Comment on "Time-dependent general quantum quadratic Hamiltonian system"

Yujun Zheng*
School of Physics and Microelectronics, Shandong University, Jinan 250100, China
(Received 29 August 2006; published 15 December 2006)

It is shown that the propagator and wave function recently found by Yeon *et al.* [Phys. Rev. A **68**, 052108 (2003)] are not complete and the results are not correct for constant coefficients of a time-dependent general quantum quadratic Hamiltonian for the case of $\gamma=0$. We presented the correct evolution operator, propagator, and wave function of the general time-dependent quantum quadratic Hamiltonian system.

DOI: 10.1103/PhysRevA.74.066101 PACS number(s): 03.65.Ge

Yeon *et al.* [1] studied the time-dependent general quantum quadratic Hamiltonian system by using canonical transformation. In Ref. [1], the original time-dependent general quadratic Hamiltonian is written as

$$\mathcal{H}_q = \frac{1}{2}[a(t)p^2 + b(t)(qp + pq) + c(t)q^2] + d(t)p + e(t)q + g(t), \tag{1}$$

where q is the canonical coordinate and p is its canonical conjugate momentum. $a(t)$, $b(t)$, $c(t)$, $d(t)$, $e(t)$, and $g(t)$ are the functions of time. It is assumed that the coefficients $a(t)$, $b(t)$, $c(t)$, $d(t)$, $e(t)$, and $g(t)$ are real and differentiable with respect to time t, and $a(t)\neq 0$.

To get the "general analytical solution of propagator," Yeon *et al.* took the original Hamiltonian (1) into the new Hamiltonian

$$\mathcal{H}_Q = \frac{1}{2}P^2 + \frac{1}{2}\gamma(t)Q^2 - \zeta(t) \tag{2}$$

by employing the canonical transformation. Yeon *et al.* obtained the general solutions of the propagators and wavefunctions of the original Hamiltonian (1) from the new Hamiltonian (2) in the condition of the "sinking term" $\zeta(t)=0$, and then they showed the results for the constant coefficients in Hamiltonian (1).

It is obviously that the general results of propagators and wave functions depend on this "sinking term" $\zeta(t)$. Or the propagators and wave functions are related to both the coefficient $g(t)$ and the other coefficients. Furthermore, the transformations (50) and (52) in Ref. [1] are clearly wrong for the case of $\gamma=0$, and the results for this case are only right for the special case $bd-ae=0$. The authors of Ref. [1] gave the results of propagators and wave functions in a "mixed representation," i.e., they gave their results in the both the parameters in the original Hamiltonian (1) and the eigenvalues of the new Hamiltonian (2).

In the present work, we obtain the correct solutions by employing the Lie algebraic approach. We introduce the generators of a three-dimensional Lie algebra as

$$J_1 = \frac{i}{2\hbar}p^2, \quad J_2 = \frac{i}{2\hbar}q^2, \quad J_3 = \frac{i}{2\hbar}(pq+qp), \tag{3}$$

and the generators of the Heisenberg-Weyl algebra h_3 [2]

$$T_1 = \frac{i}{\sqrt{\hbar}}p, \quad T_2 = \frac{i}{\sqrt{\hbar}}q, \quad T_3 = i. \tag{4}$$

In terms of generators J_r and T_ν $(r,\nu=1,2,3)$, the Hamiltonian (1) is then rewritten as follows:

$$\mathcal{H} = -\hbar i a(t)J_1 - \hbar i b(t)J_3 - \hbar i q(t)J_2 - \sqrt{\hbar} i d(t)T_1 - \sqrt{\hbar} i e(t)T_2 - i g(t)T_3. \tag{5}$$

It is easily shown that the generators J_r and T_ν form a closed Lie algebra, then the evolution operator can be written as [3–5]

$$\mathcal{U}(t) = e^{-i/\hbar g_1(t)J_1}e^{-i/\hbar g_2(t)J_2}e^{-i/\hbar g_3(t)J_3}e^{-i/\hbar \mu_1(t)T_1} \times e^{-i/\hbar \mu_2(t)T_2}e^{-i/\hbar \mu_3(t)T_3}, \tag{6}$$

where the coefficients $g_r(t)$ and $\mu_\nu(t)$ are known as the Lagrange parameters [5]. By using the Schrödinger equation of evolution operator

