Temporal distinguishability of an *N*-photon state and its characterization by quantum interference

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We present a multimode model to describe an arbitrary *N*-photon state with a wide spectral range and some arbitrary temporal distribution. In general, some of the *N* photons are spread out in time while others may overlap and become indistinguishable. From this model, we find that the temporal (in)distinguishability of photons is related to the exchange symmetry of the multiphoton wave function. We find that simple multiphoton detection scheme gives rise to a more general photon bunching effect with the famous two-photon effect as a special case. We then send this *N*-photon state into a recently discovered multiphoton interference scheme. We calculate the visibility of the multiphoton interference scheme and find that it is related to the temporal distinguishability of the *N* photons. Maximum visibility of one is achieved for the indistinguishable *N*-photon state whereas the visibility degrades when some of the photons are separated and become distinguishable. Thus we can identify an experimentally measurable quantity that may quantitatively define the degree of indistinguishability of an *N*-photon state. This presents a quantitative demonstration of the complementary principle of quantum interference.

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I. INTRODUCTION

The coherence properties of an optical field are best described by the field correlation function in space and time [1]. The most commonly used quantity to characterize the coherence property of an optical field is the coherence time or coherence length for temporal coherence. Roughly speaking, the coherence length of an optical field is the distance within which the field can be described as a single uninterrupted wave train. In other words, any two points within the coherence length will have a fixed phase relationship. However, this description is primarily concerned with wave aspect of an optical field and is based on the interference effect observed in intensity or single photon interference effect. More specifically in terms of the quantum coherence theory [2], it is related to the field correlation function of

$$\Gamma(\tau) = \langle \hat{E}^{(-)}(t+\tau)\hat{E}^{(+)}(t)\rangle, \tag{1}$$

where

$$[\hat{E}^{(-)}]^{\dagger} = \hat{E}^{(+)}(t) = \frac{1}{\sqrt{2\pi}}\int d\omega \hat{a}(\omega) e^{-i\omega t} \tag{2}$$

for a quasimonochromatic field [3] and the average is over the quantum state of the field. The visibility of the single-photon interference fringes is simply the absolute value of the normalized field correlation function [4]

$$\gamma(\tau) = \Gamma(\tau)/\Gamma(0). \tag{3}$$

However, this description becomes rudimentary when we start to deal with the cases involving more than one photon in quantum information. One may use a higher order correlation function such as the intensity correlation function [2]

$$\Gamma^{(N)}(t_1,t_2,\ldots,t_N) = \langle \hat{E}^{(-)}(t_1)\cdots\hat{E}^{(-)}(t_N)\hat{E}^{(+)}(t_N)\cdots\hat{E}^{(+)}(t_1)\rangle \tag{4}$$

which is related to an *N*-photon coincidence measurement. However, this function does not provide any information about photon entanglement, i.e., quantum superposition of different states.

The realization of multiparticle entanglement is paramount in achieving most of the tasks in quantum computing and quantum information processing [5,6]. While there are many ways to create entangled multiparticle states, the straightforward method is to start from independent single photons [7]. Knill, Laflamme, and Milburn [8] have shown that quantum computing can be realized with single photons and some linear optical elements via multiphoton interference. *N*-photon entanglement is thus produced from single-photon states. This is one of the primary reasons behind the big rush in creating light sources with a single photon on demand [9,10].

While most of the analysis is based on the single mode model, i.e., all the photons in one single temporal mode, this is, on the other hand, impossible to achieve in experiment. The multimode nature of light inevitably reduces the effect of photon interference and leads to degradation in information processing. One often uses the fidelity quantity of quantum states to characterize the degradation. But this description has emphasis only on the end result of the process and spares the true culprit of the process, that is, the multimode nature of light.

For monochromatic field of only one frequency component, the field can be represented by an infinite wave train. Photons can appear anywhere in this wave train and are indistinguishable from each other. They will produce maximum effects of entanglement. However, when many frequency components are excited, an optical field is no longer monochromatic and the wave train becomes finite with a length of the order of the coherence length of the field. With multiple photons, we generally cannot use a single wave packet to describe them. We cannot assign separate wave packets to describe each photon, either. This is because of the possibility of multiphoton entanglement. Thus, an issue is raised about how to describe the different situations of temporal distribution of photons and distinguish these situations experimentally.

Recently, this issue was addressed in the four-photon case [11,12] for distinguishing a genuine four-photon polarization entangled state from a state made of two well separated pairs of photons. The difference lies in the multiphoton interference: an entangled four-photon state will give rise to the strongest multiphoton interference effect whereas two well separate pairs produce less interference effect. This is consistent with the complementary principle of quantum mechanics which states that quantum interference is a result of indistinguishability of the paths but if the paths are distinguishable, the interference effect will be gone. Partial distinguishability will lead to reduced interference effect, as described by Eq. (3) in the coherence theory for the single-photon interference. Four-photon interference experiments were performed to distinguish an entangled four-photon state from two independent pairs of photons [11–14].

However, the abovementioned interference scheme on the four-photon state cannot be generalized to arbitrary photon number. More recently, Sun *et al.* [15,16] and Resch *et al.* [17] independently constructed a quantum state projection measurement scheme and applied it to a maximally entangled N-photon state (the so-called NOON state) for the demonstration of multiphoton de Broglie wavelengths without a NOON state. It turns out that this projection measurement scheme is based on a multiphoton interference effect that depends on the temporal distribution of the photons involved. Since the scheme can be easily generalized to arbitrary photon number, it can be used to study the relation between the multiphoton interference effect and the temporal distinguishability of an N-photon state. We will show that the various scenarios of temporal distribution of photons give rise to different visibility in the multiphoton interference, which provides a direct measure of the degree of temporal distinguishability of a multiphoton state in a similar fashion to the coherence theory [Eq. (3)]. This is a quantitative investigation into the complementary principle of quantum interference.

In the following, we will first review the two-photon and four-photon cases to look for the relation between temporal distinguishability and multiphoton interference. We then will generalize to an arbitrary N-photon state and present the criteria for photon indistinguishability and distinguishability. In Sec. IV, we use quantum coherence theory to calculate the result from a direct N-photon coincidence measurement and discuss the generalized photon bunching effect. This measurement process is not sensitive to the different temporal distribution of the photons. In Sec. V, we introduce the newly constructed NOON state projection measurement and demonstrate how it can be used to characterize the degree of temporal indistinguishability for the simple three-photon case. We will generalize the discussion for three-photon case to the general $(N+1)$-photon case. In Sec. VI, we will discuss an even more general case and present the numerical results for a few special cases. We conclude with a discussion.

II. TEMPORAL DISTINGUISHABILITY FOR THE CASE OF TWO PHOTONS AND FOR THE CASE OF TWO PAIRS OF PHOTONS

The first discussion about the temporal distinguishability was by Grice and Walmsley [18], who investigated the visibility in a Hong-Ou-Mandel interferometer [19] with a two-photon state input from type-II parametric down-conversion. Later on, Atatüre *et al.* [20] performed experiment and confirmed the degradation of the two-photon interference visibility predicted in Ref. [18] due to temporal distinguishability.

In the discussion of Ref. [18], the multimode description of the two-photon state is given by

$$|\Phi_2\rangle = \int d\omega_1 d\omega_2 \Phi(\omega_1,\omega_2)\hat{a}_s^\dagger(\omega_1)\hat{a}_i^\dagger(\omega_2)|0\rangle, \tag{5}$$

where s,i denote the two correlated signal and idler photons from parametric down-conversion. For type-II process, we have $\Phi(\omega_1,\omega_2)\neq\Phi(\omega_2,\omega_1)$ due the birefringent effect of the nonlinear crystal on the ordinary and extraordinary rays. The maximum visibility in the two-photon Hong-Ou-Mandel interferometer has the form

$$\mathcal{V}_2 = M_2 \equiv \frac{\int d\omega_1 d\omega_2 \Phi^*(\omega_1,\omega_2)\Phi(\omega_2,\omega_1)}{\int d\omega_1 d\omega_2 |\Phi(\omega_1,\omega_2)|^2}. \tag{6}$$

M_2 is defined as a degree of permutation symmetry. Note that $M_2=M_2^*$ and $0\leq|M_2|\leq 1$. The visibility or the degree of permutation symmetry is one if and only if $\Phi(\omega_1,\omega_2)$ satisfies the permutation symmetry relation

$$\Phi(\omega_1,\omega_2)=\Phi(\omega_2,\omega_1). \tag{7}$$

As stated in Ref. [18], this permutation relation is a signature of spectral indistinguishability of the two photons, that is, we cannot tell the difference between the two photons through their spectra. This in turn gives temporal indistinguishability if we consider the Fourier transformation

$$G(t_1,t_2)=\frac{1}{2\pi}\int d\omega_1 d\omega_2 \Phi(\omega_1,\omega_2)e^{-i(\omega_1 t_1+\omega_2 t_2)}. \tag{8}$$

Combination of Eqs. (7), (8) gives directly the symmetric relation

$$G(t_1,t_2)=G(t_2,t_1), \tag{9}$$

for all times of t_1,t_2.

On the other hand, the visibility is zero if $\Phi(\omega_1,\omega_2)$ does not have any overlap with $\Phi(\omega_2,\omega_1)$, which is characterized by the orthogonal relation

$$\int d\omega_1 d\omega_2 \Phi^*(\omega_1,\omega_2)\Phi(\omega_2,\omega_1)=0 \tag{10}$$

or in time

$$\int dt_1 dt_2 G^*(t_1,t_2)G(t_2,t_1)=0. \tag{11}$$

This orthogonal relation indicates that the two functions $G(t_1,t_2),G(t_2,t_1)$ have no overlap.

