit measure more subtle properties of the system. A partial list includes the purity [4], the Von Neumann entropy [5], the tangle [6], the concurrence [6], and the trace distance from another state [7]. All of these figures of merit share the property that they are nonlinear functions of the density matrix as opposed to fidelity which is linear in the density matrix. Whereas the fidelity can be measured straightforwardly as an expectation value, these nonlinear functions cannot be measured in this way. Instead, the figure of merit is usually computed from the density matrix. This presents an experimental problem because the number of measurements required to determine the density matrix rises exponentially with the size of the quantum system.

A means of circumventing this problem was proposed by Brun [8] who showed how these nonlinear functions can be measured very efficiently with nonclassical joint measurements, at least in the special case that they are polynomial in the density matrix. Brun demonstrated that an mth degree polynomial function of the density matrix can be written as the expectation value of a joint measurement performed on m copies of the system described by the same density matrix ρ. Since the density matrix describing these m copies can be written $\rho_m = \rho^{\otimes m}$, expectations values that are linear in ρ_m are mth order polynomials in ρ. In particular, purity, defined as

$$P = \mathrm{Tr}\{\rho^2\}, \qquad (1)$$

being quadratic in ρ, can be measured directly as a joint expectation value on two copies of a system. Even nonpolynomial functions like the Von Neumann entropy can be measured by approximating them with a truncated Taylor series in the density matrix.

The purity is a useful figure of merit in many situations. It is directly related to the thermodynamic temperature of the ensemble which can be easily calculated from it. It can also be used as a measure of entanglement of a particle with other systems, since for a set of interacting systems each individual system will appear pure when the overall state is separable, completely impure when the state is maximally entangled, and partially pure when the state is partially entangled. In this paper we will discuss applying Brun's technique to the measurement of purity.

Joint measurements can result in an enormous reduction in the resources required to measure nonlinear figures of merit. While complete characterization scales exponentially with the size of the system, the joint measurement used in the Brun technique is fixed by the degree of the polynomial defining the figure of merit. A 10-qubit system requires over one million measurements to measure the density matrix, but only a single joint measurement on pairs of copies of the system to measure the purity. The joint measurement method for measuring purity was also applied experimentally in a nuclear magnetic resonance system by Du *et al.* [9], and in
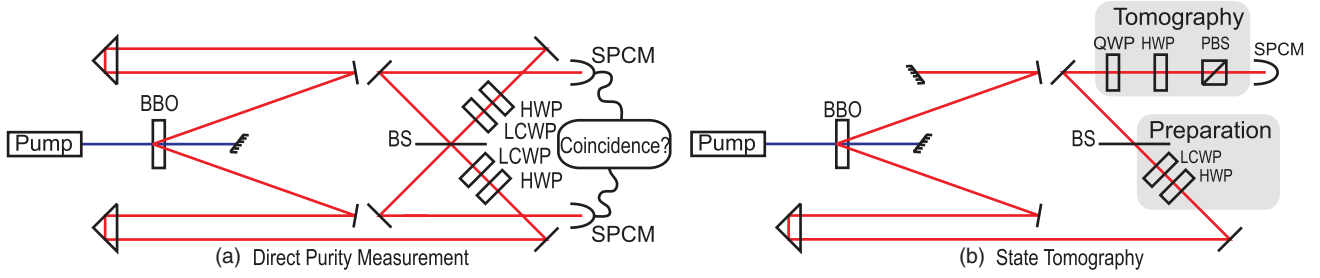
FIG. 1. (Color online) Experimental implementation of (a) the direct purity measurement and (b) quantum state tomography. Labels designate a 50/50 beam splitter (BS), a nonlinear β-barium borate (BBO) crystal, half waveplates (HWP), liquid-crystal variable waveplates (LCWP), single photon counting modules (SPCM) and a polarizing beam splitter (PBS). A type-I spontaneous parametric down conversion (SPDC) crystal produces pairs of H-polarized photons. In (a), the same state ρ is prepared in both arms and the beam splitter acts as a singlet state filter. In (b), a state is prepared in one of the arms and polarimetry is used to measure the density matrix of the state using a quarter waveplate, a half waveplate, and a polarizing beam splitter.

an entangled photon system by Bovino *et al.* [10]. Both these systems were limited in the range of mixed states that they were able to generate. Recently, another figure of merit, the concurrence, has also been directly measured in a photonic system [11]. Here we study the application of this technique to a broad range of pure and mixed states and find that the effectiveness of the approach depends crucially on the details of how the state is prepared.