$$i\hbar\frac{\partial}{\partial t}\mathcal{U}(t) = \mathcal{H}(t)\mathcal{U}(t), \tag{7}$$

with the initial condition $\mathcal{U}(t=0)=I$. By employing the Campbell-Baker-Hausdorff formula, we have the Lagrange parameters $g_r(t)$ and $\mu_\nu(t)$ defining equations

$$\frac{\partial g_1(t)}{\partial t} = \frac{ic(t)}{\hbar}\left\{\frac{\hbar^2[b^2(t)-a(t)c(t)]}{c^2(t)} + \left[g_1(t) + \frac{\hbar b(t)}{ic(t)}\right]^2\right\},$$

$$\frac{\partial g_2(t)}{\partial t} = -\left[\frac{2ic(t)g_1(t)}{\hbar} + 2b(t)\right]g_2(t) - i\hbar c(t),$$

$$\frac{\partial g_3(t)}{\partial t} = c(t)g_1(t) - i\hbar b(t),$$

$$\frac{\partial \mu_1(t)}{\partial t} = -i\sqrt{\hbar}d(t)e^{-ig_3(t)/\hbar} + \frac{1}{\sqrt{\hbar}}e(t)g_1(t)e^{-ig_3(t)/\hbar},$$

*Electronic address: yzheng@sdu.edu.cn

$$\frac{\partial\mu_2(t)}{\partial t} = -i\sqrt{\hbar}e(t)\left(1+\frac{g_1(t)g_2(t)}{\hbar^2}\right)e^{ig_3(t)/\hbar} - \frac{1}{\sqrt{\hbar}}d(t)g_2(t)e^{ig_3(t)/\hbar},$$

$$\frac{\partial\mu_3(t)}{\partial t} = \frac{1}{\sqrt{\hbar}}e(t)\mu_1(t)\left(1+\frac{g_1(t)g_2(t)}{\hbar^2}\right)e^{ig_3(t)/\hbar} - \frac{i}{\hbar\sqrt{\hbar}}d(t)\mu_1(t)g_2(t)e^{ig_3(t)/\hbar} - ig(t), \tag{8}$$

with the initial conditions $g_r(t=0)=0$ and $\mu_\nu(t=0)=0$ $(r,\nu=1,2,3)$.

Once we know the evolution operator of the system, we could extract all the information of the system. Specially, the propagator and wave function of the system in the coordinate representation are as, respectively,

$$\begin{aligned} K(q,t;q',0) = \langle q|\mathcal{U}(t)|q'\rangle = \frac{1}{\sqrt{-2\pi g_1(t)}}e^{i/2\hbar g_3(t)}\exp\left[\frac{1}{2g_1(t)}q^2 + \left(\frac{1}{2g_1(t)}+\frac{g_2(t)}{2\hbar^2}\right)e^{2i/\hbar g_3(t)}q'^2 - \frac{e^{i/\hbar g_3(t)}}{g_1(t)}qq' - \frac{i\mu_1(t)e^{i/\hbar g_3(t)}}{\sqrt{\hbar}g_1(t)}q\right] \\ \times \exp\left[\left(\frac{\mu_2(t)}{\hbar\sqrt{\hbar}} + \frac{i\mu_1(t)g_2(t)}{\hbar^2\sqrt{\hbar}}e^{2i/\hbar g_3(t)} + \frac{i\mu_1(t)}{\sqrt{\hbar}g_1(t)}e^{2i/\hbar g_3(t)}\right)q'\right] \\ \times \exp\left[\frac{\mu_3(t)}{\hbar} - \left(\frac{\mu_1^2(t)}{2\hbar g_1(t)}e^{2i/\hbar g_3(t)} + \frac{g_3(t)\mu_1^2(t)}{2\hbar^3}e^{2i/\hbar g_3(t)}\right)\right], \end{aligned} \tag{9}$$