At this point, it is not easy to see what is the physical meaning of Eq. (11). However, if we go back to Eq. (7) and introduce a nonsymmetric factor of $e^{i\omega_2 T}$, we find that the equivalent $\Phi(\omega_1,\omega_2)$ in Eq. (5) in this case will be

$\Phi'(\omega_1,\omega_2)\equiv\Phi(\omega_1,\omega_2)e^{i\omega_2 T}$, which is not symmetric with respect to ω_1,ω_2 even if $\Phi(\omega_1,\omega_2)$ is symmetric. This extra phase can be introduced by acting the evolution operator $\hat{U}(T)=\exp(-i\omega_2\hat{a}_i^\dagger\hat{a}_i T)$ on the state in Eq. (5) for an extra free propagation time T of the idler photon. This then creates a time delay T between the two photons. Then the visibility in Eq. (6) becomes

$$\mathcal{V}_2(T)=\frac{\int d\omega_1 d\omega_2 \Phi^*(\omega_1,\omega_2)\Phi(\omega_2,\omega_1)e^{i(\omega_1-\omega_2)T}}{\int d\omega_1 d\omega_2 |\Phi(\omega_1,\omega_2)|^2}. \quad (12)$$

Notice that if the delay is large enough $[T\gg T_c\sim 1/\Delta\omega_{\mathrm{PDC}}$ with $\Delta\omega_{\mathrm{PDC}}$ as the range of $\Phi(\omega_1,\omega_2)]$, we will have $\mathcal{V}_2(T)=0$ or $\Phi'(\omega_1,\omega_2)$ satisfies Eq. (10). Since T is the relative delay between the two photons before they meet at the beam splitter of the Hong-Ou-Mandel interferometer, we may believe that there is a large enough delay between the two photons so that the two photons become distinguishable in time when they arrive at the beam splitter. So the orthogonal relation in Eq. (10) or (11) corresponds to the situation when the two photons are well separated in time and form two non-overlapping and distinguishable wave packets.

Therefore, the visibility in the Hong-Ou-Mandel interferometer in Eq. (6) is a direct measure of temporal distinguishability of the two photons. This is very much similar to the role of the field correlation function γ of Eq. (3) in defining optical coherence of a field.

For the four-photon case, temporal distinguishability between two pairs of photons was first studied by Ou, Rhee, and Wang [13,14] in a similar scheme as the Hong-Ou-Mandel interferometer but with four photons. It was found that the visibility in four-photon interference is directly related to a quantity $\mathcal{E}/\mathcal{A}$, which is a measure of the temporal distinguishability of photon pairs from parametric downconversion. When $\mathcal{E}/\mathcal{A}\ll 1$, the pairs are well separated from each other corresponding to the so-called 2×2 case, but when $\mathcal{E}/\mathcal{A}=1$, the two pairs are overlap in time and form an indistinguishable four-photon state corresponding to the 4×1 case.

From the definition of the quantities $\mathcal{E}$ and $\mathcal{A}$ in Ref. [14], we rewrite them as

$$\mathcal{E}=\int d\omega_1 d\omega_2 d\omega_1' d\omega_2' \Phi^*(\omega_1,\omega_2)\Phi^*(\omega_1',\omega_2') \times \Phi(\omega_1',\omega_2)\Phi(\omega_1,\omega_2'), \quad (13)$$

and

$$\mathcal{A}=\int d\omega_1 d\omega_2 d\omega_1' d\omega_2' |\Phi(\omega_1,\omega_2)\Phi(\omega_1',\omega_2')|^2, \quad (14)$$

where $\Phi(\omega_1,\omega_2)$ is the two-photon wave function in Eq. (5).

On the other hand, the four-photon state from Ref. [14] has the form of

$$|\Phi_4\rangle=\int d\omega_1 d\omega_2 d\omega_1' d\omega_2' \Phi_4(\omega_1,\omega_2;\omega_1',\omega_2') \times \hat{a}_s^\dagger(\omega_1)\hat{a}_i^\dagger(\omega_2)\hat{a}_s^\dagger(\omega_1')\hat{a}_i^\dagger(\omega_2')|0\rangle, \quad (15)$$

where $\Phi_4(\omega_1,\omega_2;\omega_1',\omega_2')\equiv\Phi(\omega_1,\omega_2)\Phi(\omega_1',\omega_2')$. Then, we can rewrite the expression for $\mathcal{E}$ and $\mathcal{A}$ in Eqs. (13) and (14), and obtain the quantity $\mathcal{E}/\mathcal{A}$ as

$$\frac{\mathcal{E}}{\mathcal{A}}=\frac{\int d\omega_1 d\omega_2 d\omega_1' d\omega_2' \Phi_4^*(\omega_1,\omega_2,\omega_1',\omega_2')\Phi_4(\omega_1',\omega_2;\omega_1,\omega_2')}{\int d\omega_1 d\omega_2 d\omega_1' d\omega_2' |\Phi_4(\omega_1,\omega_2;\omega_1',\omega_2')|^2}. \quad (16)$$

Recall that this quantity is a measure of the temporal distinguishability of two pairs of photons. But from Eq. (16), we find that this quantity is again dependent on the permutation of the wave function $\Phi_4(\omega_1,\omega_2;\omega_1',\omega_2')$ similar to that in Eq. (6), and it is one if and only if we have the permutation symmetry of

$$\Phi_4(\omega_1,\omega_2,\omega_1',\omega_2')=\Phi_4(\omega_1',\omega_2;\omega_1,\omega_2'). \quad (17)$$

Therefore, from the discussion on the meaning of the quantity $\mathcal{E}/\mathcal{A}$, we find that the symmetry relation in Eq. (17) corresponds to the case when the two pairs completely overlap in time and become temporally indistinguishable (the 4×1 case), whereas the orthogonal relation

$$\int d\omega_1 d\omega_2 d\omega_1' d\omega_2' \Phi_4^*(\omega_1,\omega_2,\omega_1',\omega_2')\Phi_4(\omega_1',\omega_2;\omega_1,\omega_2')=0 \quad (18)$$

leads to the case of completely separated pairs of photons (the 2×2 case).

From the experiments and analysis on four-photon interference with two pairs of photons by parametric down-conversion [13–15], we find that the visibility is not zero even for $\mathcal{E}/\mathcal{A}=0$. This can be attributed to the existence of two-photon interference since we usually have two-photon indistinguishability with exchange symmetry in Eq. (7). Note that $\mathcal{E}/\mathcal{A}$ concerns the permutation symmetry between two different pairs, i.e., exchange between the group of $\{\omega_1,\omega_2\}$ and the group of $\{\omega_1',\omega_2'\}$. The exchange within each group is symmetric due to Eq. (7). Next we will generalize Eqs. (7), (17) and Eqs. (10), (18) to an arbitrary N-photon case and relate them to the visibility of some N-photon interference experiment.

III. DESCRIPTION OF A TEMPORALLY DISTRIBUTED N-PHOTON STATE

Now we can generalize Eqs. (7), (10) of the two-photon case and Eqs. (17), (18) of the two-pair case to arbitrary N case. An arbitrary N-photon state of wide spectral range can be generally described by

$$|\Phi_N\rangle = \mathcal{N}^{-1/2}\int d\omega_1 d\omega_2\cdots d\omega_N \Phi(\omega_1,\ldots,\omega_N) \times \hat{a}^\dagger(\omega_1)\hat{a}^\dagger(\omega_2)\cdots\hat{a}^\dagger(\omega_N)|0\rangle, \quad (19)$$

where the normalization factor $\mathcal{N}$ is given by

$$\mathcal{N} = \int d\omega_1 d\omega_2\cdots d\omega_N \Phi^*(\omega_1,\ldots,\omega_N) \times \sum_P \Phi(P\{\omega_1,\ldots,\omega_N\}), \quad (20)$$

where P is the permutation operator on the indices of $1,2,\ldots,N$. and the sum is over all possible permutation. There are totally $N!$ terms. So the value of $\mathcal{N}$ ranges from I to $N!I$ with $I=\int d\omega_1 d\omega_2\cdots d\omega_N|\Phi(\omega_1,\ldots,\omega_N)|^2$. The maximum value of $N!I$ is reached when

$$\Phi(\omega_1,\ldots,\omega_N) = \Phi(P\{\omega_1,\ldots,\omega_N\}) \quad (21)$$

for all P. Similar to Eqs. (7), (17), this corresponds to a case when the N photons are indistinguishable in time. We refer to this case as the $N\times 1$ case, meaning that all N photons are in one indistinguishable temporal mode. This single-mode description of an N-photon state is more vivid in the special case when $\Phi(\omega_1,\ldots,\omega_N)$ is factorized as $\phi(\omega_1)\phi(\omega_2)\cdots\phi(\omega_N)$ and the N-photon state simply becomes

$$|\Phi_N\rangle = \frac{1}{N!}\hat{A}(\phi)^{\dagger N}|0\rangle = |N\rangle_\phi \quad (22)$$

with

$$\hat{A}(\phi) = \int d\omega\phi(\omega)\hat{a}(\omega) \quad \left(\int d\omega|\phi(\omega)|^2 = 1\right). \quad (23)$$

Note that $\hat{A}(\phi)$ satisfies $[\hat{A},\hat{A}^\dagger]=1$ and represents the annihilation operator for a single temporal mode characterized by $\phi(\omega)$. The single-photon state $|1\rangle_\phi$ has a single-photon detection probability of $|g(\tau)|^2$ with a temporal shape of

$$g(\tau) = \frac{1}{\sqrt{2\pi}}\int d\omega\phi(\omega)e^{-i\omega t} \quad (24)$$

and normalization relation

$$\int d\tau|g(\tau)|^2 = 1. \quad (25)$$

The other extreme case of $\mathcal{N}=I$ requires $\Phi(\omega_1,\ldots,\omega_N)$ to be orthogonal to all the permuted functions $\Phi(P\{\omega_1,\ldots,\omega_N\})$ in the similar ways in Eqs. (10), (18) and thus corresponds to the situation when all photons are well separated in time. We refer to this case as the $1\times N$ case, meaning that each photon is in its separate temporal mode and there are totally N independent modes.

For the situations in between the two extreme cases, the value of $\mathcal{N}$ is between I and $N!I$. For example, assume that the spectral amplitude $\Phi(\{\omega\})$ has partial permutation symmetry, that is,

$$\Phi(\omega_1,\ldots,\omega_N) = \Phi(P_{\{n_i\}}\{\omega_1,\ldots,\omega_N\}), \quad (26)$$

where the permutation $P_{\{n_i\}}$ only applies to a subgroup of $\{\omega_1,\omega_2,\ldots,\omega_N\}$. In the meantime, it also satisfies the orthogonal relations

$$\int d\omega_1\cdots d\omega_N \Phi^*(\omega_1,\ldots,\omega_N)\Phi(P_{ij}\{\omega_1,\ldots,\omega_N\}) = 0 \quad (27)$$

for permutation P_{ij} between different subgroups ($\{n_i\}$ and $\{n_j\}$, $i\neq j$) defined in Eq. (26). Then it can be easily shown that $\mathcal{N}=n_1!n_2!\ldots n_k!I$. In the simple case when $\Phi(\omega)$ can be factorized as

$$\Phi(\omega_1,\ldots,\omega_N) = \phi_1(\omega_1)\ldots\phi_1(\omega_{n_1})\phi_2(\omega_{n_1+1}) \times \ldots\phi_2(\omega_{n_1+n_2})\ldots\phi_k(\omega_N) \quad (28)$$

with the orthogonal relations

$$\int d\omega_1 d\omega_2 \phi_i^*(\omega_1)\phi_j^*(\omega_2)\phi_i(\omega_2)\phi_j(\omega_1) = 0 \quad (i\neq j), \quad (29)$$

the N-photon state in Eq. (19) becomes

$$|\Phi_N\rangle = \frac{1}{n_1!}|n_1\rangle_{\phi_1}\frac{1}{n_2!}|n_2\rangle_{\phi_2}\ldots\frac{1}{n_k!}|n_k\rangle_{\phi_k}. \quad (30)$$

This is the situation when the N photons are divided into k subgroups with $n_i(i=1,2,\ldots,k)$ photons in each group in a single temporal mode characterized by ϕ_i. This situation is denoted as the $n_1+\cdots+n_k$ case.