For an ensemble of single qubits described by a density matrix

$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}, \qquad (2)$$

the formula $P = \mathrm{Tr}[\rho^2]$ can be expanded as a second-order polynomial in the density matrix elements: $P = \rho_{00}^2 + \rho_{01}\rho_{10} + \rho_{10}\rho_{01} + \rho_{11}^2$. According to Brun's result, the purity will be equal to the expectation value of a two-particle operator **A** made by replacing each quadratic term $\rho_{ij}\rho_{kl}$ in the sum with the two particle projector $|i\rangle\langle j| \otimes |k\rangle\langle l|$ such that

$$\mathbf{A} = |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 1| \otimes |1\rangle\langle 0| + |1\rangle\langle 0| \otimes |0\rangle\langle 1|$$
$$+ |1\rangle\langle 1| \otimes |1\rangle\langle 1|.$$

Inserting the two-photon polarization identity operator $\mathbb{I}_4$ and grouping tensor products into two photon states, we obtain

$$\mathbf{A} = \mathbb{I}_4 - |01\rangle\langle 01| - |10\rangle\langle 10| + |01\rangle\langle 10| + |10\rangle\langle 01| \qquad (3)$$

$$= \mathbb{I}_4 - 2|\psi^-\rangle\langle\psi^-|, \qquad (4)$$

where $|\psi^\pm\rangle \equiv (|01\rangle \pm |10\rangle)/\sqrt{2}$. Applying $P = \langle\mathbf{A}\rangle = \mathrm{Tr}\{\mathbf{A}\rho\otimes\rho\}$, we obtain

$$P = \mathrm{Tr}\{(\mathbb{I}_4 - 2|\psi^-\rangle\langle\psi^-|)\rho\otimes\rho\}$$

$$= 1 - 2\langle\psi^-|\rho\otimes\rho|\psi^-\rangle. \qquad (5)$$

Thus the purity can be obtained by a single measurement on two particles, namely a projection onto the singlet state $|\psi^-\rangle$. This fact can be intuitively understood by realizing that a projection onto $|\psi^-\rangle$ implements a measurement of permutation symmetry, since the two qubit space divides into a symmetric subspace spanned by $\{|00\rangle, |11\rangle, |\psi^+\rangle\}$ and the or-

thogonal, antisymmetric state $|\psi^-\rangle$. If a state ρ is pure, then the state $\rho \otimes \rho$ is manifestly permutation invariant, and therefore has no antisymmetric component. In the other limit of a completely mixed state $\rho = (\frac{1}{2})\mathbb{I}_2$, $\rho \otimes \rho = (\frac{1}{4})\mathbb{I}_4$ with a projection onto the singlet state (and any other state) of 0.25 and hence a purity of 0.5. For states of intermediate purity $\rho \otimes \rho$ can be decomposed into antisymmetric and symmetric parts, the relative weights of which determine the purity.

In quantum optics, $|\psi^-\rangle\langle\psi^-|$ can be measured using the Hong-Ou-Mandel (HOM) effect [12]. The effect occurs when two photons are made indistinguishable in all physical properties except polarization and then sent into the two input ports of a beam splitter so as to arrive at the same time. Photons in a permutation-symmetric polarization state such as $|HH\rangle$ or $|\psi^+\rangle$ will always leave the beam splitter in the same port, whereas if the incoming photons are in the permutation-antisymmetric state $|\psi^-\rangle$ then they will always leave the beam splitter in opposite ports. If we measure the rate of coincident firings of detectors at the two output ports we will have filtered for the singlet state $|\psi^-\rangle$, thereby, through Eq. (5), measuring the purity. This technique of using the HOM effect as a singlet state filter has been employed in many important quantum optics experiments including the demonstration of teleportation [13]. Its applicability for this task is discussed by Mitchell *et al.* [14] who fully characterized the filtering process using quantum process tomography and by Kim and Grice [15] who note some of its limitations.

