Pairing fluctuations and the superfluid density through the BCS-BEC crossover

E. Taylor,[1] A. Griffin,[1] N. Fukushima,[2] and Y. Ohashi[2,3]
[1]*Department of Physics, University of Toronto, Toronto, Ontario, M5S 1A7, Canada*
[2]*Institute of Physics, University of Tsukuba, Tsukuba, Ibaraki 305, Japan,*
[3]*Faculty of Science and Technology, Keio University, Hiyoshi, Yokohama, 223, Japan*
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We derive an expression for the superfluid density of a uniform two-component Fermi gas through the BCS-BEC crossover in terms of the thermodynamic potential in the presence of an imposed superfluid flow. Treating the pairing fluctuations in a Gaussian approximation following the approach of Nozières and Schmitt-Rink, we use this definition of ρ_s to obtain an explicit result which is valid at finite temperatures and over the full BCS-BEC crossover. It is crucial that the BCS gap Δ, the chemical potential μ, and ρ_s all include the effect of fluctuations at the same level in a self-consistent manner. We show that the normal fluid density $\rho_n \equiv n - \rho_s$ naturally separates into a sum of contributions from Fermi BCS quasiparticles (ρ_n^F) and Bose collective modes (ρ_n^B). The expression for ρ_n^F is just Landau's formula for a BCS Fermi superfluid but now calculated over the BCS-BEC crossover. The expression for the Bose contribution ρ_n^B is more complicated and only reduces to Landau's formula for a Bose superfluid in the extreme BEC limit, where all the fermions have formed stable Bose pairs and the Bogoliubov excitations of the associated molecular Bose condensate are undamped. In a companion paper, we present numerical calculations of ρ_s using an expression equivalent to the one derived in this paper, over the BCS-BEC crossover, including unitarity, and at finite temperatures.

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I. INTRODUCTION

The superfluid density ρ_s is a fundamental signature in all superfluid systems [1]. It describes the part of the system which does not respond to an external rotation (transverse perturbation). Moreover, it is an essential parameter which enters in the two-fluid hydrodynamics of a superfluid, as first discussed by Landau in 1941 [2]. The superfluid density ρ_s is quite different from the condensate density n_c. In particular, it can be shown that at $T=0$, the entire system is superfluid ($\rho_s=n$), in stark contrast to n_c [3].

In this paper, we define and derive an expression for the superfluid density for a two-component Fermi gas in the BCS-BEC crossover region at finite temperatures. Our analysis is limited to a uniform gas. The calculation of ρ_s is based on the Leggett mean-field BCS model of the BCS-BEC crossover, extended to include the effects of pairing fluctuations associated with the dynamics of the bound states using the approach of Nozières and Schmitt-Rink (NSR) [4]. The NSR approximation has also been used to calculate the thermodynamic properties in the BCS-BEC crossover at both $T=0$ and finite temperatures. As shown in detail by Hu *et al.* [5,6], this approximation gives results that are in good agreement with quantum Monte Carlo calculations [7–9]. Their work gives us confidence in using the NSR approximation to calculate the superfluid density at finite temperatures in the BCS-BEC crossover, apart from a small region just below T_c where the fluctuations require a more careful treatment.

We note that in the superfluid involved with the BCS-BEC crossover, ρ_s will always refer to the number of fermions which participate in the superfluid motion, not the number of Bose pairs. Thus at $T=0$, $\rho_s=n$, where n is the number density of fermions and hence, $m\rho_s$ is the total mass of the system.

In the present paper, we define the superfluid density by imposing a "phase twist" on the Cooper pair order parameter, endowing the Cooper pair condensate with a finite superfluid velocity $\mathbf{v}_s$. Following the approach of Ref. [10], ρ_s is formulated in terms of the second derivative of the thermodynamic potential of the superfluid with respect to v_s. We show that the normal fluid density $\rho_n \equiv n-\rho_s$ naturally separates into a sum of a Fermi quasiparticle contribution arising from the standard BCS static mean-field approximation plus a Bose contribution arising from the dynamics of the pairing order parameter. The latter contribution is treated within a Gaussian approximation for the fluctuations around the static BCS order parameter describing the Bose-condensed pair (Cooper) states [11,12]. We use a single-channel model appropriate for a broad Feshbach resonance [13], which means that one deals with an interacting Fermi gas with a varying s-wave scattering length a_s. Apart from this, our microscopic model is identical to that used in earlier work on the collective modes in the BCS-BEC crossover at finite temperatures [12].

As noted earlier, it is important to keep in mind the distinction between the superfluid density ρ_s and the condensate density n_c, related to the average occupancy of the Cooper pair state. Numerical results for both n_c and ρ_s are presented in our companion paper as a function of both T and a_s [14]. We note that n_c has recently been calculated at $T=0$ in the BCS-BEC crossover using a quantum Monte Carlo simulation [15].

Although such *ab initio* calculations of ρ_s have been done for superfluid ^{4}He as a function of temperature [3,16], a similar calculation of ρ_s in a Fermi superfluid with a Feshbach resonance has only been carried out very recently [9]. Using a restricted path integral Monte Carlo technique, the authors of Ref. [9] calculated the superfluid density at unitarity as a function of temperature in order to determine the superfluid transition temperature. While it is a fundamental property of superfluids, there have been no experimental measurements of ρ_s in the BCS-BEC crossover. As we discuss briefly in the concluding section, ρ_s plays a crucial role in two-fluid hydrodynamics. This collisional domain should be accessible at finite temperatures near unitarity in the BCS-BEC crossover [17].

We also discuss the equivalence of different formal expressions and definitions for the superfluid density within a given microscopic model. We argue that relating the normal fluid density ρ_n to the thermodynamic potential $\Omega(\mathbf{v}_s)$ in the presence of a finite superfluid flow v_s gives a very elegant way of separating out the Fermi BCS quasiparticle contribution ρ_n^F and the Bose collective mode contribution ρ_n^B arising from pairing fluctuations. When expanded out in terms of products of single-particle BCS Green's functions (see Appendix B), our expression for ρ_n^B is extremely complex and not physically transparent.

We show that the Fermi contribution ρ_n^F to the normal fluid density is always given by the well-known Landau formula in terms of Fermi BCS quasiparticle excitations. Only the values of Δ and μ appearing in the energy spectrum of these excitations change as one sweeps through the BCS-BEC crossover. In the BEC limit, the fact that μ is large and negative means that the Fermi quasiparticles are frozen out by a large effective energy gap over the relevant temperature scale $k_BT \sim k_BT_c \ll |\mu|$ and consequently, ρ_n^F becomes negligible.

In contrast, the Bose fluctuation contribution ρ_n^B to the normal fluid density becomes increasingly dominant as we go from the BCS region to the BEC region, where the dynamics of tightly-bound molecules dominate the thermodynamics. Far into the BEC region, the Bose fluctuations reduce to the usual Bogoliubov excitations calculated in the Popov approximation, which allows for a thermal depletion of the condensate density $n_c(T)$. We show in detail how our general expression for the Bose fluctuation contribution ρ_n^B to the normal fluid density reduces in the extreme BEC limit ($|\mu| \gg k_BT_c$) to the expected Landau expression [2] for the normal fluid density in terms of undamped Bogoliubov-Popov excitations. This reduction in the BEC limit has recently been proven by Andrenacci, Pieri, and Strinati [18] based on a direct diagrammatic evaluation of an expression for ρ_n^B defined in terms of a transverse velocity response function [19]. However, we find that there are additional terms in our expression for ρ_n which are not included in the diagrammatic analysis of Ref. [18]. These terms are negligible in the extreme BEC limit of strongly-bound pair states, but become important closer to unitarity where the s-wave scattering length a_s becomes very large.

The present paper concentrates on the formal definition of the superfluid density ρ_s and the derivation of an explicit (but still formal) expression for a specific microscopic model which includes contributions from the Fermi BCS quasiparticles and the Bose pairing fluctuations. We concentrate on the structure of these two contributions to ρ_n and the underlying physics of the pairing fluctuations which give rise to ρ_n^B. A companion paper by the authors [14] presents the results of extensive numerical calculations of our expression for ρ_s, as a function of both the temperature and s-wave scattering length a_s. In such calculations, it is important to use the renormalized values of BCS gap Δ and the Fermi chemical potential μ within a Gaussian approximation that includes the effects of the same pairing fluctuations which describe the Bose collective mode contribution ρ_n^B to the normal fluid density.

II. FORMAL EXPRESSION FOR THE SUPERFLUID DENSITY

Our expression for the superfluid density is based on the equilibrium thermodynamic potential for a current-carrying superfluid. Thus, our starting point is the partition function

$$\mathcal{Z} = \int \mathcal{D}[\psi,\bar{\psi}]e^{-S[\psi,\bar{\psi}]} \tag{1}$$

expressed as a functional integral over fermionic Grassmann fields ψ and $\bar{\psi}$ [20]. The imaginary-time action $S[\psi,\bar{\psi}]$ is given by

$$S[\psi,\bar{\psi}] = \int_0^\beta d\tau\left[\int d\mathbf{r}\sum_\sigma \bar{\psi}_\sigma(x)\partial_\tau\psi_\sigma(x) + H\right], \tag{2}$$

where $\beta=1/k_BT$. Here, we use the notation $x=(\mathbf{r},\tau)$ where $\mathbf{r}$ denotes spatial coordinates and $\tau=it$ is the imaginary time variable. We set $\hbar=1$ throughout. H is the usual BCS pairing Hamiltonian,

$$H = \int d\mathbf{r}\sum_\sigma \bar{\psi}_\sigma(x)\left(\frac{\hat{\mathbf{p}}^2}{2m} - \mu\right)\psi_\sigma(x) - U\int d\mathbf{r}\bar{\psi}_\uparrow(x)\bar{\psi}_\downarrow(x)\psi_\downarrow(x)\psi_\uparrow(x). \tag{3}$$

U is the parameter characterizing the s-wave scattering interaction between fermions in the two different hyperfine states, denoted by the spin indices $\sigma=\uparrow,\downarrow$. From the Lippmann-Schwinger equation for the two-body scattering problem, U is related to the s-wave scattering length a_s by [21]

$$\frac{1}{U} = -\frac{m}{4\pi a_s} + \sum_{\mathbf{k}}(2\varepsilon_{\mathbf{k}})^{-1}, \tag{4}$$

where $\varepsilon_{\mathbf{k}}=\mathbf{k}^2/2m$. Throughout this paper, we take the volume to be unity. Our analysis is restricted to uniform gases.