For simplicity of later argument, let us consider another special kind of N-photon state with

$$\Phi(\omega_1,\ldots,\omega_N) = \phi(\omega_1)e^{i\omega_1 T_1}\cdots\phi(\omega_N)e^{i\omega_N T_N}. \quad (31)$$

With this Φ, the N-photon state can be viewed as direct product of N identical single photon wave packets

$$|N\rangle_T = |T_1\rangle\otimes|T_2\rangle\otimes\cdots\otimes|T_N\rangle, \quad (32)$$

with

$$|T_i\rangle = \int d\omega\phi(\omega)e^{i\omega T_i}\hat{a}^\dagger(\omega)|0\rangle. \quad (33)$$

This state can be viewed as from single-photon sources such as quantum dots (see below for details). However, the quantum state in Eq. (32) is not normalized. Substituting Eq. (31) into Eq. (20), we have the normalization factor as

$$\mathcal{N} = \int d\omega_1 d\omega_2\cdots d\omega_N\left[\prod_k|\phi(\omega_k)|^2 e^{-i\omega_k T_k}\right] \times \sum_P P\left[\exp\left\{\sum_m i\omega_m T_m\right\}\right]. \quad (34)$$

When $T_1=T_2=\cdots=T_N$, we recover the case when all N photons are in one single temporal mode with $\mathcal{N}=N!$ ($N\times 1$). On the other hand, if $|T_i-T_j|\gg 1/\Delta\omega(i\neq j)$ with $\Delta\omega$ as the bandwidth of $\phi(\omega)$, we have $\mathcal{N}=1$. This is the case

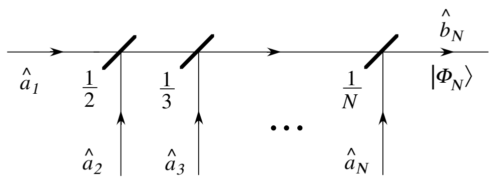

FIG. 1. Generation of an N-photon state from single-photon states by beam splitters.

when all the photons are well separated from each other ($1\times N$).

The N-photon state in Eq. (19) describes a state when all photons are in one spatial and polarization mode. They only differ in spectral mode. In practice, although $N=2$ case can be easily obtained from degenerate parametric down-conversion, such a state with $N>2$ is not easy to produce directly. It can be produced indirectly from single-photon states with a set of beam splitters as shown in Fig. 1, where the single-photon sources are, for example, quantum dots. The quantum state for the input fields has the general form of

$$|\Psi_N\rangle_{\rm in} = \int d\omega_1 d\omega_2 \cdots d\omega_N \Phi(\omega_1, \ldots, \omega_N) \times \hat{a}_1^\dagger(\omega_1)\hat{a}_2^\dagger(\omega_2)\cdots\hat{a}_N^\dagger(\omega_N)|0\rangle \tag{35}$$

with the normalization relation

$$\int d\omega_1 d\omega_2 \cdots d\omega_N \Phi^*(\omega_1, \ldots, \omega_N)\Phi(\omega_1, \ldots, \omega_N) = 1. \tag{36}$$

Here $\hat{a}_j^\dagger (j=1,\ldots,N)$ is the creation operator for each input mode. Photons are possible to exit at any of the N output ports. To produce a state of the form in Eq. (19), however, we only consider the possibility when all N photon's exit at one port, say, b_N port. It is straightforward using the beam splitter theory to show that the projected state is

$$\mathbb{P}|\Psi_N\rangle_{\rm out} = \frac{1}{N^{N/2}} \int d\omega_1 d\omega_2 \cdots d\omega_N \Phi(\omega_1, \ldots, \omega_N) \times \hat{b}_N^\dagger(\omega_1)\hat{b}_N^\dagger(\omega_2)\cdots\hat{b}_N^\dagger(\omega_N)|0\rangle, \tag{37}$$

which is in the form of Eq. (19). This state is not normalized because it is a projected state with the probability of projection as $P(|\Phi_N\rangle) = \|\mathbb{P}|\Psi_N\rangle_{\rm out}\|^2 = \mathcal{N}/N^N$. The delay factors $\{e^{i\omega_j T_j}\}$ in Eq. (31) can be easily introduced on individual mode $\hat{a}_j$ before the beam splitters via the free-field evolution operator $\hat{U}_j(T_j) = \exp(-i\omega_j \hat{a}_j^\dagger \hat{a}_j T_j)$.

More generally, to include different spatial and polarization modes, the N-photon state has the following shape:

$$|\Phi_N\rangle = \mathcal{N}_k^{-1/2} \int d\omega_1^{(1)} \cdots d\omega_{n_1}^{(1)} \cdots d\omega_1^{(k)} \cdots d\omega_{n_k}^{(k)} \Phi(\{\omega^{(1)}\}, \ldots, \{\omega^{(k)}\}) \hat{a}_1^\dagger(\omega_1^{(1)}) \cdots \hat{a}_1^\dagger(\omega_{n_1}^{(1)}) \cdots \hat{a}_k^\dagger(\omega_{n_k}^{(k)})|0\rangle, \tag{38}$$

where $\{\omega^{(1)}\} = \omega_1^{(1)}, \ldots, \omega_{n_1}^{(1)}$, etc. The normalization factor $\mathcal{N}_k$ takes the form of

$$\mathcal{N}_k = \int d\{\omega^{(1)}\} \cdots d\{\omega^{(k)}\} \Phi^*(\{\omega^{(1)}\}, \ldots, \{\omega^{(k)}\}) \sum_{P_1, \ldots, P_k} \Phi(P_1\{\omega^{(1)}\}, \ldots, P_k\{\omega^{(k)}\}). \tag{39}$$

$\mathcal{N}_k$ now ranges from I to $n_1! \ldots n_k! I$. The special case when $\Phi(\{\omega^{(1)}\}, \ldots, \{\omega^{(k)}\})$ factorizes is similar as before.

IV. DIRECT N-PHOTON MEASUREMENT: PHOTON BUNCHING EFFECT FOR N PHOTONS

Next, we consider an N-photon joint measurement with the joint probability density given from the quantum coherence theory in Eq. (4). The average is over the quantum state of the system given in Eq. (19) for an arbitrary N-photon state. For simplicity, we first apply it to the state in Eq. (32).

To carry out the quantum average, it is easier to first find the N-photon detection probability amplitude

$$C^{(N)}(t_1, t_2, \ldots, t_N) = \langle 0|\hat{E}^{(+)}(t_N) \cdots \hat{E}^{(+)}(t_1)|\Phi_N\rangle. \tag{40}$$

Then $\Gamma^{(N)}(t_1, t_2, \ldots, t_N) = |C^{(N)}(t_1, t_2, \ldots, t_N)|^2$. From Eq. (2) for the field operator and Eq. (31) for Φ, it is straightforward to obtain

$$C^{(N)}(t_1, t_2, \ldots, t_N) = \sum_P P[g(t_1 - T_1) \cdots g(t_N - T_N)], \tag{41}$$

where the permutation operation P is on $t_1 t_2 \cdots t_N$ and there are $N!$ terms in the sum.

The overall probability of detecting N photons together (N-photon coincidence) is an integral of $\Gamma^{(N)}(t_1, t_2, \ldots, t_N)$ over all times $t_1, \ldots, t_N$:

$$P_N = \int dt_1 \cdots dt_N \Gamma^{(N)}(t_1, t_2, \ldots, t_N) = \int dt_1 \cdots dt_N \left| \sum_P P[g(t_1 - T_1) \cdots g(t_N - T_N)] \right|^2. \tag{42}$$

In the extreme case when $T_1 = T_2 = \cdots = T_N$, we obtain $P_N(N\times 1) = (N!)^2 I$ while in the other extreme case when $|T_i - T_j| \gg 1/\Delta\Omega$, we have $P_N(1\times N) = N! I$. Therefore, we seem to have

$$P_N(N \times 1) = N!\, P_N(1 \times N) \tag{43}$$

or

$$P_N(N \times 1)/P_N(1 \times N) = N!\,, \tag{44}$$

that is, the N-photon detection probability is N! larger in the case of N identical photons than in the case of N separated photons. This can be thought of as the Bosonic photon bunching effect for N photons. The case of $N=2$ gives the familiar photon bunching factor of 2.

However, as we know, the N-photon state in Eq. (32) is not normalized. With the normalization factor considered, we have instead

$$P_N(N \times 1) = P_N(1 \times N) = N!\,. \tag{45}$$

For the case in between the two extreme cases, we may evaluate Eq. (42) as

$$P_N = \int dt_1 \cdots dt_N \sum_{P'} P'[g^*(t_1 - T_1) \cdots g^*(t_N - T_N)] \times \sum_P P[g(t_1 - T_1) \cdots g(t_N - T_N)]. \tag{46}$$

Since the sum is over all permutations, the integral does not change if we make the variable change $\{t_1 \cdots t_N\} \to P'\{t_1 \cdots t_N\}$, i.e.,

$$\begin{aligned} P_N &= \sum_{P'} \int dt_1 \cdots dt_N g^*(t_1 - T_1) \cdots g^*(t_N - T_N) \\ &\quad \times \sum_P P[g(t_1 - T_1) \cdots g(t_N - T_N)] \\ &= N! \int dt_1 \cdots dt_N g^*(t_1 - T_1) \cdots g^*(t_N - T_N) \\ &\quad \times \sum_P P[g(t_1 - T_1) \cdots g(t_N - T_N)]. \end{aligned} \tag{47}$$

It can be further shown that

$$\int dt_1 \cdots dt_N g^*(t_1 - T_1) \cdots g^*(t_N - T_N) \times \sum_P P[g(t_1 - T_1) \cdots g(t_N - T_N)] = \mathcal{N}, \tag{48}$$

where $\mathcal{N}$ is given in Eq. (34). Thus we have

$$P_N = N!\,\mathcal{N}. \tag{49}$$

For a normalized N-photon state, we have $P_N = N!$ in any case as in Eq. (45).