The experimental implementation of the purity measurement was carried out using the apparatus shown in Fig. 1(a). A β-barium borate crystal cut for type-I phase matching was pumped with a 28-mW, 405-nm diode laser, producing pairs of horizontally polarized photons over a wide bandwidth. The photons were sent through 10-nm interference filters to reduce spectral differences between them. A corner prism mounted on a motorized translation stage allowed control of the relative delay in the two arms. The photons passed through a half waveplate and a liquid crystal variable waveplate with its axis aligned with the horizontal. The photons then arrived at the beam splitter which implemented the joint measurement before passing to the detector. Ideally the beam splitter should act as a perfect singlet state filter and never allow the photons to leave in opposite ports when the input

polarization states are the same. In reality, the visibility of the two-photon interference was limited to 0.90 ± 0.03 by the unequal reflection and transmission coefficients of the beam splitter and by small alignment errors. As a result, the measurement actually implemented was $\mathbf{P}_{actual} = 0.10I + 0.90|\psi^-\rangle\langle\psi^-|$ rather than $\mathbf{P}_{ideal} = |\psi^-\rangle\langle\psi^-|$. The tabulated purities were calculated by taking account of this modified measurement: $\langle\mathbf{P}_{ideal}\rangle = (\langle\mathbf{P}_{actual}\rangle - 0.1)/0.9$.

This is not quite equivalent to the measurement that might be made with an ideal singlet state filter. Since in practice $\langle\mathbf{P}_{actual}\rangle$ represents a measured expectation value obtained from a finite number of measurements it will have some shot noise associated with it. Consequently the estimate of $\langle\mathbf{P}_{ideal}\rangle$ will also have some shot noise and thus for a pure state there is a finite probability of obtaining a value for the singlet state projection of less than zero and hence a purity greater than one. Similarly for mixed states there is a finite probability of obtaining a purity less than $1/2$ with a finite number of measurements. The probability of obtaining these unphysical results goes to zero as $1/\sqrt{N}$ in the limit of a large number of measurements N, and it is only in this limit that our expression for the purity may be considered exact. This is in keeping with the well-known fact in statistics that an exact estimate of an expectation value requires an infinite number of measurements.

Impure states were created by applying random polarization phase shifts with the liquid crystal waveplates. Thus polarization was correlated to a pseudorandom number generator rather than to some traced-over degree of freedom. As far as polarization measurements are concerned, there is no observable difference between impure states generated with this technique and those generated by some more complicated method such as loss of a single photon from an entangled pair [16], although, as will be discussed later, there is a difference between this approach and the generation of decoherence by coupling to another degree of freedom of the photon.

For comparison, the purity was also determined by measuring the density matrix using quantum state tomography and applying the formula $P = \mathrm{Tr}\{\rho^2\}$. Quantum state tomography was performed by blocking one of the photons and performing projective measurements on the other photon. Background counts due to detector dark counts and residual light were subtracted from the data before reconstructing the density matrix.

Table I shows the results of state tomography and direct purity measurement for a variety of states of varying purity. A number of nondecohering and decohering preparation processes were performed. If no preparation was done the state was left in $|H\rangle$ as in Fig. 2(a) with essentially unit purity measured with both methods. Figure 2(b) shows the state after a unitary rotation to the state $|+\rangle$ where we have used the notation $|\pm\rangle \equiv |H\rangle \pm |V\rangle/\sqrt{2}$. This rotation has no effect on the purity as measured with either method since purity is an invariant under unitary operations. Figure 2(c) shows the completely mixed state $\rho = (1/2)\mathbb{I}_2$ obtained by randomly applying a phase shift of 0 or π with the liquid crystal waveplates to the state $|+\rangle$. The purity measured with either method is consistent with the theoretical value of 0.5 or com-

TABLE I. The purities measured for five states using the direct joint purity measurement and a full characterization followed by a calculation of $\mathrm{Tr}\{\rho^2\}$. The stated errors arise from counting statistics and the statistics associated with the random state selection.

State	Purities		
	Direct	Tomographic	Theoretical
$|H\rangle$	1.00 ± 0.03	1.00 ± 0.01	1
$|+\rangle$	0.99 ± 0.03	0.98 ± 0.01	1
Equal mixture $|+\rangle, |-\rangle$	0.52 ± 0.01	0.50 ± 0.01	0.5
Equal mixture $|+\rangle, |-\rangle, |R\rangle$	0.568 ± 0.008	0.56 ± 0.01	$5/9 \approx 0.5556$
$|H\rangle + e^{i\phi}|V\rangle$, $\phi \in [0, \pi]$	0.72 ± 0.01	0.70 ± 0.01	$0.5 + \dfrac{2}{\pi^2} \approx 0.7026$

pletely mixed. Figure 2(d) shows a state made by randomly applying either a 0, π, or $\pi/2$ phase shift to the state $|+\rangle$. The purity is consistent with the theoretical value of $5/9$. Finally, Fig. 2(e) shows a state created by selecting randomly from the continuum of phases in the range $[0, \pi]$. Again, the measured purity was consistent with the theoretical value of $0.5 + 2/\pi^2$.