The Bose pairing field $\Delta(x)$ that includes fluctuations about the mean-field static BCS order parameter Δ is introduced through the Hubbard-Stratonovich transformation,

$$e^{U\int d\tau\int d\mathbf{r}\bar{\psi}_\uparrow\bar{\psi}_\downarrow\psi_\downarrow\psi_\uparrow} = \int \mathcal{D}[\Delta,\Delta^*]\exp\left\{-\int_0^\beta d\tau\int d\mathbf{r}\left[\frac{|\Delta(x)|^2}{U} - (\Delta^*(x)\psi_\downarrow\psi_\uparrow + \Delta(x)\bar{\psi}_\uparrow\bar{\psi}_\downarrow)\right]\right\}. \tag{5}$$

With this identity, the partition function becomes

$$\mathcal{Z}=\int \mathcal{D}[\psi,\bar{\psi}]\mathcal{D}[\Delta,\Delta^*]\exp\left\{-\int_0^\beta d\tau\int d\mathbf{r}\left[\sum_\sigma \bar{\psi}_\sigma(x)\left(\partial_{au}+\frac{\hat{\mathbf{p}}^2}{2m}-\mu\right)\psi_\sigma(x)-\Delta^*(x)\psi_\downarrow\psi_\uparrow-\Delta(x)\bar{\psi}_\uparrow\bar{\psi}_\downarrow+\frac{|\Delta(x)|^2}{U}\right]\right\}$$
$$=\int \mathcal{D}[\psi,\bar{\psi}]\mathcal{D}[\Delta,\Delta^*]\exp\left\{-\int_0^\beta d\tau\int d\mathbf{r}\left[\Psi^\dagger[-\mathbf{G}^{-1}]\Psi+\frac{|\Delta(x)|^2}{U}\right]\right\}, \tag{6}$$

where we have introduced the Nambu spinors

$$\Psi^\dagger=(\bar{\psi}_\uparrow\psi_\downarrow),\quad \Psi=\begin{pmatrix}\psi_\uparrow\\ \bar{\psi}_\downarrow\end{pmatrix}, \tag{7}$$

and $\mathbf{G}^{-1}$ is the inverse of the 2×2 matrix Nambu-Gorkov BCS Green's function,

$$\mathbf{G}^{-1}(x,x')=\begin{pmatrix}-\partial_\tau-\dfrac{\hat{\mathbf{p}}^2}{2m}+\mu & \Delta(x)\\ \Delta^*(x) & -\partial_\tau+\dfrac{\hat{\mathbf{p}}^2}{2m}-\mu\end{pmatrix}\delta(x-x'). \tag{8}$$

The integration over the Grassmann fields ψ in Eq. (6) can be performed in straightforward fashion to give

$$\mathcal{Z}=\int \mathcal{D}[\Delta,\Delta^*]e^{-S_{\text{eff}}}, \tag{9}$$

where [21]

$$S_{\text{eff}}[\Delta,\Delta^*]=\int_0^\beta d\tau\int d\mathbf{r}\frac{|\Delta(x)|^2}{U}-\text{Tr}\ln[-\mathbf{G}^{-1}]. \tag{10}$$

The trace in Eq. (10) is performed over space and imaginary time variables, in addition to the Nambu indices. We have used the standard identity $\ln\text{Det}\,\mathbf{A}=\text{Tr}\ln\mathbf{A}$.

The key function of interest in this paper is the thermodynamic potential Ω, defined by

$$\Omega=-k_BT\ln\mathcal{Z}. \tag{11}$$

All thermodynamic quantities of interest can be calculated once Ω is given in some microscopic approximation. We make use of the fact that ρ_s can also be obtained from the thermodynamic potential of a current-carrying superfluid. To impose a current, one applies a "phase twist" [10] to the order parameter $\Delta(x)$:

$$\Delta(x)\to\Delta(x)e^{i\mathbf{Q}\cdot\mathbf{r}}. \tag{12}$$

The superfluid velocity $\mathbf{v}_s$ associated with this imposed phase twist is

$$\mathbf{v}_s=\frac{\mathbf{Q}}{M}, \tag{13}$$

where $M=2m$ is the Cooper-pair mass. Treating $\mathbf{Q}$ as small, the superfluid density is obtained from the lowest-order change in the free energy of the system ($F=\Omega+\mu N$) due to the added kinetic energy of the imposed superfluid flow [10]. This extra kinetic energy is

$$\Delta F=F(\mathbf{Q})-F(\mathbf{0})\approx\frac{Q^2}{2}\left(\frac{\partial^2F(\mathbf{Q})}{\partial Q^2}\right)_{Q\to 0}\equiv\frac{1}{2}\rho_s m v_s^2, \tag{14}$$

with

$$\rho_s\equiv 4m\left(\frac{\partial^2F(\mathbf{Q})}{\partial Q^2}\right)_{Q\to 0}. \tag{15}$$

Note that the superfluid density defined here is the superfluid number density and not the superfluid mass density used in discussions of two-fluid hydrodynamics. As can be seen from Eq. (14), $\rho_s m$ is the total mass involved in the superfluid flow, with m being the Fermi atom mass.

For a fixed number of fermions N,

$$\left(\frac{\partial^2F}{\partial Q^2}\right)_{Q\to 0}=\frac{\partial^2\Omega}{\partial Q^2}+N\frac{\partial^2\mu}{\partial Q^2}. \tag{16}$$

Microscopically, Ω can be expressed as a functional of the mean-field gap Δ, the chemical potential μ, and the phase twist Q. In addition to an explicit Q dependence, Ω also depends on the phase twist implicitly through the gap $\Delta(Q)$ and the chemical potential $\mu(Q)$. Using these facts, we can write Eq. (16) as follows:

$$\left(\frac{\partial^2F}{\partial Q^2}\right)_{Q\to 0}=\left(\frac{\partial^2\Omega}{\partial Q^2}\right)_{\Delta,\mu}+\left(\frac{\partial\Omega}{\partial\Delta}\right)_\mu\frac{\partial^2\Delta}{\partial Q^2}+\left(\frac{\partial\Omega}{\partial\mu}\right)_\Delta\frac{\partial^2\mu}{\partial Q^2}+N\frac{\partial^2\mu}{\partial Q^2}$$
$$=\left(\frac{\partial^2\Omega}{\partial Q^2}\right)_{\Delta,\mu}+\left(\frac{\partial\Omega}{\partial\Delta}\right)_\mu\left[\frac{\partial^2\Delta}{\partial Q^2}-\left(\frac{\partial\Delta}{\partial\mu}\right)\frac{\partial^2\mu}{\partial Q^2}\right]. \tag{17}$$

In obtaining the last line of Eq. (17), we have made use of the number equation

$$N\equiv-\left(\frac{\partial\Omega}{\partial\mu}\right)=-\left(\frac{\partial\Omega}{\partial\mu}\right)_\Delta-\left(\frac{\partial\Omega}{\partial\Delta}\right)_\mu\left(\frac{\partial\Delta}{\partial\mu}\right). \tag{18}$$

The evaluation of the derivatives at $Q=0$ is left implicit on the right-hand side of Eq. (17) and we have also made use of the fact that the first-order corrections to μ and Δ vanish: $(\partial\mu/\partial Q)_{Q\to 0}=(\partial\Delta/\partial Q)_{Q\to 0}=0$. Separating the mean-field and fluctuation contributions, $\Omega=\Omega_{\text{mf}}+\delta\Omega$, where $\Omega_{\text{mf}}\equiv\Omega(\Delta(x)\to\Delta)$, we can write the derivative of the thermodynamic potential with respect to Δ as follows:

$$\left(\frac{\partial\Omega}{\partial\Delta}\right)_\mu = \left(\frac{\partial\Omega_{\mathrm{mf}}}{\partial\Delta}\right)_\mu + \left(\frac{\partial\delta\Omega}{\partial\Delta}\right)_\mu . \tag{19}$$

The first term in Eq. (19) vanishes since, by definition, $\partial\Omega(\Delta(x)\to\Delta)/\partial\Delta=0$. As argued in Ref. [12], the second term in Eq. (19) is a higher-order correction, beyond the Gaussian theory we use to evaluate Ω in Sec. III. Thus, for the sake of consistency we ignore this contribution and set the second term in Eq. (17) equal to zero. Our final expression for the superfluid density is thus

$$\rho_s = 4m\left(\frac{\partial^2\Omega(\mathbf{Q})}{\partial Q^2}\right)_{Q\to 0} = \frac{1}{m}\left(\frac{\partial^2\Omega(\mathbf{v}_s)}{\partial v_s^2}\right)_{v_s=0} . \tag{20}$$

In Eq. (20) and elsewhere, the constancy of Δ and μ in taking derivatives with respect to Q is left implicit. This formula is the basis for our discussion of ρ_s in this paper.