For the multispatial and polarization state in Eq. (38), we may find P_N after some lengthy manipulation as that leads to Eq. (49)):

$$P_N = n_1! \cdots n_k!\,\mathcal{N}_k \tag{50}$$

for the un-normalized state and $P_4 = n_1! \cdots n_k!$ for the normalized state.

Hence, it is impossible to characterize different cases of temporal entanglement with just simple direct multiphoton detection for the normalized state. Furthermore, even for the un-normalized state, we cannot explore the temporal indistinguishability among different spatial and polarization modes with multiphoton detection, for $\mathcal{N}_k$ depends only on the permutation symmetry within photons in one spatial and polarization mode.

Before we proceed further, it is interesting to evaluate the multiphoton detection rates in some special cases. For example, for the single-photon detection rate P_1, we have

$$P_1 = \int dt \langle \Phi_N | \hat{E}^\dagger(t) \hat{E}(t) | \Phi_N \rangle. \tag{51}$$

With some manipulation, it can be shown that $P_1 = N\mathcal{N}$ for an un-normalized N-photon state and $P_1 = N$ for a normalized N-photon state.

The reason that we still discuss the un-normalized case of an N-photon state is because we encounter this kind of state in practice when a projection measurement is involved such as that in Fig. 1. Consider, for example, a multiphoton state from degenerate parametric down-conversion, which, for small η, has the form of [15]

$$|\Phi_{\mathrm{PDC}}\rangle = C\left(|0\rangle + \eta|\Phi_{\mathrm{2D}}\rangle + \frac{\eta^2}{2}|\Phi_{\mathrm{4D}}\rangle + \cdots\right), \tag{52}$$

with

$$|\Phi_{\mathrm{2D}}\rangle = \int d\omega_1 d\omega_2 \Phi(\omega_1, \omega_2) \hat{a}^\dagger(\omega_1) \hat{a}^\dagger(\omega_2) |0\rangle \tag{53}$$

and

$$|\Phi_{\mathrm{4D}}\rangle = \int d\omega_1 d\omega_2 d\omega_1' d\omega_2' \Phi(\omega_1, \omega_2) \Phi(\omega_1', \omega_2') \times \hat{a}^\dagger(\omega_1) \hat{a}^\dagger(\omega_2) \hat{a}^\dagger(\omega_1') \hat{a}^\dagger(\omega_2') |0\rangle. \tag{54}$$

Here C in Eq. (52) is a normalization factor but because $|\eta| \ll 1$, $|C| \approx 1$ no matter what function $\Phi(\omega_1, \omega_2)$ is. Two-photon and four-photon detections project the state to $\eta|\Phi_{\mathrm{2D}}\rangle$ and $\eta^2|\Phi_{\mathrm{4D}}\rangle/2$, respectively, which are not normalized. From Eqs. (20), (49), we then have

$$P_2 = 2|\eta|^2 \int d\omega_1 d\omega_2 [|\Phi(\omega_1, \omega_2)|^2 + \Phi^*(\omega_1, \omega_2) \Phi(\omega_2, \omega_1)]. \tag{55}$$

The last term is related to the permutation symmetry or the degree of two-photon temporal distinguishability and can be viewed as a two-photon bunching effect. For a state from parametric down-conversion in the degenerate case as in Eq. (52), we usually have the symmetry $\Phi(\omega_1, \omega_2) = \Phi(\omega_2, \omega_1)$ so that

$$P_{2D} = 4|\eta|^2 \int d\omega_1 d\omega_2 |\Phi(\omega_1,\omega_2)|^2. \tag{56}$$

Similarly for four-photon case, we have

$$P_{4D} = 48|\eta|^2(\mathcal{A} + 2\mathcal{E}) = 3P_2^2(1 + 2\mathcal{E}/\mathcal{A}), \tag{57}$$

where $\mathcal{E},\mathcal{A}$ are given in Eqs. (13), (14), respectively. The dependence on $\mathcal{E}/\mathcal{A}$ indicates that the extra term in Eq. (57) is a pair bunching effect—a generalized photon bunching effect for a multiphoton state. Direct measurement by Sun *et al.* [16] confirmed the four-photon bunching effect in Eq. (57).

Another example is from nondegenerate parametric down-conversion in type-II $\chi^{(2)}$ medium. The quantum state is similar to that in Eq. (52) [14]:

$$|\Phi_{\rm NPDC}\rangle = C\left(|0\rangle + \eta|\Phi_{\rm 2N}\rangle + \frac{\eta^2}{2}|\Phi_{\rm 4N}\rangle + \cdots\right), \tag{58}$$

with

$$|\Phi_{\rm 2N}\rangle = \int d\omega_1 d\omega_2 \Phi(\omega_1,\omega_2)\hat{a}_H^\dagger(\omega_1)\hat{a}_V^\dagger(\omega_2)|0\rangle \tag{59}$$

and

$$|\Phi_{\rm 4N}\rangle = \int d\omega_1 d\omega_2 d\omega_1' d\omega_2' \Phi(\omega_1,\omega_2)\Phi(\omega_1',\omega_2') \times \hat{a}_H^\dagger(\omega_1)\hat{a}_V^\dagger(\omega_2)\hat{a}_H^\dagger(\omega_1')\hat{a}_V^\dagger(\omega_2')|0\rangle. \tag{60}$$

From Eq. (50), it is straightforward to have

$$P_{\rm 2N} = |\eta|^2 \int d\omega_1 d\omega_2 |\Phi(\omega_1,\omega_2)|^2 \tag{61}$$

and

$$P_{\rm 4N} = 2|\eta|^2(\mathcal{A} + \mathcal{E}) = 2P_2^2(1 + \mathcal{E}/\mathcal{A}). \tag{62}$$

Although there is no photon bunching at two-photon detection, we still have the pair bunching effect that depends on the $\mathcal{E}/\mathcal{A}$ quantity.

V. *N*-PHOTON INTERFERENCE FROM AN *N*-PHOTON STATE

As seen in the previous section, a direct N-photon detection scheme cannot characterize the temporal indistinguishability in the general case. Therefore, we need to seek another method. Since the direct result of photon indistinguishability is the interference effect, our scheme will be an N-photon interference scheme. As a matter of fact, a Hong-Ou-Mandel interferometer [19] has already been used to measure two-photon indistinguishability from a type-II nondegenerate parametric down-conversion [18,20]. Our method proposed in the following will be a generalization of the Hong-Ou-Mandel interferometer from a two-photon case to an arbitrary N-photon case.

A. NOON state projection as a measure for distinguishability

The NOON state projection measurement was recently proposed to demonstrate an N-photon de Broglie wavelength without the need for a NOON state [15]. It was demonstrated for N=4 with states from parametric down-conversion [16] and N=6 for a coherent state [17] experimentally. The scheme is depicted in Fig. 2 where the input is an arbitrary N-photon state of two polarization modes in the form of

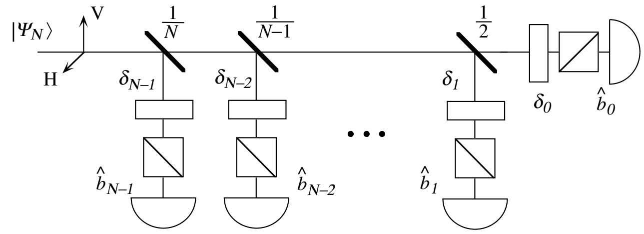

FIG. 2. A NOON-state projection measurement. δ_k=$2k\pi/N$ is the phase difference between H and V. $\hat{b}_k \propto \hat{E}_H - \hat{E}_V e^{i\delta_k}$.

$$|\Psi_N\rangle = \sum_{k=0}^{N} c_k|N-k,k\rangle. \tag{63}$$

The N-photon coincidence probability from the N detectors is proportional to

$$P_N \propto |\langle \text{NOON}|\Psi_N\rangle|^2. \tag{64}$$

If the input state is of the form of $|N-k,k\rangle (k\neq 0,N)$, the output of the projection is zero. From the construction of the NOON state, we find this orthogonal projection is a result of N-photon interference and thus it can be used to characterize the temporal indistinguishability by the visibility in the interference. We will demonstrate this in the following sections.

B. Three-photon case

Let us start with a three-photon state of the form $|2_H,1_V\rangle$. So the three-photon NOON state projection measurement should yield null three-photon coincidence in the ideal case when all three photons are in one temporal mode. However, there may be some delay between the vertical photon and the two horizontal photons due to birefringence. Furthermore, the two horizontal photons may also be separated from each other. To account for the three scenarios described above, we cannot use the single-mode state of $|2_H,1_V\rangle$ and have to resort to the multimode model discussed in Sec. III.

A multimode three-photon polarization state for $|2_H,1_V\rangle$ has the form of

$$|\Phi_3\rangle = \int d\omega_1 d\omega_2 d\omega_3 \Phi(\omega_1,\omega_2,\omega_3) \times \hat{a}_H^\dagger(\omega_1)\hat{a}_H^\dagger(\omega_2)\hat{a}_V^\dagger(\omega_3)|0\rangle. \tag{65}$$

For simplicity of argument, we take $\Phi(\omega_1,\omega_2,\omega_3)$ in the form of Eq. (31):

$$\Phi(\omega_1,\omega_2,\omega_3) = \phi(\omega_1)e^{i\omega_1 T_1}\phi(\omega_2)e^{i\omega_2 T_2}\phi(\omega_3)e^{i\omega_3 T_3}. \tag{66}$$

We will use the un-normalized state because in practice, the state in Eq. (65) can be generated by superposing a weak coherent state $|\alpha\rangle$ with a two-photon state $|\eta\rangle$=$|0\rangle$

$+\eta|1_H,1_V\rangle$ from nondegenerate parametric down-conversion:

$$|\Phi_3\rangle = |\alpha\rangle_H|\eta\rangle \approx |0\rangle + \alpha|1_H,0_V\rangle + (\alpha^2/\sqrt{2})|2_H,0_V\rangle + \eta|1_H,1_V\rangle + (\alpha^3/\sqrt{6})|3_H,0_V\rangle + \eta\alpha|2_H,1_V\rangle, \quad (67)$$

where the states are in a single temporal mode and we only write out states up to three photons. A three-photon coincidence measure similar to that in the N-photon NOON state projection will only have contributions from the last two terms. By making the coherent state weak enough so that $|\eta| \gg |\alpha|^2$, we are left with only $|2_H,1_V\rangle$ term. Since $|\alpha|,|\eta| \ll 1$, the three-photon state is not normalized.