While the joint measurement technique correctly measures the purity in these cases, caution must be exercised in using it since the method depends crucially on the assumption that the couplings to the environment that cause the reductions in purity in the two copies of the state be uncorrelated. In the case where the couplings to the environment are perfectly correlated, any given two copies of the state will be in the same pure state at any given moment, and the direct purity measurement will indicate unit purity. In this experiment, correlated decoherence was achieved by applying phase shifts to the two photons with a liquid crystal waveplate so as to always give the two photons the same birefringent phase, even as the value of this phase varied randomly over time. In this situation the density matrix for either individual photon is mixed since the phase shifts are random, but at any given moment the two photons are in the same state, and hence their singlet state projection is always zero.

The purity was measured directly for the three mixed states [Figs. 2(c)–2(e)] generated with the mixing completely correlated for the two copies of the system. All measured purities were consistent with unit purity. This demonstrates that the joint measurement technique relies on the assumption that the decoherence processes are independent for all the particles being jointly measured and fails to work when this assumption does not hold.

The quantum optics implementation of this technique depends in another way on the details of how mixed photon states are prepared. A common technique for creating impure photon polarization states is to create a correlation between polarization and some other photon degree of freedom such as frequency or spatial mode [16] which is subsequently ignored or traced over in the measurement. However, for the HOM effect to perform as a polarization singlet state filter,
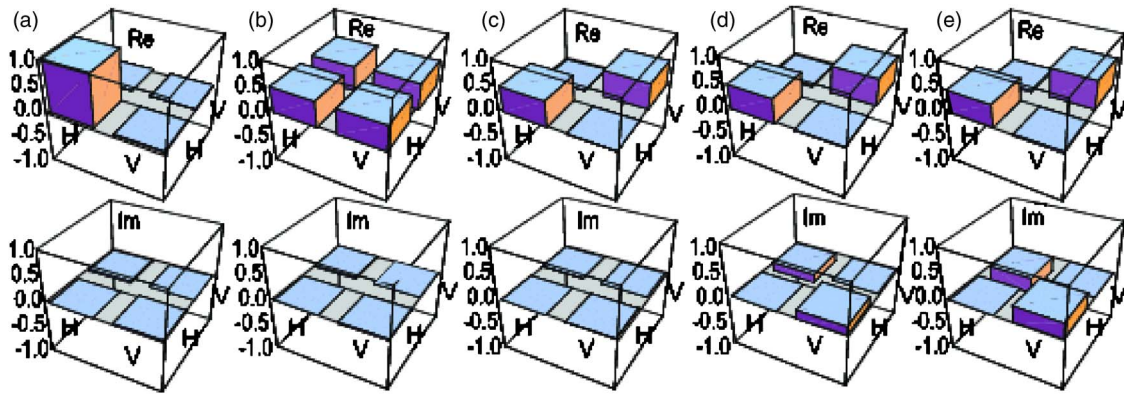
FIG. 2. (Color online) Experimentally obtained density matrices for various single-photon polarization states. (a) Pure horizontal $|H\rangle$, (b) pure diagonal $|+\rangle$, (c) equal mixture of $|+\rangle$ and $|-\rangle$, (d) equal mixture of $|+\rangle$, $|-\rangle$ and $|R\rangle = (|H\rangle - i|V\rangle)/\sqrt{2}$, and (e) mixture of states of the form $(|H\rangle + e^{i\phi}|V\rangle)/\sqrt{2}$ with ϕ distributed equally over $[0, \pi]$.

the photons must be indistinguishable in all degrees of freedom other than polarization, and this will not be the case if extra degrees of freedom become correlated to polarization.

To demonstrate this fact the same experiment was done using the method of [16] to create impurity. In this method, impurity is obtained by creating a frequency-dependent birefringent phase shift (or equivalently, introducing a constant group delay between the horizontal and vertical components). Since the photon measurement is insensitive to frequency, this correlation is traced over, resulting in a loss of coherence between the horizontal and vertical components of the density matrix. In order to create the maximally mixed state $\rho = (1/2)\mathbb{I}_2$ a 20 mm piece of quartz was introduced into each beam with the fast and slow axes of the two crystals relatively rotated by $90°$ so that the vertical polarization component was advanced in one arm relative to the horizontal component and delayed in the other. Diagonally polarized light was then sent into each arm. State tomography on the individual photons resulted in the expected density matrix $\rho = (1/2)\mathbb{I}_2$, but the HOM visibility did not result in the correct value for the purity. As the path length difference in the two arms was scanned, rather than observing a single HOM dip yielding a purity of 0.5 as in Fig. 3(a), two dips