We note that by ignoring terms proportional to $(\partial\Omega/\partial\Delta)$ in Eqs. (17) and (18), the number equation we use to define the superfluid density reduces to

$$N = -\left(\frac{\partial\Omega}{\partial\mu}\right)_\Delta . \tag{21}$$

This expression keeps Δ fixed, meaning that derivatives of the form $(\partial\Delta/\partial\mu)$ do not enter into the resulting equation for N, in contrast to Eq. (18), which includes higher order corrections. For our calculations to be consistent, the chemical potential used to evaluate our expression for ρ_s must also be calculated using Eq. (21), as done in Refs. [11,12]. The contribution of the higher order term $(\partial\Omega/\partial\Delta)$ to the calculation of equilibrium thermodynamic quantities has been discussed in some recent papers [5,6,22]. In particular, in the context of the BCS-BEC crossover problem, Refs. [5,6] make use of the full number equation given by Eq. (18) to obtain results that are in excellent agreement with Monte Carlo simulations at both $T=0$ [7] and finite T [8]. We defer further remarks on this to Sec. VI. However, it appears from Ref. [6] that this derivative brings in the effect of cubic and quartic fluctuations [23] which have the effect of renormalizing the strength of the effective interaction between stable Cooper pairs [24].

In Appendix A, we review the arguments demonstrating the equivalence of Eq. (20) and the usual definition of ρ_s in terms of the transverse current correlation function [19].

III. THERMODYNAMIC POTENTIAL FOR A CURRENT-CARRYING SUPERFLUID

In order to calculate the thermodynamic potential for a current-carrying superfluid, the phase twist is applied to the order parameter that enters the inverse Green's function $\mathbf{G}^{-1}$ in the action given by Eq. (10). To remove the phase from the order parameter, we apply the unitary transformation $\tilde{\mathbf{G}}^{-1}=\mathbf{U}^{-1}\mathbf{G}^{-1}\mathbf{U}$ [25–27], where

$$\mathbf{U} \equiv \begin{pmatrix} e^{i\mathbf{Q}\cdot\mathbf{r}/2} & 0 \\ 0 & e^{-i\mathbf{Q}\cdot\mathbf{r}/2}\end{pmatrix}. \tag{22}$$

Owing to the invariance of $\mathrm{Tr}\ln[-\mathbf{G}^{-1}]$ with respect to the action of a unitary transformation of $\mathbf{G}^{-1}$, the effective action with a phase-twisted order parameter can be written

$$S_{\mathrm{eff}}[\Delta,\Delta^*,\mathbf{Q}] = \int_0^\beta d\tau \int d\mathbf{r}\frac{|\Delta(x)|^2}{U} - \mathrm{Tr}\ln[-\tilde{\mathbf{G}}^{-1}], \tag{23}$$

where $(\hat{\mathbf{p}}\equiv -i\nabla)$,

$$\tilde{\mathbf{G}}^{-1}(x,x') = \begin{pmatrix} -\partial_\tau - \frac{(\hat{\mathbf{p}}-\mathbf{Q}/2)^2}{2m} + \mu & \Delta(x) \\ \Delta^*(x) & -\partial_\tau + \frac{(\hat{\mathbf{p}}+\mathbf{Q}/2)^2}{2m} - \mu \end{pmatrix} \times \delta(x-x'). \tag{24}$$

The thermodynamic potential for a current-carring superfluid can be evaluated from this action, using Eqs. (9) and (11), once some approximation is introduced so that the functional integration in Eq. (9) can be carried out. Following the standard prescription, we expand the action in powers of fluctuation about the mean-field BCS pairing field: $\Delta(x)=\Delta+\Lambda(x)$; $\tilde{\mathbf{G}}^{-1}=\tilde{\mathbf{G}}_0^{-1}+\Sigma$, where $\tilde{\mathbf{G}}_0^{-1}=\tilde{\mathbf{G}}^{-1}(\Delta(x)\to\Delta)$ and

$$\Sigma = \begin{pmatrix} 0 & \Lambda(x) \\ \bar{\Lambda}(x) & 0\end{pmatrix}\delta(x-x'). \tag{25}$$

Clearly $\Lambda(x)$ corresponds to the fermionic self-energies due to coupling to Bose collective modes involving pair fluctuations in the Cooper pair channel.

Using the expansion

$$\begin{aligned}\mathrm{Tr}\ln[-\tilde{\mathbf{G}}^{-1}] &= \mathrm{Tr}\ln[-\tilde{\mathbf{G}}_0^{-1}(1+\tilde{\mathbf{G}}_0\Sigma)] \\ &= \mathrm{Tr}\ln[-\tilde{\mathbf{G}}_0^{-1}] + \mathrm{Tr}\ln[1+\tilde{\mathbf{G}}_0\Sigma] \\ &= \mathrm{Tr}\ln[-\tilde{\mathbf{G}}_0^{-1}] + \sum_{n=1}\mathrm{Tr}[(\tilde{\mathbf{G}}_0\Sigma)^n](-1)^{n+1}/n,\end{aligned} \tag{26}$$

we expand Eq. (23) up to quadratic order in the Bose fluctuation field Λ to obtain the Gaussian action, $S_{\mathrm{Gauss}}\equiv S^{(0)}+S^{(2)}$. Fourier-transforming, the mean-field $S^{(0)}$ and fluctuation $S^{(2)}$ contributions are given by

$$S^{(0)} = \beta\frac{\Delta^2}{U} - \sum_k \mathrm{tr}\ln[-\tilde{\mathbf{G}}_0^{-1}(k)] \tag{27}$$

and

$$\begin{aligned} S^{(2)} &= \beta\sum_k \frac{|\Lambda_k|^2}{U} + \frac{1}{2}\sum_{k,q}\mathrm{tr}[\tilde{\mathbf{G}}_0(k)\Sigma(-q)\tilde{\mathbf{G}}_0(k+q)\Sigma(q)] \\ &\equiv \frac{1}{2}\sum_q \Lambda^\dagger\tilde{\mathbf{M}}\Lambda. \end{aligned} \tag{28}$$

In Eq. (27), $q\equiv(\mathbf{q},i\nu_m)$ and $k\equiv(\mathbf{k},i\omega_m)$ are 4-vectors denoting the momenta $\mathbf{q}$ and $\mathbf{k}$ as well as the Bose and Fermi Matsubara frequencies $\nu_m=2\pi m/\beta$ and $\omega_n=2\pi(n+1)/\beta$, respectively, where m, n are integers. In momentum-frequency space, the mean-field (denoted by the subscript "0") Nambu-

Gorkov BCS Green's function $\tilde{\mathbf{G}}_0(k)$ for the current-carrying BCS superfluid is defined by its inverse,

$$\tilde{\mathbf{G}}_0^{-1}(k) = \left(i\omega_n - \frac{\mathbf{k}\cdot\mathbf{Q}}{2m}\right) - \left(\xi_{\mathbf{k}} + \frac{Q^2}{8m}\right)\hat{\tau}_3 + \Delta\hat{\tau}_1. \quad (29)$$

Here, $\xi_{\mathbf{k}} \equiv \mathbf{k}^2/2m - \mu$, while $\hat{\tau}_1$, $\hat{\tau}_3$ are Pauli spin matrices. We have assumed that the mean-field order parameter $\Delta = \Delta^*$ is real. In the last line of Eq. (28), we have defined the spinor $\Lambda^\dagger \equiv (\bar{\Lambda}(q), \Lambda(-q))$, and the matrix elements of the inverse 2×2 matrix pair fluctuation propagator $\tilde{\mathbf{M}}$ for a current-carrying superfluid are given by

$$\frac{\tilde{M}_{11}(q)}{\beta} = \frac{\tilde{M}_{22}(-q)}{\beta} = \frac{1}{U} + \frac{1}{\beta}\sum_k \tilde{G}_{0,11}(k+q)\tilde{G}_{0,22}(k) \quad (30)$$

and

$$\frac{\tilde{M}_{12}(q)}{\beta} = \frac{\tilde{M}_{21}(q)}{\beta} = \frac{1}{\beta}\sum_k \tilde{G}_{0,12}(k+q)\tilde{G}_{0,12}(k). \quad (31)$$

Here, $\tilde{G}_{0,ij}$ denotes the ijth element of the matrix mean-field BCS Green's function defined by Eq. (29).

Substituting $S_{\text{eff}} \approx S^{(0)} + S^{(2)}$ into Eq. (9) and performing the Gaussian integration over the Bose fluctuation fields $(\bar{\Lambda}, \Lambda)$, the thermodynamic potential for a current-carrying superfluid reduces to

$$\Omega(\mathbf{Q}) = \frac{\Delta^2}{U} - \frac{1}{\beta}\sum_k \text{tr}\ln[-\tilde{\mathbf{G}}_0^{-1}(k)] + \frac{1}{2\beta}\sum_q \ln\det\tilde{\mathbf{M}}(q)$$
$$\equiv \Omega^F(\mathbf{Q}) + \Omega^B(\mathbf{Q}). \quad (32)$$

This formula will be used to calculate ρ_s in Eq. (20) and thus plays a key role in the rest of this paper. The first two terms of Eq. (31) comprise the mean-field contribution from Fermi BCS quasiparticles,

$$\Omega^F(\mathbf{Q}) = \frac{\Delta^2}{U} - \frac{1}{\beta}\sum_k \text{tr}\ln[-\tilde{\mathbf{G}}_0^{-1}(k)]. \quad (33)$$

We emphasize that the values of Δ and μ in Eq. (29) evaluated using our Gaussian theory are strongly renormalized from their mean-field values by the effects of fluctuations in the Cooper pair field and the associated collective modes, as described by the NSR theory [4,11,12]. The values of these microscopic parameters for a current-carrying superfluid are obtained by self-consistently solving the gap equation, $(\partial S^{(0)}/\partial\Delta)=0$, together with the number equation $N = -(\Omega(\mathbf{Q})/\partial\mu)_\Delta$, where $S^{(0)}$ is given by Eq. (27) and $\Omega(\mathbf{Q})$ is given by Eq. (32). Recall that our expression for the superfluid density, given by Eq. (20), leaves Δ and μ fixed, so we only require the values of these quantities in the current-free state, found from $\partial S^{(0)}(\mathbf{Q}=\mathbf{0})/\partial\Delta = 0$ and $N = -(\Omega(\mathbf{0})/\partial\mu)_\Delta$. Further details of this calculation are given in Ref. [14].