$$\begin{aligned} \psi(q,t) = \mathcal{U}(t)\psi(q,0) = \frac{Ne^{-i/2\hbar g_3(t)}}{\sqrt{1+\frac{g_1(t)g_2(t)}{\hbar^2} - \alpha^2 g_1(t)e^{-2i/\hbar g_3(t)}}}\exp\left[\frac{-\hbar^2\alpha^2 e^{-2/i\hbar g_3(t)} + g_2(t)}{2\hbar^2 + 2g_1(t)g_2(t) - 2\hbar^2\alpha^2 g_1(t)e^{-2i/\hbar g_3(t)}}q^2\right] \\ \times \exp\left[\frac{i\hbar\sqrt{\hbar}\alpha^2\mu_1(t)e^{-i/\hbar g_3(t)} + \sqrt{\hbar}\mu_2(t)e^{-i/\hbar g_3(t)} + i\hbar^2 k e^{-i/\hbar g_3(t)}}{\hbar^2 + g_1(t)g_2(t) - \hbar^2\alpha^2 g_1(t)e^{-2i/\hbar g_3(t)}}q\right] \\ \times \exp\left[\frac{2\hbar^2\sqrt{\hbar}kg_1(t)e^{-2i/\hbar g_2(t)} - 2i\hbar\mu_2(t)g_1(t)e^{-2i/\hbar g_3(t)} + 2\mu_1(t)g_1(t)g_2(t) + 2\hbar^2\mu_1(t)}{2\hbar^2\sqrt{\hbar} + 2\sqrt{\hbar}g_1(t)g_2(t) - 2\hbar^2\sqrt{\hbar}\alpha^2 g_1(t)e^{-2i/\hbar g_3(t)}}k\right] \\ \times \exp\left[\frac{-i\mu_1(t)\mu_2(t) - \hbar g_1(t)\mu_2^2(t) - 2i\mu_1(t)\mu_2(t)g_1(t)g_2(t) + \hbar^3\alpha^2\mu_1^2(t) + \hbar\alpha^2 g_1(t)g_2(t)\mu_1^2(t)}{2\hbar^4 + 2\hbar^2 g_1(t)g_2(t) - 2\hbar^4\alpha^2 g_1(t)e^{-2i/\hbar g_3(t)}}\right]\exp\left[\frac{\mu_3(t)}{\hbar}\right]. \end{aligned} \tag{10}$$

The initial wave function is chosen as a wave packet, namely,

$$\psi(q,t=0) = Ne^{ikq-1/2\alpha^2 q^2}, \tag{11}$$

where N is normalization constant, and k and α are the parameters of the wave packet.

We here show the explicit results of the constant coefficients of the quadratic system in Eq. (1). It is obvious that the solution of the Lagrange parameters in Eq. (8) depends on the sign of the expression $\gamma=ac-b^2$. It can partition three cases according to the relations between the coefficients a, b, and c in Eq. (8): $\gamma>0$, $\gamma<0$, and $\gamma=0$. The Lagrange parameters $g_r(t)$ and $\mu_\nu(t)$, from Eq. (8), can be obtained. The propagators $K(q,t;q',0)$, from Eq. (9), are as follows.

The case $\gamma>0$ (here we let $\omega=\sqrt{ac-b^2}$),

$$\begin{aligned} K(q,t;q',0) = \sqrt{\frac{\omega}{2\pi i\hbar a\sin\omega t}}\exp\left[-\frac{ib}{2\hbar a}(q^2-q'^2) - \frac{id}{\hbar a}(q-q')\right] \\ \times\exp\left\{\frac{i\omega}{2\hbar a}\cot\omega t\left[(q^2+q'^2) + \frac{2(ae-bd)}{\omega^2}(q+q') + \frac{2(ae-bd)^2}{\omega^4}\right]\right\} \\ \times\exp\left\{-\frac{i\omega}{\hbar a\sin\omega t}\left[qq' + \frac{ae-bd}{\omega^2}(q+q') + \frac{(ae-bd)^2}{\omega^4}\right]\right\}\exp\left\{\frac{i}{\hbar}t\left[\frac{d(cd-eb)+e(ae-bd)}{2\omega^2} - g\right]\right\}. \end{aligned} \tag{12}$$

The case $\gamma<0$ (for the case, we let $\beta=\sqrt{b^2-ac}$),

$$K(q,t;q',0)=\sqrt{\frac{\beta}{2\pi i\hbar a\sinh\beta t}}\exp\left[-\frac{ib}{2\hbar a}(q^2-q'^2)-\frac{id}{\hbar a}(q-q')\right]$$
$$\times\exp\left\{\frac{i\beta}{2\hbar a}\coth\beta t\left[(q^2+q'^2)-\frac{2(ae-bd)}{\beta^2}(q+q')+\frac{2(ae-bd)^2}{\beta^4}\right]\right\}$$
$$\times\exp\left\{-\frac{i\beta}{\hbar a\sinh\beta t}\left[qq'-\frac{ae-bd}{\beta^2}(q+q')+\frac{(ae-bd)^2}{\beta^4}\right]\right\}\exp\left\{\frac{i}{\hbar}t\left[-\frac{d(cd-be)+e(ae-bd)}{2\beta^2}-g\right]\right\}. \quad (13)$$