For the scenarios presented in the beginning, we can relate them to different values of T_1, T_2, T_3. So $T_1=T_2=T_3$ is for the case of three photons all in one single temporal mode. When $|T_3-T_1| \gg 1/\Delta\omega, |T_3-T_2| \gg 1/\Delta\omega$, the V photon is far from the two H photons. When $|T_2-T_1| \gg 1/\Delta\omega$, the two H photons are far apart.

For the projection measurement in Fig. 2 with $N=3$, we have the electric field operators at three detectors as

$$\hat{E}_0(t) = [\hat{E}_H(t) - \hat{E}_V(t)]/\sqrt{6},$$

$$\hat{E}_1(t) = [\hat{E}_H(t) - e^{i2\pi/3}\hat{E}_V(t)]/\sqrt{6},$$

$$\hat{E}_2(t) = [\hat{E}_H(t) - e^{i4\pi/3}\hat{E}_V(t)]/\sqrt{6}. \quad (68)$$

To find the three-photon coincidence probability, we first calculate the time correlation function

$$\Gamma^{(3)}(t_1,t_2,t_3) = \langle \hat{E}_0^\dagger(t_3)\hat{E}_1^\dagger(t_2)\hat{E}_2^\dagger(t_1)\hat{E}_2(t_1)\hat{E}_1(t_2)\hat{E}_0(t_3)\rangle. \quad (69)$$

It is easy to calculate $\hat{E}_2(t_1)\hat{E}_1(t_2)\hat{E}_0(t_3)|\Phi_3\rangle$:

$$\hat{E}_2(t_1)\hat{E}_1(t_2)\hat{E}_0(t_3)|\Phi_3\rangle = \frac{-1}{6\sqrt{6}}(\hat{E}_H\hat{E}_V\hat{E}_H e^{i2\pi/3} + \hat{E}_V\hat{E}_H\hat{E}_H + \hat{E}_H\hat{E}_H\hat{E}_V e^{i4\pi/3})|\Phi_3\rangle. \quad (70)$$

Here we dropped the terms that have no contribution. The order of the operators is kept for the time variables $t_3t_2t_1$. With the state in Eq. (65) and Φ_3 in Eq. (66), it is straightforward to find

$$\hat{E}_2(t_1)\hat{E}_1(t_2)\hat{E}_0(t_3)|\Phi_3\rangle = \frac{-1}{6\sqrt{6}}\{[G(t_1,t_2,t_3) + G(t_2,t_1,t_3)]e^{i4\pi/3} + [G(t_1,t_3,t_2) + G(t_3,t_1,t_2)]e^{i2\pi/3} + [G(t_2,t_3,t_1) + G(t_3,t_2,t_1)]\}|0\rangle, \quad (71)$$

where

$$G(t_1,t_2,t_3) = \frac{1}{\sqrt{(2\pi)^3}}\int d\omega_1 d\omega_2 d\omega_3 \Phi(\omega_1,\omega_2,\omega_3) \times e^{-i(\omega_1 t_1+\omega_2 t_2+\omega_3 t_3)} = g(t_1-T_1)g(t_2-T_2)g(t_3-T_3) \quad (72)$$

with $g(\tau)$ given in Eq. (24). The three-photon joint detection probability is an integral of the correlation function in Eq. (69) over all time variables:

$$P_3 = \int dt_1 dt_2 dt_3 \Gamma^{(3)}(t_1,t_2,t_3). \quad (73)$$

We are now ready to discuss the three scenarios presented in the beginning of this section. The interference effect is best measured by the visibility which is usually defined as the relative depth of modulation as compared to the situation when the interference effect is zero. In the three scenarios, we find the situation when the V photon is far apart from the two H photons corresponds to the case of no interference, which sets the reference line for evaluating the visibility defined by

$$\mathcal{V}_3 = \frac{|P_3 - P_3(T_3=\infty)|}{P_3(T_3=\infty)}. \quad (74)$$

Experimentally, we can scan T_3 from $T_3=\infty$ until we observe the dip in P_3 and use Eq. (74) to calculate the visibility.

Depending on the separation between the two H photons, we actually only have two distinct cases: (i) the two H photons are completely indistinguishable with $T_1=T_2\equiv T$ and (ii) the two H photons are well separated and distinguishable in time with $|T_1-T_2| \gg 1/\Delta\omega$.

In case (i) with $T_1=T_2\equiv T$, we have the exchange symmetry $G(t_1,t_2,t_3)=G(t_2,t_1,t_3)$ and Eq. (73) becomes after the time integral

$$P_3 = 2\frac{\mathcal{A}_3 - \mathcal{E}_3(\Delta T)}{36} \quad (75)$$

with $\Delta T = T_3 - T$ and

$$\mathcal{A}_3 \equiv \int d\omega_1 d\omega_2 d\omega_3 |\Phi(\omega_1,\omega_2,\omega_3)|^2 = \left(\int d\omega|\phi(\omega)|^2\right)^3, \quad (76)$$

$$\mathcal{E}_3(\tau) \equiv \int d\omega|\phi(\omega)|^2 \left|\int d\omega|\phi(\omega)|^2 e^{-i\omega\tau}\right|^2. \quad (77)$$

Note that $\mathcal{E}_3(0)=\mathcal{A}_3$ and $\mathcal{E}_3(\infty)=0$. So from Eq. (74), we have the visibility for case (i) as

$$\mathcal{V}_3(\mathrm{i}) = \frac{|P_3(\Delta T=0) - P_3(\Delta T=\infty)|}{P_3(\Delta T=\infty)} = 1. \quad (78)$$

The 100% visibility corresponds to the single-mode discussion before.

In case (ii) with $|T_1-T_2| \gg 1/\Delta\omega$, there is no overlap between $G(t_1,t_2,t_3)$ and $G(t_2,t_1,t_3)$ so that $\int dt_1 dt_2 dt_3 G^*(t_1,t_2,t_3)G(t_2,t_1,t_3)=0$. We obtain after the time integral

$$P_3 = \frac{\mathcal{A}_3 - \mathcal{E}_3(\Delta T_1)/2 - \mathcal{E}_3(\Delta T_2)/2}{36} \quad (79)$$

with $\Delta T_1 = T_3 - T_1$ and $\Delta T_2 = T_3 - T_2$. So, we will then have two dips with half depth when T_3 scans through T_1 and T_2. The visibility of each dip is then 50%.

In summary, we find that the scenario when the two H photons are separated have a visibility of 50% while when the two H photons are in one temporal mode, the interference visibility becomes 100%. Therefore, we can distinguish the two different scenarios in the three-photon case by measuring the visibility in the NOON state projection measurement. Recent experiment by Liu *et al.* [21] realized the two scenarios described above and confirmed the corresponding visibility. Next, we will generalize this result to an N-photon state.

C. $(N+1)$-photon case

Let us now generalize the conclusion in the previous section to the case of $|1_H,N_V\rangle$ with an arbitrary integer N. The most general scenario in this case is when the single horizontal photon (H) is indistinguishable from m vertical photons (V) while other $N-m$ V photons are well separated in time from the $m+1$ photons [the case of $1HmV+(N-m)V$ or $1HmV$ for short]. The multimode description of this state has the form of

$$|\Phi(1HmV)\rangle = \int d\omega_1 d\omega_2 \cdots d\omega_{N+1}\Phi(\omega_1,\ldots,\omega_N;\omega_{N+1}) \times \hat{a}_V^\dagger(\omega_1)\ldots\hat{a}_V^\dagger(\omega_N)\hat{a}_H^\dagger(\omega_{N+1})|\text{vac}\rangle, \tag{80}$$

with

$$\Phi(\omega_1,\ldots,\omega_N;\omega_{N+1}) = \phi(\omega_1)e^{i\omega_1T_1}\ldots\phi(\omega_N)e^{i\omega_NT_N} \times\phi(\omega_{N+1})e^{i\omega_{N+1}T_{N+1}}. \tag{81}$$

Here we take Φ in the form of Eq. (31) for ease of calculation.

When m H photons are in the same temporal mode with the V photon, we have $T_1=\cdots=T_m=T_{N+1}\equiv T$. But the other $N-m$ V photons are well separated from these $m+1$ photons. This leads to $|T_j-T_k|\gg 1/\Delta\omega$ with $j=1,2,\ldots,m,N+1$ and $k=m+1,\ldots,N$ and the orthogonal relation

$$\int dt_1dt_2 g^*(t_1-T_j)g^*(t_2-T_k)g(t_1-T_k)g(t_2-T_j)=0. \tag{82}$$

Now we are ready to evaluate the joint $(N+1)$-photon probability P_{N+1} in the NOON-state projection measurement scheme with an input state of $|\Phi(1HmV)\rangle$ in Eq. (80). P_{N+1} is a time integral of the correlation function from $(N+1)$ detectors

$$\Gamma^{(N)}(t_1,t_2,\ldots,t_N) = \langle\Phi(1HmV)|\hat{E}_{N+1}^\dagger(t_{N+1})\cdots\hat{E}_1^\dagger(t_1) \times\hat{E}_1(t_1)\cdots\hat{E}_{N+1}(t_{N+1})|\Phi(1HmV)\rangle, \tag{83}$$

with

$$\hat{E}_j(t)\propto\hat{E}_V(t)-\hat{E}_H(t)e^{i\delta_j}+\cdots, \tag{84}$$

where

$$\hat{E}_{H,V}(t)=\frac{1}{\sqrt{2\pi}}\int d\omega\hat{a}_{H,V}(\omega)e^{-i\omega t}. \tag{85}$$

It is easy to first evaluate $\hat{E}_1(t_1)\cdots\hat{E}_{N+1}(t_{N+1})\,|\Phi(1HmV)\rangle$. After expanding the product, we find only $N+1$ nonzero terms of the form

$$-\sum_{k=1}^{N+1}e^{i\delta_k}\hat{E}_V(t_1)\cdots\hat{E}_H(t_k)\cdots\hat{E}_V(t_{N+1})|\Phi(1HmV)\rangle. \tag{86}$$