The contribution from the Bose collective modes in Eq. (32) is

$$\Omega^B(\mathbf{Q}) = \frac{1}{2\beta}\sum_{\mathbf{q},i\nu_m} \ln\det\tilde{\mathbf{M}}(\mathbf{q},i\nu_m). \quad (34)$$

The collective modes will be shown to play an increasingly important role in ρ_n as one goes from the BCS to the BEC regimes. The spectrum $\omega_{\mathbf{q}}$ of the collective modes is determined from

$$\det\tilde{\mathbf{M}}(\mathbf{q}, i\nu_m \to \omega_{\mathbf{q}} + i0^+) = \tilde{M}_{11}(q)\tilde{M}_{11}(-q) - \tilde{M}_{12}^2(q) = 0, \quad (35)$$

where $i\nu_m \to \omega_{\mathbf{q}} + i0^+$ denotes the usual analytic continuation from imaginary Bose frequencies. In most of the BCS-BEC crossover, these collective modes will be damped at finite temperatures (i.e., $\omega_{\mathbf{q}}$ has an imaginary part).

IV. SUPERFLUID DENSITY IN THE BCS-BEC CROSSOVER

In this section, we derive an explicit expression for the superfluid density in the crossover starting from the formula given by Eq. (20) for the model defined in Sec. III. From Eqs. (29) and (32), one sees that the thermodynamic potential for a superfluid with a finite superfluid velocity $\mathbf{v}_s = \mathbf{Q}/M$ is equivalent to the thermodynamic potential for a current-free superfluid $(\mathbf{v}_s = \mathbf{0})$, but where the chemical potential and Matsubara frequencies are now Doppler-shifted [26]:

$$\mu \to \mu - Q^2/8m \equiv \tilde{\mu}, \quad (36)$$

$$i\omega_n \to i\omega_n - \mathbf{k}\cdot\mathbf{Q}/2m \equiv i\tilde{\omega}_n, \quad (37)$$

$$i\nu_m \to i\nu_m - \mathbf{q}\cdot\mathbf{Q}/2m \equiv i\tilde{\nu}_m. \quad (38)$$

Equation (36) is more transparent when it is written as an expression for the chemical potential of pairs in the presence of a supercurrent: $\mu_B = 2\tilde{\mu} = 2\mu - Q^2/2M$. The shift in the chemical potential is simply the kinetic energy per Cooper pair, $Q^2/2M$. Considering separately the effects of the shifts to the chemical potential and the Matsubara frequencies, we can write the second-order derivative of Ω with respect to Q (keeping Δ and μ fixed) as follows:

$$\frac{\partial^2\Omega}{\partial Q^2} = \left(\frac{\partial^2\tilde{\mu}}{\partial Q^2}\right)\frac{\partial\Omega}{\partial\tilde{\mu}} + 2\left(\frac{\partial\tilde{\mu}}{\partial Q}\right)\frac{\partial^2\Omega}{\partial\tilde{\mu}\partial Q} + \left(\frac{\partial^2\Omega}{\partial Q^2}\right)_{\tilde{\mu}}$$
$$= -\frac{1}{4m}\frac{\partial\Omega}{\partial\tilde{\mu}} - \frac{Q}{2m}\frac{\partial^2\Omega}{\partial\tilde{\mu}\partial Q} + \left(\frac{\partial^2\Omega}{\partial Q^2}\right)_{\tilde{\mu}}. \quad (39)$$

Evaluated at $Q=0$, the middle term in Eq. (39) vanishes and Eq. (20) reduces to

$$\rho_s = -\left(\frac{\partial\Omega}{\partial\tilde{\mu}}\right)_{Q\to 0} + 4m\left(\frac{\partial^2\Omega}{\partial Q^2}\right)_{\tilde{\mu},Q\to 0} = n + 4m\left(\frac{\partial^2\Omega}{\partial Q^2}\right)_{\tilde{\mu},Q\to 0}. \quad (40)$$

In the last line, we have made use of the number equation $n = -(\partial\Omega/\partial\tilde{\mu})_{\mu,\Delta,Q\to 0} = -(\partial\Omega/\partial\mu)_\Delta$. Since $n \equiv \rho_s + \rho_n$, Eq. (40) gives us the following expression for the normal fluid density:

$$\rho_n = -4m\left(\frac{\partial^2\Omega}{\partial Q^2}\right)_{\tilde{\mu},Q\to 0}. \tag{41}$$

Carrying out the summation over Fermi Matsubara frequencies in Eq. (33), the mean-field BCS quasiparticle contribution to the thermodynamic potential in the presence of a current becomes [28]

$$\Omega^F(\mathbf{Q}) = \frac{\Delta^2}{U} + \sum_{\mathbf{k}}(\tilde{\xi}_{\mathbf{k}} - \tilde{E}_{\mathbf{k}}) - \frac{2}{\beta}\sum_{\mathbf{k}} \ln[1 + e^{-\beta(\mathbf{k}\cdot\mathbf{Q}/2m+\tilde{E}_{\mathbf{k}})}], \tag{42}$$

where the single-particle quasiparticle energies are given by $\tilde{E}_{\mathbf{k}} = \sqrt{\tilde{\xi}_{\mathbf{k}}^2 + \Delta^2}$ with $\xi_{\mathbf{k}} \equiv \mathbf{k}^2/2m - \tilde{\mu}$, where $\tilde{\mu}$ is the Doppler-shifted chemical potential defined in Eq. (36).

Summing over the fermion Matsubara frequencies in Eqs. (30) and (31), the matrix elements of the inverse matrix propagator for pair fluctuations in the current-carrying superfluid are given by

$$\begin{aligned}\frac{\tilde{M}_{11}(q)}{\beta} = \frac{\tilde{M}_{22}(-q)}{\beta} = \frac{1}{U} + \sum_{\mathbf{k}}\Bigg[&(f^+_{\mathbf{k}} - f^-_{\mathbf{k+q}})\frac{v^2_{\mathbf{k}}v^2_{\mathbf{k+q}}}{i\nu_m - \mathbf{q}\cdot\mathbf{Q}/2m + \tilde{E}_{\mathbf{k}} + \tilde{E}_{\mathbf{k+q}}} \\ &+ (f^-_{\mathbf{k}} - f^+_{\mathbf{k+q}})\frac{u^2_{\mathbf{k}}u^2_{\mathbf{k+q}}}{i\nu_m - \mathbf{q}\cdot\mathbf{Q}/2m - \tilde{E}_{\mathbf{k}} - \tilde{E}_{\mathbf{k+q}}} \\ &+ (f^+_{\mathbf{k}} - f^+_{\mathbf{k+q}})\frac{v^2_{\mathbf{k}}u^2_{\mathbf{k+q}}}{i\nu_m - \mathbf{q}\cdot\mathbf{Q}/2m + \tilde{E}_{\mathbf{k}} - \tilde{E}_{\mathbf{k+q}}} \\ &+ (f^-_{\mathbf{k}} - f^-_{\mathbf{k+q}})\frac{u^2_{\mathbf{k}}v^2_{\mathbf{k+q}}}{i\nu_m - \mathbf{q}\cdot\mathbf{Q}/2m - \tilde{E}_{\mathbf{k}} + \tilde{E}_{\mathbf{k+q}}}\Bigg]\end{aligned} \tag{43}$$

and

$$\begin{aligned}\frac{\tilde{M}_{12}(q)}{\beta} = \frac{\tilde{M}_{21}(q)}{\beta} = \sum_{\mathbf{k}}\Bigg[&(f^-_{\mathbf{k+q}} - f^+_{\mathbf{k}})\frac{u_{\mathbf{k}}v_{\mathbf{k}}u_{\mathbf{k+q}}v_{\mathbf{k+q}}}{i\nu_m - \mathbf{q}\cdot\mathbf{Q}/2m + \tilde{E}_{\mathbf{k}} + \tilde{E}_{\mathbf{k+q}}} \\ &+ (f^+_{\mathbf{k+q}} - f^-_{\mathbf{k}})\frac{u_{\mathbf{k}}v_{\mathbf{k}}u_{\mathbf{k+q}}v_{\mathbf{k+q}}}{i\nu_m - \mathbf{q}\cdot\mathbf{Q}/2m - \tilde{E}_{\mathbf{k}} - \tilde{E}_{\mathbf{k+q}}} \\ &+ (f^+_{\mathbf{k}} - f^+_{\mathbf{k+q}})\frac{u_{\mathbf{k}}v_{\mathbf{k}}u_{\mathbf{k+q}}v_{\mathbf{k+q}}}{i\nu_m - \mathbf{q}\cdot\mathbf{Q}/2m + \tilde{E}_{\mathbf{k}} - \tilde{E}_{\mathbf{k+q}}} \\ &+ (f^-_{\mathbf{k}} - f^-_{\mathbf{k+q}})\frac{u_{\mathbf{k}}v_{\mathbf{k}}u_{\mathbf{k+q}}v_{\mathbf{k+q}}}{i\nu_m - \mathbf{q}\cdot\mathbf{Q}/2m - \tilde{E}_{\mathbf{k}} + \tilde{E}_{\mathbf{k+q}}}\Bigg],\end{aligned} \tag{44}$$

where

$$f^{\pm}_{\mathbf{p}} \equiv f(\mathbf{p}\cdot\mathbf{Q}/2m \pm \tilde{E}_{\mathbf{p}}) \tag{45}$$

are the Fermi distribution functions. Here, $u_p = \sqrt{(1+\tilde{\xi}_{\mathbf{p}}/\tilde{E}_{\mathbf{p}})/2}$ and $v_{\mathbf{p}} = \sqrt{(1-\tilde{\xi}_{\mathbf{p}}/\tilde{E}_{\mathbf{p}})/2}$ are the usual Bogoliubov quasiparticle amplitudes. Recall that the normal fluid density is evaluated at fixed $\tilde{\mu}$ and consequently, the dependence of $\tilde{E}_{\mathbf{p}}$ on Q can be ignored for the sake of calculating ρ_n in Eq. (41). The expressions given by Eqs. (43) and (44) reduce to the standard expressions in the literature [11] for $M_{ij}(\mathbf{q}, i\nu_m)$ when $\mathbf{v}_s = \mathbf{0}$.