The case $\gamma=0$,

$$K(q,t;q',0)=\frac{1}{\sqrt{2\pi i\hbar at}}\exp\left[-\frac{ib}{2\hbar a}(q^2-q'^2)+\frac{i}{2\hbar at}(q-q')^2-\frac{id}{\hbar a}(q-q')\right]$$
$$\times\exp\left[-\frac{i(ae-bd)t}{2\hbar a}(q+q')+\frac{id^2}{2\hbar a}t-\frac{i(ae-bd)^2}{24\hbar a}t^3-\frac{ig}{\hbar}t\right]. \quad (14)$$

The wave function for the case of $\gamma=0$, from Eq. (10), is as follows:

$$\psi(q,t)=\frac{N}{\sqrt{1+bt+i\hbar at\alpha^2}}\exp\left[-\frac{\frac{1}{2}\alpha^2(1-bt)+\frac{ic}{2\hbar}t}{1+bt+i\hbar at\alpha^2}q^2\right]\exp\left[\frac{-\frac{i}{\hbar}[el+(be-cd)t^2/2]+[dt+(ae-bd)t^2/2]\alpha^2+ik}{1+bt+i\hbar at\alpha^2}q\right]$$
$$\times\exp\left[\frac{ik[-dt+(ae-bd)t^2/2]-k^2i\hbar at/2}{1+bt+i\hbar at\alpha^2}\right]\exp\left[\frac{\left(ag-\frac{d^2}{2}\right)t^2+\frac{1}{24}(ae-bd)^2t^4}{1+bt+i\hbar at\alpha^2}\alpha^2\right]$$
$$\times\exp\left[-\frac{i}{a\hbar}\frac{agt+\left(abg-\frac{ade}{2}\right)t^2+\frac{1}{24}(ae-bd)^2(2t^3+qt^4)}{1+bt+i\hbar at\alpha^2}\right]. \quad (15)$$

It is obvious that the case $\gamma=0$ is the limit of the cases $\gamma>0$ and $\gamma<0$. If we take the limits of $\omega\to 0$ or $\beta\to 0$ in the first two cases, we would get the results of $\gamma=0$. Furthermore, the case of $\gamma<0$ will be the case of $\gamma>0$, if we make the replacement of $i\beta\to\omega$. Here we can also consider a special case: *the linear potential*, namely, $b=c=d=g=0$. It belongs to the case of $\gamma=0$, the above propagator (14) and wave function (15) can go back to the previous results [6–8].

In the work we obtained exact analytical solutions of evolution operators for the general time-dependent quadratic Hamiltonian system, and then we calculated the propagators and wave functions by the Lie algebraic approach. The solutions of the time-dependent general quadratic system obtained in the work are of generality. In general, for the mixed case of the quadratic system, namely, the sign of $\gamma=a(t)c(t)-b^2(t)$ could not be determined, it is not easy to get the analytical solutions of the Lagrange parameters. However, we can solve Eq. (8) numerically.

This work was supported by the National Science Foundation of China (Grant No. 10674083).

[1] K. H. Yeon, C. I. Um, and T. F. George, Phys. Rev. A **68**, 052108 (2003).
[2] B. G. Wybourne, *Classical Groups for Physicists* (Wiley, New York, 1974).
[3] A. R. P. Rau and R. A. Wendell, Phys. Rev. Lett. **89**, 220405 (2002).
[4] J. Wei and E. Norman, Proc. Am. Math. Soc. **15**, 327 (1964).
[5] Y. Alhassid and R. D. Levine, Phys. Rev. A **18**, 89 (1978).
[6] A. R. P. Rau and K. Unnikrishnan, Phys. Lett. A **222**, 304 (1996).
[7] J. Bauer, Phys. Rev. A **65**, 036101 (2002).
[8] S. Balasubramanian, Am. J. Phys. **69**, 508 (2001).