For the state $|\Phi(1HmV)\rangle$ in Eq. (80), we have

$$\hat{E}_V(t_1)\cdots\hat{E}_H(t_k)\cdots\hat{E}_V(t_{N+1})|\Phi(1HmV)\rangle = \mathcal{G}(P_{k,N+1}\{t_1,\ldots,t_{N+1}\})|\text{vac}\rangle, \tag{87}$$

with

$$\mathcal{G}(t_1,\ldots,t_N;t_{N+1})=\sum_P G(P\{t_1,\ldots,t_N\};t_{N+1}) \tag{88}$$

and

$$G(t_1,\ldots,t_N;t_{N+1})=\prod_{s=1}^{N+1}g(t_s-T_s), \tag{89}$$

where $P_{k,N+1}$ exchanges t_k with t_{N+1} and P is a permutation of $t_1,\ldots,t_N$. For the case of $1VmH$, we have

$$G(t_1,\ldots,t_{N+1})=g(t_{N+1}-T)\prod_{s=1}^{m}g(t_s-T)\prod_{l=m+1}^{N}g(t_l-T_l), \tag{90}$$

so that $G(t_1,\ldots,t_{N+1})$ has exchange symmetry in $t_1,\ldots,t_m,t_{N+1}$. The overall $(N+1)$-photon coincidence probability is then given by

$$P_{N+1}(1HmV)\propto\int dt_1\cdots dt_{N+1}\left|\sum_{k=1}^{N+1}e^{i\delta_k}\mathcal{G}(P_{k,N+1}\{t_1,\ldots,t_{N+1}\})\right|^2 = \sum_{k,j}e^{i(\delta_k-\delta_j)}\int dt_1\cdots dt_{N+1}\mathcal{G}(P_{k,N+1}\{t_1,\ldots,t_{N+1}\})\mathcal{G}^*(P_{j,N+1}\{t_1,\ldots,t_{N+1}\}). \tag{91}$$

Diagonal terms of $k=j$ in the double sum are all same because the integration is over all time variables

$$\int dt_1\cdots dt_{N+1}|\mathcal{G}(P_{k,N+1}\{t_1,\ldots,t_{N+1}\})|^2=\int dt_1\cdots dt_{N+1}|\mathcal{G}(t_1,\ldots,t_{N+1})|^2. \quad (92)$$

Furthermore,

$$\int dt_1\cdots dt_{N+1}|\mathcal{G}(t_1,\ldots,t_{N+1})|^2=\int dt_1\cdots dt_N dt_{N+1}\left|\sum_P P[g(t_1-T_1)\cdots g(t_N-T_N)]g(t_{N+1}-T_{N+1})\right|^2$$
$$=\int dt_1\cdots dt_N\left|\sum_P P[g(t_1-T_1)\cdots g(t_N-T_N)]\right|^2, \quad (93)$$

where we used the normalization relation in Eq. (25). From Eqs. (46)–(49), we find that it is simply $N!\mathcal{N}$ with $\mathcal{N}$ given in Eq. (34). So the diagonal terms of $k=j$ in Eq. (91) are summed to be $(N+1)N!\mathcal{N}$.

The cross terms in the double sum in Eq. (91) are given by

$$\sum_{k\neq j}e^{i(\delta_k-\delta_j)}\int dt_1\cdots dt_{N+1}\mathcal{G}(P_{k,N+1}\{t_1,\ldots,t_{N+1}\})\mathcal{G}^*(P_{j,N+1}\{t_1,\ldots,t_{N+1}\}). \quad (94)$$

Let us consider one arbitrary term in the sum. The time integral part can be rewritten as

$$\sum_P\int dt_1\cdots dt_{N+1}G(P\{t_1,\ldots,t_{k-1},t_{N+1},t_{k+1},\ldots,t_N\};t_k)\sum_{P'}G^*(P'\{t_1,\ldots,t_{j-1},t_{N+1},t_{j+1},\ldots,t_N\};t_j). \quad (95)$$

Since $k\neq j$, the variable set $\{t_1,\ldots,t_{k-1},t_{N+1},t_{k+1},\ldots,t_N\}$ is different from $\{t_1,\ldots,t_{j-1},t_{N+1},t_{j+1},\ldots,t_N\}$ only at t_j and t_k. For those Ps such that $P\{t_1,\ldots,t_{k-1},t_{N+1},t_{k+1},\ldots,t_N\}$ moves t_j to the first m positions in the variable set $\{t_1,\ldots,t_N\}$, the symmetry between $t_1,\ldots,t_m$ and t_{N+1} in the function $G(t_1,\ldots,t_N;t_{N+1})$ in Eq. (90) will make $G(P\{t_1,\ldots,t_{k-1},t_{N+1},t_{k+1},\ldots,t_N\};t_k)=G(P\{t_1,\ldots,t_{j-1},t_{N+1},t_{j+1},\ldots,t_N\};t_j)$. There are totally $m(N-1)!$ such permutations and they all lead the time integral to

$$\int dt_1\cdots dt_{N+1}G(t_1,\ldots,t_{j-1},t_{N+1},t_{j+1},\ldots,t_N;t_j)\sum_{P'}G^*(P'\{t_1,\ldots,t_{j-1},t_{N+1},t_{j+1},\ldots,t_N\};t_j). \quad (96)$$

By Eq. (48), it is simply $\mathcal{N}$.

For the other permutations that move t_j to the position of $t_{m+1},\ldots,t_N$, it cannot be interchanged with t_k because $T\neq T_s(s=m+1,\ldots,N)$. Furthermore, by the orthogonal relation in Eq. (82), the time integral is simply zero. Therefore, the cross terms are equal to

$$\int dt_1\cdots dt_{N+1}\sum_{k\neq j}e^{i(\delta_k-\delta_j)}\mathcal{G}(P_{1k}\{t_1,\ldots,t_{N+1}\})\mathcal{G}^*(P_{1j}\{t_1,\ldots,t_{N+1}\})=m(N-1)!\,\mathcal{N}\sum_{k\neq j}e^{i(\delta_k-\delta_j)}. \quad (97)$$

But because $\sum_k e^{i\delta_k}=0$, we have

$$\sum_{k\neq j}e^{i(\delta_k-\delta_j)}=\left(\sum_{k,j}-\sum_{k=j}\right)e^{i(\delta_k-\delta_j)}=\sum_k e^{i\delta_k}\sum_j e^{-i\delta_j}-(N+1)=-(N+1). \quad (98)$$

So the final result is

$$P_{N+1}(1HmV)\propto\mathcal{N}(N+1)(N-1)!\,(N-m)=(N+1)!\,\mathcal{N}\left(1-\frac{m}{N}\right). \quad (99)$$

For the generalized Hong-Ou-Mandel interferometer, we scan the delay of the H-photon relative to the V photons. When it does not overlap with any of the V photons, no interference occurs and P_{N+1} is a straight line which corresponds to $m=0$ in Eq. (99) with $P_{N+1}(\infty)=(N+1)!\mathcal{N}$. The value in Eq. (99) corresponds to the case when the delay is zero between the m V photons and the one H photon and a local maximum interference is achieved. So the visibility is

$$\mathcal{V}_{N+1}(1HmV)\equiv\frac{P_{N+1}(\infty)-P_{N+1}(1HmV)}{P_{N+1}(\infty)}=\frac{m}{N}. \quad (100)$$

Note that this visibility only depends on N and m, i.e., the total number N of V photons and the number m of V photons that overlap with the single H photon. It is independent of the normalization factor $\mathcal{N}$ or how the other $N-m$ photons distribute in time.

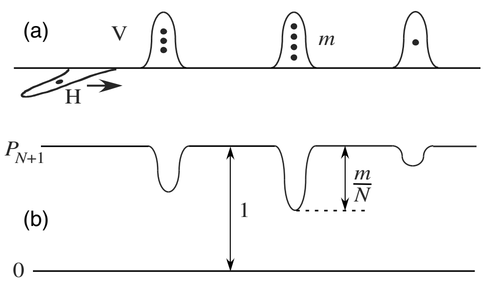

FIG. 3. (a) A temporal distribution with well separated groups of V photons and (b) the corresponding normalized P_{N+1} as the position of the H photon is scanned.

So for a temporal distribution of well separated groups of V photons shown in Fig. 3(a), as we scan the location of the single H photon, we will have more dips of various visibility [Fig. 3(b)] and the visibility is m/N when the single H photon overlaps with the group of m V photons that are in one temporal mode and are well separated from other V photons.

In general for a temporal distribution with m partially overlapping V photons, the visibility will be a value less than m/N. Therefore, the experimentally measurable visibility of the dips can be used to characterize the degree of temporal indistinguishability of an N photon state.

VI. THE GENERAL CASE OF $|k_H,N_V\rangle$ WITH $k>1$

For a more general case of input state of $|k_H,N_V\rangle$ with $k>1$, there are many scenarios for the temporal distribution of the photons. We will start with the four-photon case of $k=N=2$.

A. Four-photon case of $|2_V,2_H\rangle$

This situation was discussed in Ref. [15] for 4×1 case and $2HV\times 2HV$ case. It was shown that $\mathcal{V}_4(4\times 1)=1$ and $\mathcal{V}_4(2\times 2)=1/3$. But there are other scenarios such as $2H1V+1V$ and $1H1V+1H+1V$. We will consider a simpler model to include these two scenarios so as to complete the distinguishability discussion in the four-photon case.