The distribution functions appearing in Eqs. (43) and (44) involve Doppler-shifted Fermi quasiparticle energies: $\mathbf{p}\cdot\mathbf{Q}/2m \pm \tilde{E}_{\mathbf{p}}$. The shift $\mathbf{p}\cdot\mathbf{Q}/2m$ reflects the fact that additional Fermi quasiparticles will be excited when the superfluid velocity is finite since thermal equilibrium is defined with respect to the stationary lab frame [26].

Using the thermodynamic potential in Eq. (32), the normal fluid density ρ_n is given by the sum of Fermi quasiparticle and Bose collective mode contributions:

$$\rho_n = \rho_n^F + \rho_n^B, \tag{46}$$

where

$$\rho_n^F = -\frac{m}{\beta}\sum_k \left(\frac{\mathbf{k}\cdot\hat{\mathbf{Q}}}{m}\right)^2 \mathrm{tr}[\mathbf{G}_0(k)\mathbf{G}_0(k)] \tag{47}$$

and

$$\rho_n^B = -\frac{2m}{\beta}\sum_q \frac{1}{(\det\tilde{\mathbf{M}})^2}\left[\det\tilde{\mathbf{M}}\left(\frac{\partial^2\det\tilde{\mathbf{M}}}{\partial Q^2}\right)_{\tilde{\mu}} - \left(\frac{\partial\det\tilde{\mathbf{M}}}{\partial Q}\right)^2_{\tilde{\mu}}\right]_{Q\to 0}. \tag{48}$$

Here, $\hat{\mathbf{Q}} = \mathbf{Q}/|\mathbf{Q}|$. The expression for the Bose contribution ρ_n^B is very compactly given in terms of the determinant of the inverse fluctuation propagator $\det\tilde{\mathbf{M}}$, the zeros of which give the spectrum of the Bose collective modes. The simplicity of this expression for ρ_n^B is lost when expanded in terms of products of current-free BCS Green's function (see Appendix B). One can show after a little work that the result given by Eq. (48) is identical to that obtained in Ref. [14] based on a calculation of the current response to a superfluid flow.

The normal fluid density ρ_n^F due to Fermi BCS quasiparticles given in Eq. (47) is readily identified as the long-wavelength, static limit $(q\to 0)$ of the BCS current-current correlation function (multiplied by $-m$). Carrying out the Matsubara frequency sum in the usual way, Eq. (47) reduces to

$$\begin{aligned}\rho_n^F &= -\frac{2}{m}\int\frac{d^3\mathbf{k}}{(2\pi)^3}(\mathbf{k}\cdot\hat{\mathbf{Q}})^2\frac{\partial f(E_{\mathbf{k}})}{\partial E_{\mathbf{k}}} \\ &= \frac{2}{3m}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\mathbf{k}^2\left(-\frac{\partial f(E_{\mathbf{k}})}{\partial E_{\mathbf{k}}}\right).\end{aligned} \tag{49}$$

This is the well-known Landau formula for the normal fluid density of a uniform weak-coupling BCS superfluid, arising from thermally excited Fermi BCS quasiparticles [29]. In our case, it is valid for the entire BCS-BEC crossover, taking into account that the quasiparticle spectrum depends on Δ and μ which are renormalized from their mean-field BCS values by the inclusion of the effects of Bose fluctuations

[6,11,12]. Note that the Landau formula given by Eq. (49) also results by using Eq. (42) in Eq. (41).

In the BCS limit, it is well known (see p. 163 in Ref. [29]) that Eq. (49) gives

$$\rho_n^F = n\left(\frac{2\pi\Delta}{k_BT}\right)^{1/2} e^{-\Delta_0/k_BT} \quad (50)$$

at low temperatures ($k_BT \ll \Delta$). In the strong-coupling BEC limit, however, the fermions form bound pairs with a large binding energy [21]: $E_{\rm binding}=-1/ma_s^2$. As a result of this large binding energy, Fermi quasiparticle excitations, which involve the breakup of pairs, become completely frozen out over the experimentally relevant temperature scale $k_BT \sim k_BT_c \ll |E_{\rm binding}|$. In this region, one can show that Eq. (49) gives

$$\rho_n^F = \frac{1}{3}\left(\frac{mk_BT}{2\pi^3}\right)^{3/2} e^{-|E_{\rm binding}|/2k_BT}. \quad (51)$$

Thus, the Fermi contribution the the normal fluid density clearly vanishes in the BEC limit of tightly bound pairs where $|E_{\rm binding}| \gg k_BT$.

Equation (48) describes the contributions to the normal fluid density from fluctuations $\delta\Delta$ of the Bose pairing field. In general, the Bose pair excitations are damped at finite temperatures, coupling to the continuum of BCS quasiparticle states. As a result, the Bose fluctuations will have a finite lifetime and ρ_n^B will not reduce to the usual Landau formula involving Bose excitations [see Eq. (75)]. In the BEC limit, however, the pair binding energy becomes very large and BCS quasiparticles are strongly suppressed, as also seen in Eq. (51). As a result, damping will not occur. In this limit, we expect that our expression for ρ_n^B will be given by Landau's formula for a Bose superfluid. In the next section, we give the details of this proof.

V. THE NORMAL FLUID DENSITY IN THE BEC LIMIT

Close to unitarity and on the BCS side of the BCS-BEC crossover, Landau damping of the Bose collective modes described in Sec. IV arises due to scattering processes that involve BCS Fermi quasiparticles: $\tilde{\omega}_{\mathbf{q}}+\tilde{E}_{\mathbf{k}}=\tilde{E}_{\mathbf{k+q}}$. When such damping occurs, the collective modes strongly hybridize with BCS Fermi quasiparticles and the concept of well-defined, long-lived Bose excitations breaks down. In this region, we do not expect the normal fluid density ρ_n^B to be given by a Landau formula for the Bose excitations. In the extreme BEC limit, where the Fermi quasiparticles are frozen out, they no longer contribute to Landau damping of the Bose collective modes. Thus, in the BEC limit, the normal fluid is comprised of a gas of well-defined Bose excitations and one expects ρ_n^B will reduce to the usual Landau expression for Bose excitations in this limit [30]. In this section, we show how this result emerges from our formalism (which is valid in the entire BCS-BEC crossover) in the BEC limit.

Deep in the BEC region, the chemical potential becomes increasingly large and negative. In the strong-coupling limit where $\Delta, Q^2/8m, k_BT \ll |\mu|$, the BCS gap equation for the current-carrying superfluid ($Q=Mv_s$),

$$\frac{\Delta}{U} = \frac{1}{\beta}\sum_k \tilde{G}_{0,12}(k), \quad (52)$$

can be solved analytically. This gives $\tilde{\mu} \equiv \mu - Q^2/8m = -1/(2ma_s^2)$, which is one-half the molecular binding energy [21]. When $|\tilde{\mu}| \gg k_BT$, the BCS quasiparticles are frozen out ($f_{\mathbf{p}}^+ \to 0$; $f_{\mathbf{p}}^- \to 1$) and the BCS quasiparticle contribution ρ_n^F, given by Eq. (49), vanishes.

In the low-energy regime $\omega_{\mathbf{q}} \ll |\tilde{\mu}|$, the spectrum of Bose excitations is expected to have the form $\sqrt{(c\mathbf{q})^2+(\mathbf{q}^2/2m^*)^2}$. To extract the contribution of these modes to the normal fluid density ρ_n^B, we set $f^+=0$ and $f^-=1$ in Eqs. (30) and (31) and then expand the inverse fluctuation propagator matrix elements in powers of q. This procedure gives

$$\frac{\tilde{M}_{11}(q)}{\beta} \simeq A + B|\mathbf{q}|^2 + C(i\nu_m - \mathbf{q}\cdot\mathbf{Q}/2m)^2 + D(i\nu_m - \mathbf{q}\cdot\mathbf{Q}/2m), \quad (53)$$

and

$$\frac{\tilde{M}_{12}(q)}{\beta} \simeq A + F|\mathbf{q}|^2 + G(i\nu_m - \mathbf{q}\cdot\mathbf{Q}/2m)^2. \quad (54)$$

We note that outside the BEC region, where $k_BT \sim \mathcal{O}(|\tilde{\mu}|)$, we cannot set $f^+=0$, $f^-=1$. Consequently, the terms in the inverse fluctuation propagator responsible for Landau damping, given by the last two lines in Eq. (43) and Eq. (44), cannot be neglected. In this case, it is well known that one cannot carry out an expansion in powers of $\mathbf{q}$ and $i\tilde{\nu}_m$, as in Eqs. (53) and (54), since these terms are singular in the long wavelength, zero frequency limit [31,32]. This means that the expansions in Eqs. (53) and (54) are not valid in the unitarity or BCS regions.