were observed as in Fig. 3(b), either of which would have implied a purity of 0.25. The discrepancy can be accounted for by noting that after the delay is introduced, a complete description of the state of each photon involves not only polarization but also the relative delay between the horizontal and vertical components. The state space becomes four-dimensional, being spanned by $\{|H, \text{early}\rangle, |H, \text{late}\rangle, |V, \text{early}\rangle, |V, \text{late}\rangle\}$. Coincidences are suppressed when either both photons arrive late with the same polarization or both arrive early with the same polarization. When either the polarizations or the arrival times are different the photons cause a coincidence 50% of the time. Since out of 16 possible combinations of arrival time and polarization only four are removed by singlet state filtering the dip visibility is 25% as observed. This result demonstrates the danger of thinking of the HOM effect as acting as a polarization singlet state filter. Rather the HOM effect is a symmetry filter over all the properties of the photon, even those that one would prefer that it ignore.

For completeness we also looked at the case where the crystals were oriented so as to delay the same polarization in the two arms. In that case the interaction with the "environment" is correlated and the photons in the two arms are al-
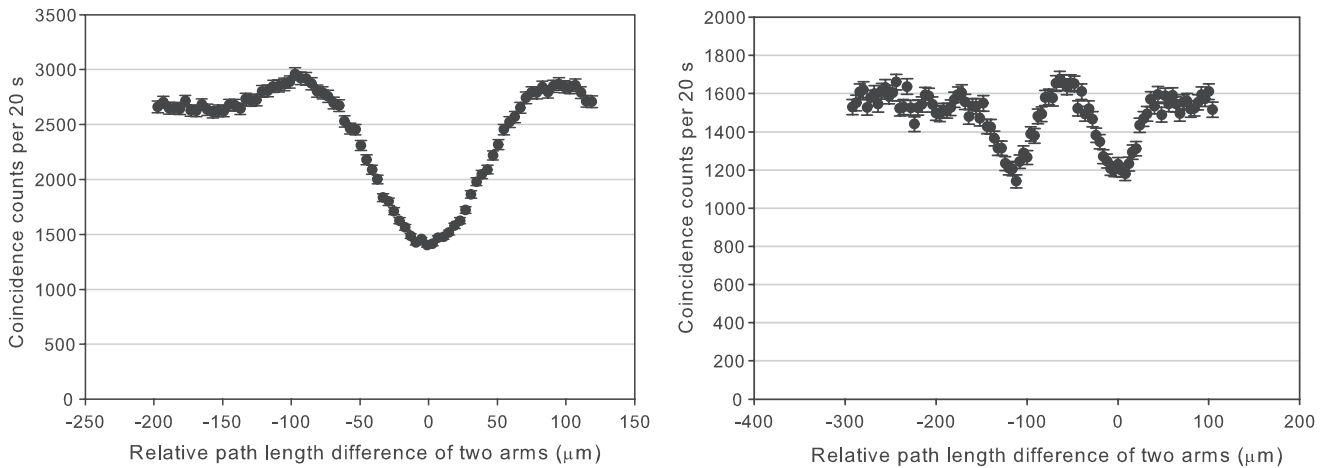

FIG. 3. (a) The HOM dip expected for a maximally mixed qubit state. (b) Observed data when such a state is created by introducing a large birefringent group delay between horizontal and vertical polarizations.

ways in the same state in the four-dimensional state space. Just as when the correlated mixed states were created with liquid crystals waveplates, the joint measurement of the purity for correlated delays gives a purity of one even though the polarization density matrix measured with tomography is mixed.

We have demonstrated the direct measurement of the purity of a single qubit through a joint measurement on two particles drawn from an ensemble. Such measurements are much more efficient at determining nonlinear figures of merit such as purity than the common method of measuring the density matrix and calculating the figure of merit from it, particularly in large Hilbert spaces. Furthermore, we have shown that the method depends critically on the properties of the decoherence process generating a mixture of states. Joint measurements on a mixed state where the decoherence is correlated between the measured photons generate the same experimental signature as no decoherence at all, while a widely used technique for preparing decohered states fails because of the behavior of the HOM effect when used as a singlet state filter. These caveats aside, joint measurements still provide an efficient means of characterizing quantum states that may find application in thermometry, quantum error correction, and the characterization of quantum devices.

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