For simplicity, we will again only discuss an unnormalized independent four-photon state of the form

$$|\Phi_4\rangle = \int d\omega_1 d\omega_2 d\omega_3 d\omega_4 \Phi(\omega_1,\omega_2,\omega_3,\omega_4) \times \hat{a}_H^\dagger(\omega_1)\hat{a}_H^\dagger(\omega_2)\hat{a}_V^\dagger(\omega_3)a_V^\dagger(\omega_4)|0\rangle, \quad (101)$$

with $\Phi(\omega_1,\omega_2,\omega_3,\omega_4)$ in the form of Eq. (31):

$$\Phi(\omega_1,\omega_2,\omega_3,\omega_4) = \phi(\omega_1)e^{i\omega_1 T_1}\phi(\omega_2)e^{i\omega_2 T_2} \times \phi(\omega_3)e^{i\omega_3 T_3}\phi(\omega_4)e^{i\omega_4 T_4}. \quad (102)$$

For the NOON state projection measurement with $N=4$, the field operators at the four detectors are related to the input field operators as

$$\hat{E}_0(t) = [\hat{E}_H(t) - \hat{E}_V(t)]/2 + \cdots,$$
$$\hat{E}_1(t) = [\hat{E}_H(t) + \hat{E}_V(t)]/2 + \cdots,$$
$$\hat{E}_2(t) = [\hat{E}_H(t) - i\hat{E}_V(t)]/2 + \cdots,$$
$$\hat{E}_3(t) = [\hat{E}_H(t) + i\hat{E}_V(t)]/2 + \cdots, \quad (103)$$

where we omit the vacuum modes. The four-photon detection probability at the four detectors is related to the following correlation function:

$$\Gamma^{(4)}(t_1,t_2,t_3,t_4) = \langle \hat{E}_0^\dagger(t_4)\hat{E}_1^\dagger(t_3)\hat{E}_2^\dagger(t_2)\hat{E}_3^\dagger(t_1) \times \hat{E}_3(t_1)\hat{E}_2(t_2)\hat{E}_1(t_3)\hat{E}_0(t_4)\rangle. \quad (104)$$

Again, we first calculate $\hat{E}_3(t_1)\hat{E}_2(t_2)\hat{E}_1(t_3)\hat{E}_0(t_4)|\Phi_4\rangle$. For this, we expand $\hat{E}_3(t_1)\hat{E}_2(t_2)\hat{E}_1(t_3)\hat{E}_0(t_4)$:

$$\hat{E}_3(t_1)\hat{E}_2(t_2)\hat{E}_1(t_3)\hat{E}_0(t_4) = [(VVHH - HHVV) + i(VHVH + HVHV) - i(HVVH + VHHV)]/16, \quad (105)$$

where $H=\hat{E}_H$, $V=\hat{E}_V$ and we keep the time ordering. For the state $|\Phi_4\rangle$ in Eq. (101), we have

$$HHVV|\Phi_4\rangle = [G(t_1,t_2,t_3,t_4) + G(t_2,t_1,t_3,t_4) + G(t_1,t_2,t_4,t_3) + G(t_2,t_1,t_4,t_3)]|0\rangle, \quad (106)$$

$$VVHH|\Phi_4\rangle = [G(t_3,t_4,t_1,t_2) + G(t_4,t_3,t_1,t_2) + [G(t_3,t_4,t_2,t_1) + G(t_4,t_3,t_2,t_1)]|0\rangle, \quad (107)$$

$$HVHV|\Phi_4\rangle = [G(t_1,t_3,t_2,t_4) + G(t_3,t_1,t_2,t_4) + G(t_1,t_3,t_4,t_2) + G(t_3,t_1,t_4,t_2)]|0\rangle, \quad (108)$$

$$VHVH|\Phi_4\rangle = [G(t_2,t_4,t_1,t_3) + G(t_2,t_4,t_3,t_1) + G(t_4,t_2,t_1,t_3) + G(t_4,t_2,t_3,t_1)]|0\rangle, \quad (109)$$

$$HVVH|\Phi_4\rangle = [G(t_1,t_4,t_2,t_3) + G(t_1,t_4,t_3,t_2) + G(t_4,t_1,t_2,t_3) + G(t_4,t_1,t_3,t_2)]|0\rangle, \quad (110)$$

$$VHHV|\Phi_4\rangle = [G(t_2,t_3,t_1,t_4) + G(t_2,t_3,t_4,t_1) + G(t_3,t_2,t_1,t_4) + G(t_3,t_2,t_4,t_1)]|0\rangle \quad (111)$$

with

$$G(t_1,t_2,t_3,t_4) = \frac{1}{(2\pi)^2}\int d\omega_1 d\omega_2 d\omega_3 d\omega_4 \Phi(\omega_1,\omega_2,\omega_3,\omega_4) \times e^{-i(\omega_1 t_1+\omega_2 t_2+\omega_3 t_3+\omega_4 t_4)}. \quad (112)$$

For the Φ function given in Eq. (102), the above G function is simply

TABLE I. Visibility for two H photons and two V photons input.

$2H2V$	$2H1V+1V$	$1HV+1HV$	$1HV+H+V$
1	2/3	1/3	1/3

$$G(t_1,t_2,t_3,t_4) = g(t_1 - T_1)g(t_2 - T_2)g(t_3 - T_3)g(t_4 - T_4). \tag{113}$$

Four-photon coincidence probability is proportional to a time integral of the correlation function $\Gamma^{(4)}$:

$$P_4 = \int dt_1 dt_2 dt_3 dt_4 \Gamma^{(4)}(t_1,t_2,t_3,t_4). \tag{114}$$

Next, we will evaluate P_4 for various scenarios of photon distinguishability. To describe the four scenarios discussed in the beginning of this section, we introduce three delay parameters $\Delta T, \Delta T_V, \Delta T_H$ so that $T_2 = T_1 + \Delta T_H, T_3 = T_1 + \Delta T, T_4 = T_3 + \Delta T_V$. Therefore, ΔT is for the delay between the H photons and the V photons and $\Delta T_{H(V)}$ for the delay between the two $H(V)$ photons. When $\Delta T = \pm\infty$, there is no overlap between the H and V photons and no interference occurs. This sets up the baseline for evaluating the visibility of interference. We start with the 4×1 case.

(i) $\Delta T_H = 0 = \Delta T_V$. There is an exchange symmetry between t_1, t_2 and between t_3, t_4 in $G(t_1,t_2,t_3,t_4)$ with

$$G(t_1,t_2,t_3,t_4) = g(t_1 - T_1)g(t_2 - T_1)g(t_3 - T_1 - \Delta T) \times g(t_4 - T_1 - \Delta T). \tag{115}$$

So we have

$$\hat{E}_3(t_1)\hat{E}_2(t_2)\hat{E}_1(t_3)\hat{E}_0(t_4)|\Phi_4\rangle = \frac{1}{4}\{[G(t_1,t_2,t_3,t_4) - G(t_3,t_4,t_1,t_2)] + i[G(t_1,t_3,t_2,t_4) + G(t_2,t_4,t_1,t_3)] - i[G(t_1,t_4,t_3,t_2) + G(t_3,t_2,t_1,t_4)]\}|0\rangle. \tag{116}$$

After the time integral, we obtain

$$P_4(\Delta T) = \frac{1}{8}[3\mathcal{A}_4 - 4\mathcal{E}_4^{(1)}(\Delta T) + \mathcal{E}_4^{(2)}(\Delta T)] \tag{117}$$

with

$$\mathcal{A}_4 = \left(\int d\omega |\phi(\omega)|^2\right)^4, \tag{118}$$

$$\mathcal{E}_4^{(1)}(\tau) = \left(\int d\omega |\phi(\omega)|^2 e^{-i\omega\tau} \int d\omega |\phi(\omega)|^2\right)^2, \tag{119}$$

$$\mathcal{E}_4^{(2)}(\tau) = \left(\int d\omega |\phi(\omega)|^2 e^{-i\omega\tau}\right)^4. \tag{120}$$

Note that $\mathcal{E}_4^{(1)}(0) = \mathcal{E}_4^{(2)}(0) = \mathcal{A}_4$ and $\mathcal{E}_4^{(1)}(\infty) = \mathcal{E}_4^{(2)}(\infty) = 0$. As we scan the relative delay ΔT between the H and V photons, the four-photon coincidence will show an interference dip all the way to zero when $\Delta T = 0$, which corresponds to the case of $T_1 = T_2 = T_3 = T_4$ or the 4×1 case. So the visibility is 100% for the 4×1 case.

TABLE II. Visibility for two H photons and three V photons input.

$2H3V$	$2H2V$ $+V$	$2H1V$ $+2V$	$1H3V$ $+H$	$1H2V$ $+HV$	$1H2V$ $+H+V$	$HV+V$ $+HV$	$HV+V$ $+H+V$
1	5/6	1/2	3/4	5/12	1/2	1/3	1/4

(ii) $\Delta T_H = 0$ but $\Delta T_V \gg 1/\Delta\omega$. In this case, the two V photons are well separated and we have

$$G(t_1,t_2,t_3,t_4) = g(t_1 - T_1)g(t_2 - T_1)g(t_3 - T_1 - \Delta T) \times g(t_4 - T_1 - \Delta T_V - \Delta T). \tag{121}$$

When $\Delta T = 0$, there is an exchange symmetry between $\{t_1, t_2, t_3\}$ in $G(t_1,t_2,t_3,t_4)$. This is the $2H1V+1V$ case. But for arbitrary ΔT, there is only a permutation symmetry between t_1, t_2 in $G(t_1,t_2,t_3,t_4)$. Then we have

$$\hat{E}_3(t_1)\hat{E}_2(t_2)\hat{E}_1(t_3)\hat{E}_0(t_4)|\Phi_4\rangle = \frac{1}{8}\{[G(t_1,t_2,t_3,t_4) + G(t_1,t_2,t_4,t_3) - G(t_3,t_4,t_1,t_2) - G(t_3,t_4,t_2,t_1)] + i[G(t_1,t_3,t_2,t_4) + G(t_1,t_3,t_4,t_2) + G(t_2,t_4,t_1,t_3) + G(t_2,t_4,t_3,t_1)] - i[G(t_1,t_4,t_3,t_2) + G(t_3,t_2,t_1,t_4,) + G(t_1,t_4,t_2,t_3) + G(t_3,t_2,t_4,t_1,)]\}|0\rangle. \tag{122}$$

When $\Delta T = \pm\infty$, there is no overlap between all the terms in Eq. (122) so that all the cross terms are zero after the time integral in Eq. (114). So we have

$$P_4(\Delta T = \pm\infty) = 3\mathcal{A}_4/16. \tag{123}$$

On the other hand, when $\Delta T = 0$, there is an exchange symmetry between $\{t_1, t_2, t_3\}$ in $G(t_1,t_2,t_3,t_4)$. So Eq. (122) becomes

$$\hat{E}_3(t_1)\hat{E}_2(t_2)\hat{E}_1(t_3)\hat{E}_0(t_4)|\Phi_4\rangle = \frac{1}{8}[G(t_1,t_2,t_3,t_4) + G(t_1,t_2,t_4,t_3) - G(t_3,t_4,t_1,t_2) - G(t_3,t_4,t_2,t_1)] \times |0\rangle \tag{124}$$

and there is no overlap between all four terms above. After the time integral, we obtain

$$P_4(\Delta T = 0) = \mathcal{A}_4/16. \tag{125}$$

So the visibility is

$$\mathcal{V}_4(2H1V + 1V) = 2/3. \tag{126}$$

for the $2H1V+1V$ case. In fact, there is another $2H1V+1V$ case when the two H-photons overlaps with the other V photon and $\Delta T = -\Delta T_V$. In this case, we have the exchange symmetry between $\{t_1, t_2, t_4\}$ in $G(t_1,t_2,t_3,t_4)$ so that

TABLE III. Visibility for two H photons and four V photons input.