Apart from the shift to the chemical potential given by Eq. (36), the expansion coefficients in Eqs. (53) and (54) are the same as for the $Q=0$ case given in Ref. [11], namely

$$A = \sum_{\mathbf{k}} \frac{\Delta^2}{4\tilde{E}_{\mathbf{k}}^3}, \quad (55)$$

$$B = \sum_{\mathbf{k}} \left[\left(2 - 3\frac{\Delta^2}{\tilde{E}_{\mathbf{k}}^2}\right)\frac{\tilde{\xi}_{\mathbf{k}}}{m} + \frac{|\mathbf{k}|^2\cos^2\phi}{m^2} \times\left(-2 + 13\frac{\Delta^2}{\tilde{E}_{\mathbf{k}}^2} - 10\frac{\Delta^4}{\tilde{E}_{\mathbf{k}}^4}\right)\right]\frac{1}{16\tilde{E}_{\mathbf{k}}^3}, \quad (56)$$

$$C = \sum_{\mathbf{k}} \left(\frac{\Delta^2}{\tilde{E}_{\mathbf{k}}^2} - 2\right)\frac{1}{16\tilde{E}_{\mathbf{k}}^3}, \quad (57)$$

$$D = -\sum_{\mathbf{k}} \frac{\tilde{\xi}_{\mathbf{k}}}{4\tilde{E}_{\mathbf{k}}^3}, \quad (58)$$

$$F=\sum_{\mathbf{k}}\left[-3\frac{\Delta^2}{\tilde{E}_{\mathbf{k}}^2}\frac{\tilde{\xi}_{\mathbf{k}}}{m}+\frac{|\mathbf{k}|^2\cos^2\phi}{m^2}\left(7\frac{\Delta^2}{\tilde{E}_{\mathbf{k}}^2}-10\frac{\Delta_0^4}{\tilde{E}_{\mathbf{k}}^4}\right)\right]\frac{1}{16\tilde{E}_{\mathbf{k}}^3}, \tag{59}$$

and

$$G=\sum_{\mathbf{k}}\left(\frac{\Delta^2}{\tilde{E}_{\mathbf{k}}^2}\right)\frac{1}{16\tilde{E}_{\mathbf{k}}^3}. \tag{60}$$

We have made use of the gap equation, given by Eq. (52), to eliminate $1/U$ from $(\beta)^{-1}\tilde{M}_{11}(q)$. In the strong-coupling BEC limit, $\Delta \ll |\tilde{\mu}|$, and we can further expand the integrands in powers of $\Delta/|\tilde{\mu}|$. To leading order, using $|\tilde{\mu}|=(2ma_s^2)^{-1}$, we find

$$A \approx \Delta^2\sum_{\mathbf{k}}\frac{1}{4\tilde{\xi}_{\mathbf{k}}^3}=\frac{\Delta^2 a_s^3 m^3}{16\pi}, \tag{61}$$

$$B \approx \sum_{\mathbf{k}}\left[\frac{1}{8m\tilde{\xi}_{\mathbf{k}}^2}-\frac{|\mathbf{k}|^2\cos^2\phi}{4m^2\tilde{\xi}_{\mathbf{k}}^3}\right]=\frac{ma_s}{32\pi}, \tag{62}$$

$$C \approx -\sum_{\mathbf{k}}\frac{1}{8\tilde{\xi}_{\mathbf{k}}^3}=-\frac{m^3a_s^3}{16\pi}, \tag{63}$$

and

$$D \approx -\sum_{\mathbf{k}}\frac{1}{4\tilde{\xi}_{\mathbf{k}}^2}=-\frac{m^2a_s}{8\pi}. \tag{64}$$

To leading order, we find $F\sim\Delta^2 a_s^5$ and $G\sim\Delta^2 a_s^7$, which are vanishingly small in the BEC limit, $a_s\to 0$. Similarly, since $C\propto(ma_s)^3$, we set this coefficient equal to zero as well. However, since $\Delta^2\propto a_s^{-1}$, one finds that $A\propto a_s^2$, and we retain A in Eqs. (53) and (54).

With coefficients given by Eqs. (61), (62), and (64), and setting $C=F=G=0$, we find

$$\det\tilde{\mathbf{M}}(\mathbf{q},i\tilde{\nu}_m)=2AB\mathbf{q}^2+B^2\mathbf{q}^4-D^2\left(i\nu_m-\frac{\mathbf{q}\cdot\mathbf{Q}}{M}\right)^2. \tag{65}$$

Since the fluctuation spectrum is given by the zeros of $\det\tilde{\mathbf{M}}(\mathbf{q},\omega_{\mathbf{q}})$, one finds

$$\omega_{\mathbf{q}}(\mathbf{v}_s)=\mathbf{q}\cdot\mathbf{v}_s+\sqrt{\frac{2AB}{D^2}\mathbf{q}^2+\frac{B^2}{D^2}\mathbf{q}^4}$$
$$=\mathbf{q}\cdot\mathbf{v}_s+\sqrt{\frac{\Delta^2a_s^2}{4}\mathbf{q}^2+\left(\frac{\mathbf{q}^2}{2M}\right)^2}. \tag{66}$$

We note that the value of Δ appearing in this expression is temperature-dependent. Since $\Delta(T)\neq 0$ is associated with the existence of a molecular Bose condensate in the BEC region of interest, the dispersion of Bose collective modes can be written in terms of the condensate density n_c. In Ref. [14], we show that the corrections δn_c to the mean-field expression for the condensate density,

$$n_{c0}(T)=\sum_{\mathbf{k}}\frac{\Delta^2(T)}{4E_{\mathbf{k}}^2}\tanh^2(\beta E_{\mathbf{k}}/2), \tag{67}$$

are negligible throughout the BCS-BEC crossover, within our NSR Gaussian approximation. Thus, we can use Eq. (67) to determine the condensate density in the BEC limit (where $|\mu|\gg k_BT$),

$$n_c(T)=\frac{\Delta^2(T)M^2a_s}{32\pi}. \tag{68}$$

It is important to emphasize that in obtaining this expression, we have only taken the limit $|\mu|/k_BT\to\infty$, where $\tanh^2(\beta E_{\mathbf{k}}/2)\to 1$. However, Δ still has a strong temperature dependence arising from the thermally excited pairing fluctuations which are not frozen out. The temperature dependence of $\Delta(T)$ is calculated within our Gaussian approximation throughout the BCS-BEC crossover in Ref. [14].

Using the result in Eq. (68), one can show that the sound velocity in Eq. (66) can be written

$$c^2\equiv\frac{\Delta(T)a_s^2}{4}=\frac{U_Mn_c(T)}{M} \tag{69}$$

for an interacting gas of bosons of mass $M=2m$. This is the standard Bogoliubov-Popov sound velocity with $U_M=4\pi a_M/M$, but with the molecular scattering length given by the mean-field result $a_M=2a_s$ [21].

In order to get the correct value of the molecular scattering length $a_M\simeq 0.6a_s$ in the BEC limit [24], one would have to include the effects of 4-body correlations which are beyond the 2-body physics contained in our Gaussian theory; i.e., we would need to expand the action to quartic order in fluctuations [23]. As pointed out by Hu *et al.* [6], the correct renormalized value of a_M emerges when one calculates μ using the number equation given in Eq. (18) that includes the contribution from $\partial\Omega/\partial\Delta$. Thus, while we do not consider it in this paper, it appears that we understand how our present calculation can be improved to get the correct value of $a_M\simeq 0.6a_s$.

Using the expression for $\det\tilde{\mathbf{M}}(\mathbf{q},i\tilde{\nu}_m)$ given by Eq. (65), it is straightforward to evaluate ρ_n^B in Eq. (48). Making use of Eq. (66), Eq. (65) reduces to

$$\det\tilde{\mathbf{M}}(\mathbf{q},i\tilde{\nu}_m)=-D^2[(i\nu_m-\mathbf{q}\cdot\mathbf{Q}/M)^2-\omega_{\mathbf{q}}^2(\mathbf{v}_s=\mathbf{0})]. \tag{70}$$

Using this expression, we find

$$\frac{1}{(\det\tilde{\mathbf{M}})^2}\left[\det\tilde{\mathbf{M}}\left(\frac{\partial^2\det\tilde{\mathbf{M}}}{\partial Q^2}\right)_{\tilde{\mu}}-\left(\frac{\partial\det\tilde{\mathbf{M}}}{\partial Q}\right)_{\tilde{\mu}}^2\right]_{Q\to 0}$$
$$=\frac{1}{D^4[(i\nu_m)^2-\omega_{\mathbf{q}}^2]^2}\left\{2D^4[(i\nu_m)^2-\omega_{\mathbf{q}}^2]\left(\frac{\mathbf{q}\cdot\hat{\mathbf{Q}}}{M}\right)^2-4D^4(i\nu_m)^2\left(\frac{\mathbf{q}\cdot\hat{\mathbf{Q}}}{M}\right)^2\right\}, \tag{71}$$

where $\omega_{\mathbf{q}}=\omega_{\mathbf{q}}(Q=0)$ is the usual Bogoliubov-Popov excitation energy in the absence of a superfluid flow, $\mathbf{v}_s=\mathbf{0}$. Using this result in Eq. (48), and recalling that ρ_n^F vanishes in the BEC limit, we obtain

$$\rho_n=\rho_n^B=\frac{M}{\beta}\sum_{\mathbf{q},i\nu_m}\left(\frac{\mathbf{q}\cdot\hat{\mathbf{Q}}}{M}\right)^2\frac{2(i\nu_m)^2+2\omega_{\mathbf{q}}^2}{(i\nu_m-\omega_{\mathbf{q}})^2(i\nu_m+\omega_{\mathbf{q}})^2}. \quad (72)$$

To bring out the physics of Eq. (72), it can also be written in terms of the transverse current correlation function for a dilute Bose gas of interacting molecules [30],

$$\rho_n^B=\frac{M}{\beta}\sum_{\mathbf{q},i\nu_m}\left(\frac{\mathbf{q}\cdot\hat{\mathbf{Q}}}{M}\right)^2\mathrm{tr}[\mathbf{D}(\mathbf{q},i\nu_m)\mathbf{D}(\mathbf{q},i\nu_m)], \quad (73)$$

where

$$\mathbf{D}(\mathbf{q},i\nu_m)=\frac{1}{(i\nu_m)^2-\omega_{\mathbf{q}}^2}\begin{pmatrix} i\nu_m+\omega_{\mathbf{q}} & 0\\ 0 & i\nu_m-\omega_{\mathbf{q}}\end{pmatrix} \quad (74)$$

is the 2×2 Bose propagator describing the Bogoliubov excitations.

Carrying out the Bose frequency sum in Eq. (73) as in Ref. [30], we find the expected result

$$\rho_n^B=-\frac{2}{M}\int\frac{d^3\mathbf{q}}{(2\pi)^3}(\mathbf{q}\cdot\hat{\mathbf{Q}})^2\frac{\partial n_B(\omega_{\mathbf{q}})}{\partial\omega_{\mathbf{q}}}$$
$$=\frac{2}{3M}\int\frac{d^3\mathbf{q}}{(2\pi)^3}\mathbf{q}^2\left(-\frac{\partial n_B(\omega_{\mathbf{q}})}{\partial\omega_{\mathbf{q}}}\right), \quad (75)$$

where $n_B(\omega)=(e^{\beta\omega}-1)^{-1}$ is the Bose distribution function. Equation (75) is precisely Landau's formula for the normal fluid density of a Bose gas described in terms of Bogoliubov excitations [30]. Recall from Sec. I that ρ_s and hence ρ_n always refers to the number of fermions. Thus, Eq. (75) is twice the usual expression [30], reflecting the fact that it is counting the number of fermions (not the number of bosons) involved with a normal fluid composed of Bogoliubov excitations of a molecular BEC. The result given in Eq. (75) for the BEC limit of the BCS-BEC expression has also been derived in Ref. [18] using a diagrammatic approach. This is discussed in Appendix B.

As we have already discussed, retaining terms in Eq. (18) that are proportional to $(\partial\Omega/\partial\Delta)$ leads to the renormalization of the molecular scattering length, from $a_M=2a_s$ to $a_M\simeq0.6a_s$. To be consistent, one must include the analogous terms in Eq. (17) and additional terms will be generated in our definition of the superfluid density given by Eq. (20): $\rho_s\to\rho_s+4m(\partial\Omega/\partial\Delta)_\mu[\partial^2\Delta/\partial Q^2-(\partial\Delta/\partial\mu)\partial^2\mu/\partial Q^2]$. In the extreme BEC limit, however, $\partial^2\Delta/\partial Q^2\to0$ as the BCS quasiparticles become frozen out, and $\partial^2\mu/\partial Q^2\to1/4m$ [as shown below Eq. (52)] so that $\rho_s\to\rho_s+n_{pf,\Delta}$, where $n_{pf,\Delta}\equiv-(\partial\Omega/\partial\Delta)_\mu(\partial\Delta/\partial\mu)$ is the correction to the number equation [6]. Using this new expression in Eq. (40), $n_{pf,\Delta}$ just adds another contribution to the total density n and Eq. (41) remains unchanged in the BEC limit. Thus, even if we retain terms in Eqs. (17) and (18) that lead to the renormalization of the molecular scattering length, in the BEC limit, ρ_n is still given by Eq. (41). Consequently, our major result in Eq. (75) still holds in the BEC limit when we include the higher-order corrections, except that a_M will now be $\simeq0.6a_s$.

For completeness, we write down the pair fluctuation contribution to the thermodynamic potential in Eq. (34) in the BEC limit. Using Eq. (70), the Bose Matsubara frequency sum can be evaluated analytically [28] and we find

$$\Omega^B(\mathbf{Q})=\frac{1}{2}\sum_{\mathbf{q}}\omega_{\mathbf{q}}+\frac{1}{\beta}\sum_{\mathbf{q}}\ln[1-e^{-\beta(\mathbf{q}\cdot\mathbf{Q}/M+\omega_{\mathbf{q}})}]. \quad (76)$$

Inserting Eq. (76) into Eq. (41) also leads to Eq. (75). The integrands in Eq. (75) are strictly only valid at small momenta, such that $\mathbf{q}^2/2M\ll\Delta$, and the divergent zero-point energy contribution to Eq. (76) must be regularized by an appropriate choice of cutoff.

A similar analysis of ρ_n^B in the BCS limit can be carried out. If we limit ourselves to the low temperature region where $k_BT\ll\Delta$, a small q expansion analogous to Eqs. (53) and (54) gives [11]

$$\det\tilde{\mathbf{M}}(\mathbf{q},i\tilde{\nu}_m)\propto\left[(i\nu_m-\mathbf{q}\cdot\mathbf{Q}/M)^2-\left(\frac{v_F\mathbf{q}}{3}\right)^2\right] \quad (77)$$

in place of Eq. (70). We emphasize, however, that the expansion leading to Eq. (77) leaves out terms responsible for Landau damping and consequently, unlike its analogue in the BEC limit given by Eq. (70), this expression is only valid at very low temperatures. Equation (77) is just the propagator for undamped Bogoliubov-Anderson (BA) modes with a Doppler-shifted dispersion $\omega_{\mathbf{q}}(\mathbf{v}_s)=\mathbf{q}\cdot\mathbf{v}_s+(v_F/\sqrt{3})\mathbf{q}$. As a simple approximation, we can neglect the effect of Landau damping on $\omega_{\mathbf{q}}$ at finite temperatures, but introduce a sharp cutoff at 2Δ, where the BA mode enters the two-particle continuum. In this way, we arrive at an expression for ρ_n^B in the BCS limit for BA phonons which is identical to Eq. (75), but with $\omega_{\mathbf{q}}=(v_F/\sqrt{3})\mathbf{q}$ and where the integration is limited to the wave vector region $\omega_{\mathbf{q}}=(v_F/\sqrt{3})\mathbf{q}<2\Delta$. However, as we noted at the beginning of this section, the finite lifetime of Bose collective modes at finite temperature means that a Landau expression like Eq. (75) is never really valid outside the extreme BEC limit, except in the limit of very low temperatures, where Landau damping can be ignored. For BA phonons, Eq. (75) gives (see p. 92 in Ref. [29])

$$\rho_n^B=n\frac{3\sqrt{3}\pi^4}{40}\left(\frac{k_BT}{\epsilon_F}\right)^4. \quad (78)$$

In the BCS region, $(k_BT_c/\epsilon_F)\ll1$, and the normal fluid contribution from the BA phonons is negligible.

VI. CONCLUSIONS

In the present paper, we have derived an explicit formula for the normal fluid density ρ_n in terms of two contributions. One is the expected contribution ρ_n^F given by Eq. (47) arising from Fermi BCS single-particle excitations. In the BEC limit, ρ_n^F vanishes since the effective quasiparticle energy gap becomes very large. Physically, the pair binding energy becomes very large and the Fermi quasiparticles, which are excitations corresponding to the breakup of these pair states, become frozen out.

The most interesting contribution to ρ_n in the BCS-BEC crossover is the contribution ρ_n^B from collective modes associated with the dynamics of the pair states. This is given in our NSR formalism by Eq. (48). Within this Gaussian approximation to the pair fluctuation propagator, as summarized in Eqs. (44) and (44), one can proceed to calculate ρ_n^B numerically. The results of such calculations are discussed in a companion paper [14] over the whole BCS-BEC crossover and as a function of temperature.

The delicate nature of the Cooper pair molecule in the unitarity region of the crossover leads to damping of the collective modes given by Eq. (35). This means that, in general, ρ_n^B is not given by a simple Landau expression such as Eq. (75). However, one does expect such a Landau formula to emerge in the extreme BEC limit where the Cooper pairs become very strongly bound and the system reduces to a weakly interacting Bose gas of stable molecules. In Sec. V, we showed how this expected result emerges naturally from our general formalism.

In deriving our key starting formula for the superfluid density given by Eq. (20), we neglected the contribution of $(\partial\Omega/\partial\Delta)$ in Eqs. (17) and (18). We argued that such a term is a higher order correction which cannot be consistently included in a Gaussian approximation on which our formal analysis and numerical calculations [14] are based. The role of the second term in Eq. (18) has been discussed in recent calculations [5,6,22]. In particular, Hu, Lui, and Drummond [6] have shown that in the BEC limit, this term in the number equation gives rise to a renormalization of the molecular scattering length from the mean-field value $a_M=2a_s$ to the exact result $a_M\simeq 0.6a_s$ [24]. This result is consistent with the calculation of Ohashi [23] who went past our NSR Gaussian pairing fluctuation approximation to include the effects of cubic and quartic fluctuations (for a diagrammatic analysis, see Refs. [33–35]). Ohashi found that these higher order effects lead to a renormalization of the effective interaction between molecules, and obtained a value for a_M close to the result of Petrov *et al.* [24].

We conclude that the neglected contribution to Eq. (18) picks up an important class of fluctuations left out of our Gaussian model, which are precisely those needed to give the correct molecular scattering length in the BEC limit. As one knows from other problems, derivatives of the Gaussian thermodynamic potential can generate results which describe an improved model. This emphasizes the usefulness of calculating ρ_s starting from the result in Eq. (20).