2H4V	2H3V +V	2H2V +2V	2H1V +3V	1H4V +H	1H3V +HV	1H3V +H+V	1H2V +1H2V	2×HV +2V	1H1V +1H+3V
1	9/10	7/10	2/5	4/5	1/2	3/5	2/5	3/10	1/5

$$P_4(\Delta T = -\Delta T_V) = \mathcal{A}_4/16, \quad (127)$$

which also gives $\mathcal{V}_4(2H1V+1V)=2/3$.

(iii) $\Delta T_H=\Delta T_V\equiv T\gg 1/\Delta\omega$. This is the $1H1V+1H1V$ or the 2×2 case when $\Delta T=0$ and we have the exchange symmetry between $\{t_1,t_3\}$ and between $\{t_2,t_4\}$. But for $\Delta T=\pm\infty$, there is no overlap between any two of the 24 terms in Eqs. (106)–(111). So we have after the time integral in Eq. (114)

$$P_4(\Delta T = \pm\infty) = 24\mathcal{A}_4/16^2 = 3\mathcal{A}_4/32. \quad (128)$$

When $\Delta T=0$, on the other hand, we have

$$\hat{E}_3(t_1)\hat{E}_2(t_2)\hat{E}_1(t_3)\hat{E}_0(t_4)|\Phi_4\rangle = \frac{i}{8}[G(t_1,t_3,t_4,t_2) + G(t_3,t_1,t_2,t_4) - G(t_1,t_4,t_3,t_2) - G(t_4,t_1,t_2,t_3)] \times|0\rangle. \quad (129)$$

The above four terms have no overlap so that we obtain

$$P_4(\Delta T = 0) = 4\mathcal{A}_4/8^2 = \mathcal{A}_4/16. \quad (130)$$

Therefore, the visibility for the 2×2 case is simply

$$\mathcal{V}_4(2\times 2) = 1/3. \quad (131)$$

(iv) $|\Delta T_H-\Delta T_V|\gg 1/\Delta\omega$ and $|\Delta T_H|,|\Delta T_V|\gg 1/\Delta\omega$. As we scan ΔT, there is an exchange symmetry only in one pair of the variables between $\{t_1,t_2\}$ and $\{t_3,t_4\}$, that is, between $\{t_1,t_3\}$ when $\Delta T=0$, or between $\{t_1,t_4\}$ when $\Delta T=-\Delta T_V$, or between $\{t_2,t_3\}$ when $\Delta T=\Delta T_H$, or between $\{t_2,t_4\}$ when $\Delta T=\Delta T_H-\Delta T_V$. This is the $(1H1V+1H+1V)$ case. In all these cases, 8 out of 24 terms in Eqs. (106)–(111) are cancelled in Eq. (105) and the remaining ones are orthogonal to each other so that we have

$$P_4(\Delta T = 0) = 16\mathcal{A}_4/16^2 = \mathcal{A}_4/16. \quad (132)$$

The situation when $\Delta T=\pm\infty$ is same as Eq. (128). Therefore the visibility is

$$\mathcal{V}_4(1H1V + 1H + 1V) = 1/3. \quad (133)$$

for the $1H1V+1H+1V$ case.

These are all likely distinct scenarios. We summarize the visibility in Table I. Although visibility is derived with a specific Φ function in Eq. (102), in general, visibility is the same regardless of the form of Φ as long as it is such that $G(t_1,t_2,t_3,t_4)$ has the required exchange symmetry in each scenario listed above. The intermediate situations will not have any symmetry in $G(t_1,t_2,t_3,t_4)$ and thus have very complicated dependence on the various permutations of $\Phi(\omega_1,\omega_2,\omega_3,\omega_4)$. Reference [15] discussed the intermediate scenario from the 2×2 case to the 4×1 case. Indeed, the visibility depends on the quantity $\mathcal{E}/\mathcal{A}$, which defines the degree of pair distinguishability. Xiang *et al.* [22] realized the 2×2 and the 4×1 cases experimentally and confirmed the visibility in Table I.

B. The special cases of $|2_H,3_V\rangle$, $|2_H,4_V\rangle$, and $|3_H,3_V\rangle$

Following the same line of derivation but in a much more complicated fashion, we may find the visibility for all the scenarios for the input states of $|2_H,3_V\rangle$, $|2_H,4_V\rangle$, and $|3_H,3_V\rangle$. We list the likely scenarios below and tabulate the visibility for each scenarios in Tables II–IV.

1. The case of $|2_H,3_V\rangle$

The case of $|2_H,3_V\rangle$ has eight different scenarios. They are (i) $|2H3V\rangle,|2H2V+1V\rangle,|2H1V+2V\rangle$ and (ii) $|1H3V+1H\rangle$, $|1H2V+1H1V\rangle,|1H2V+1H+1V\rangle$, $|1H1V+1H1V+1V\rangle$, $|1H1V+1H+2V\rangle$. Their visibilities are listed in Table II.

2. The case of $|2_H,4_V\rangle$

The case of $|2_H,4_V\rangle$ has 12 different scenarios. They are (i) $|2H4V\rangle,|2H3V+1V\rangle,|2H2V+2V\rangle,|2H1V+3V\rangle$ and (ii) $|1H4V+1H\rangle,|1H3V+1H1V\rangle,|1H3V+1H+1V\rangle$, $|1H2V+1H2V\rangle$, $|1H2V+1H1V+1V\rangle,|1H2V+1H+2V\rangle$, $|1H1V+1H1V+2V\rangle$, $|1H1V+1H+3V\rangle$.

The scenarios with different visibility are listed in Table III. $|1H2V+1H1V+1V\rangle$ and $|1H2V+1H+2V\rangle$ have the same visibility of 2/5 as $|1H2V+1H2V\rangle$.

In general, they follow the trend that smaller visibility corresponds to less photon overlapping. However, there are exceptions $1H2V+HV$ has less visibility than $1H2V+1H+V$ in Table II and $1H3V+HV$ has less visibility than $1H3V+1H+1V$ in Table III. So the runaway HV does not help when H and V overlap in these cases.

3. The case of $|3_H,3_V\rangle$

There are totally 11 different scenarios in the special case of $|3_H,3_V\rangle$: (i) $|3H3V\rangle,|3H2V+V\rangle,|3H1V+2V\rangle$; (ii) $|2H2V$

TABLE IV. Visibility for three H photons and three V photons input.

3H3V	3H2V +V	3H1V +2V	2H2V +HV	2H2V +H+V	2H1V +1H2V	HV×3	HV×2 +H+V	HV+V +H+H+V
1	9/10	3/5	3/5	7/10	2/5	2/5	3/10	1/5

$+1H1V\rangle, |2H2V+1H+1V\rangle, |2H1V+1H2V\rangle$, $|2H1V+1H1V+1V\rangle, |2H1V+1H+2V\rangle$; and (iii) $|1H1V+1H1V+1H1V\rangle$, $|1H1V+1H1V+1H+1V\rangle$, $|1H1V+1H+1V+1H+1V\rangle$.

In Table IV, we list the visibility for most of the scenarios. $|2H1V+1H1V+1V\rangle$ and $|2H1V+1H+2V\rangle$ have the same visibility of 2/5 as $|2H1V+1H2V\rangle$ and are not listed. As can be seen, anomaly occurs for $|2H2V+1H1V\rangle$ and $|2H2V+1H+1V\rangle$ where visibility is bigger for the case with less photon overlap. The scenarios of $|3H3V\rangle$, $|2H2V+1H1V\rangle$, and $|3\times HV\rangle$ were observed experimentally by Xiang *et al.*. [22] with the corresponding visibility in Table IV.

C. General formula for the visibility

The most general case is when the input state is in the form of $|k_H, N_V\rangle$ with $k \le N$. The most general scenario is when the k H photons do not overlap in time but rather are split into r temporally well separated subgroups with k_j indistinguishable photons in the jth group and $k_1+\cdots+k_r=k$. We also divide the N V photons into $r+1$ subgroups with m_j V photons overlap in time with the jth H photon group. The rest $N-m_1-\cdots-m_r$ V photons are in a separate group by themselves. The wave function for these $N+k$ photons will satisfy the permutation symmetry relation similar to Eq. (26) for the overlapping photons and the orthogonal relation similar to Eq. (27) for the well separated photons.

The derivation of the general formula for the visibility in the $(N+k)$-photon NOON-state projection measurement is very complicated and lengthy. It follows the general line of argument as that leading to Eq. (100). We will present the detailed procedure elsewhere [23] but only give the result as

$$\mathcal{V}_{N+k} = \sum_{l=1}^{k} (-1)^{l-1} \sum_{\substack{i_1 \cdots i_r \\ i_1+\cdots+i_r=l}} \left(\frac{l!}{i_1!\cdots i_r!}\right) \times \frac{C_{k_1}^{i_1}\cdots C_{k_r}^{i_r} m_1^{(i_1)}\cdots m_r^{(i_r)}}{(N+k-1)\cdots(N+k-l)}, \qquad (134)$$

where $m^{(0)}=0=m^{(m)}$, $m^{(i)} \equiv m(m-1)\cdots(m-i+1)$, and $C_N^M \equiv (N+M)!/N!M!$. For the special case of $k=1$, Eq. (134) recovers the expression in Eq. (100). Furthermore, we can easily check that the formula in Eq. (134) indeed leads to the visibility values in Tables I–IV.

VII. CONCLUSION AND DISCUSSION

The complementary principle of quantum interference is demonstrated in a quantitative way in multiphoton interference where photons can be categorized by their temporal distinguishability. The temporal indistinguishability of photons in turn can be characterized by the permutation symmetry in the multiphoton wave function while the temporal distinguishability by the orthogonality of the permuted wave functions. Generalization to other degrees of freedom such as spatial modes is straightforward. Although the above conclusions were made on photons, they should apply to any bosons as well as fermions so long as the occupation number of each mode is less than or equal to one.

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