The superfluid density was first introduced by Landau in connection with a two-fluid theory for the collisional hydrodynamics of a Bose superfluid [2]. The form of the Landau two-fluid hydrodynamics is generic for any superfluid with a two-component order parameter (amplitude and phase) [17]. The frequencies of the resulting hydrodynamic modes are given completely in terms of the equilibrium thermodynamic functions, including the superfluid density. The precise values of these equilibrium quantities depend on the nature of the dominant thermal excitations, which can be different in different superfluids. In the BCS-BEC crossover, one goes from the BCS limit, where Fermi BCS quasiparticles dominate the thermodynamics, to the BEC limit where the Bose collective modes (Bogoliubov-Popov excitations) dominate the thermodynamics.

As a result, a careful discussion of the two-fluid collective modes requires a careful analysis of the changing weights of the Fermi and Bose excitations as we pass through the BCS-BEC crossover, both for thermodynamic quantities such as the entropy and compressibility as well as the equilibrium superfluid density. The advantage of calculating ρ_s from the second derivative of the thermodynamic potential $\Omega(\mathbf{v}_s)$ calculated within a Gaussian approximation for the fluctuations, as we do in our work (see also Ref. [14]), is that all other thermodynamic functions can also be determined from $\Omega(\mathbf{v}_s=0)$. Heiselberg [36] has given an informative first study of first and second sound in the BCS-BEC crossover for a uniform gas by calculating ρ_s and other thermodynamic functions in the BEC and BCS limits and interpolating into the unitarity region ($|a_s|\to\infty$). We hope to give a more definitive discussion of first and second sound based on the variational formalism in Ref. [17] using the numerical results for ρ_s given in Ref. [14].

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APPENDIX A: SUPERFLUID DENSITY AND THE TRANSVERSE CURRENT CORRELATION FUNCTION

In this section we review the relationship between our definition of the superfluid density and the transverse response definition commonly evoked in the literature [19]. We start with the partition function expressed in terms of both Bose and Fermi fields, given by Eq. (6). Applying a phase twist to the order parameter as was done in Sec. III, Eq. (6) is written:

$$\mathcal{Z}=\int \mathcal{D}[\bar{\psi},\psi]\mathcal{D}[\Delta^*,\Delta]e^{-S[\bar{\psi},\psi,\Delta^*,\Delta,\mathbf{Q}]}, \tag{A1}$$

where

$$S[\bar{\psi},\psi,\Delta^*,\Delta,\mathbf{Q}]=\int d^4x\left[\Psi^\dagger(x)[-\tilde{\mathbf{G}}^{-1}(x,x')]\Psi(x')+\frac{|\Delta|^2}{U}\right], \tag{A2}$$

and $\mathbf{G}^{-1}(x,x')$ is given by Eq. (24). From Eq. (24), keeping Δ and μ fixed,

$$\frac{\partial \tilde{\mathbf{G}}^{-1}(x,x')}{\partial Q} = \left(\frac{\hat{\mathbf{p}}\cdot\hat{\mathbf{Q}}}{2m} - \frac{Q}{4m}\hat{\tau}_3\right)\delta(x-x') \tag{A3}$$

and

$$\frac{\partial^2 \tilde{\mathbf{G}}^{-1}(x,x')}{\partial Q^2} = -\left(\frac{1}{4m}\hat{\tau}_3\right)\delta(x-x'). \tag{A4}$$

Using Eqs. (A2)–(A4), we obtain the relations

$$\frac{\partial S}{\partial Q} = \int d^4x \Psi^\dagger(x)\left[\left(-\frac{\hat{\mathbf{p}}\cdot\hat{\mathbf{Q}}}{2m} + \frac{Q}{4m}\hat{\tau}_3\right)\delta(x-x')\right]\Psi(x') \tag{A5}$$

and

$$\frac{\partial^2 S}{\partial Q^2} = \int d^4x \Psi^\dagger(x)\left[\left(\frac{1}{4m}\hat{\tau}_3\right)\delta(x-x')\right]\Psi(x'). \tag{A6}$$

We use these expressions to obtain the relation

$$\begin{aligned}\frac{\partial^2}{\partial Q^2}e^{-S}\bigg|_{Q\to 0} &= \left[\left(\frac{\partial S}{\partial Q}\right)^2 - \frac{\partial^2 S}{\partial Q^2}\right]e^{-S}\bigg|_{Q=0}\\ &= -\left[\frac{1}{4m}\int d^4x\Psi^\dagger\hat{\tau}_3\Psi - \int d^4x d^4x'\left(\Psi^\dagger(x)\frac{\hat{\mathbf{p}}\cdot\hat{\mathbf{Q}}}{2m}\Psi(x)\right)\left(\Psi^\dagger(x')\frac{\hat{\mathbf{p}}'\cdot\hat{\mathbf{Q}}}{2m}\Psi(x')\right)\right]e^{-S}\\ &= -\left(\frac{\beta}{4m}\hat{N} - \frac{1}{4}\int d^4x d^4x' \hat{j}_z(x)\hat{j}_z(x')\right)e^{-S},\end{aligned} \tag{A7}$$

where

$$\hat{N} = \int d\mathbf{r}\sum_\sigma \bar{\Psi}_\sigma(\mathbf{r})\Psi_\sigma(\mathbf{r}) \tag{A8}$$

is the number operator, and, having arbitrarily chosen $\hat{\mathbf{Q}}=\hat{\mathbf{z}}$,

$$\hat{j}_z = \frac{1}{2mi}\sum_\sigma\left(\bar{\Psi}_\sigma\left(\frac{\partial}{\partial z}\Psi_\sigma\right) - \left(\frac{\partial}{\partial z}\bar{\Psi}_\sigma\right)\Psi_\sigma\right) \tag{A9}$$

is the z component of the current density operator.

Using Eq. (A7) along with the thermodynamic potential in the presence of a superfluid flow,

$$\Omega(\mathbf{Q}) = -k_BT\ln\int \mathcal{D}[\bar{\psi},\psi]\mathcal{D}[\Delta^*,\Delta]e^{-S[\bar{\psi},\psi,\Delta^*,\Delta,\mathbf{Q}]}, \tag{A10}$$

we obtain

$$\begin{aligned}\frac{\partial^2\Omega}{\partial Q^2}\bigg|_{Q\to 0} &= -\frac{k_BT}{\mathcal{Z}}\int \mathcal{D}[\bar{\psi},\psi]\mathcal{D}[\Delta^*,\Delta]\ \frac{\partial^2}{\partial Q^2}e^{-S}\bigg|_{Q\to 0} - k_BT\left(\frac{1}{\mathcal{Z}}\int\mathcal{D}[\bar{\psi},\psi]\mathcal{D}[\Delta^*,\Delta]\ \left(\frac{\partial S}{\partial Q}\right)e^{-S}\bigg|_{Q\to 0}\right)^2\\ &= \frac{1}{\mathcal{Z}_0}\int\mathcal{D}[\bar{\psi},\psi]\mathcal{D}[\Delta^*,\Delta]\left(\frac{1}{4m}\hat{N} - \frac{1}{4\beta}\int d^4x d^4x'\hat{j}_z(x)\hat{j}_z(x')\right)e^{-S}\\ &= \frac{N}{4m} - \frac{1}{4}\langle\hat{J}_z\hat{J}_z\rangle_0,\end{aligned} \tag{A11}$$

where $N=\langle\hat{N}\rangle_0$ and $\hat{J}_z\equiv\beta^{-1/2}\int d^4x\hat{j}_z$. The expectation value with respect to the current-free state is defined as

$$\langle\cdots\rangle_0 \equiv \frac{1}{\mathcal{Z}_0}\int\mathcal{D}[\bar{\psi},\psi]\mathcal{D}[\Delta^*,\Delta](\cdots)e^{-S[\bar{\psi},\psi,\Delta^*,\Delta,\mathbf{Q}=0]}, \tag{A12}$$

and

$$\mathcal{Z}_0 = \int\mathcal{D}[\bar{\psi},\psi]\mathcal{D}[\Delta^*,\Delta]e^{-S[\bar{\psi},\psi,\Delta^*,\Delta,\mathbf{Q}=0]}. \tag{A13}$$

Note that the second term in the first line in Eq. (A11) vanishes by symmetry, i.e., $\langle\hat{J}_z\rangle_0=0$.

Comparing Eq. (A11) with Eq. (20), we obtain the result

$$\rho_s = n - m\langle \hat{J}_z \hat{J}_z \rangle_0, \tag{A14}$$

which identifies [19]

$$\rho_n = m\langle \hat{J}_z \hat{J}_z \rangle_0 \tag{A15}$$

as the normal fluid density.

APPENDIX B: NORMAL FLUID CONTRIBUTION FROM THE BOSE FLUCTUATIONS

A central result of our paper is the formal expression in Eq. (48) for the Bose fluctuation contribution to the normal fluid density, where the $\tilde{\mathbf{M}}$ matrix elements are given in Eqs. (43) and (44). In this appendix, we "unpack" this formal result for ρ_n^B to give it more explicitly in terms of single-particle Nambu-Gorkov Green's functions of a *current-free* BCS superfluid. This will allow us to compare with other results in the literature [18].

Expanding the Q derivatives in Eq. (48), the Bose fluctuation contribution to ρ_n is given by

$$\begin{aligned}\rho_n^B = \frac{2m}{\beta}\sum_q \frac{1}{(M_{11}M_{22}-M_{12}M_{12})^2}\Bigg[& M_{11}M_{11}\frac{\partial\tilde{M}_{22}}{\partial Q}\frac{\partial\tilde{M}_{22}}{\partial Q} - 4M_{11}M_{12}\frac{\partial\tilde{M}_{22}}{\partial Q}\frac{\partial\tilde{M}_{12}}{\partial Q} + 2M_{11}M_{22}\frac{\partial\tilde{M}_{12}}{\partial Q}\frac{\partial\tilde{M}_{12}}{\partial Q} \\ & + 2M_{12}M_{12}\frac{\partial\tilde{M}_{11}}{\partial Q}\frac{\partial\tilde{M}_{22}}{\partial Q} + 2M_{12}M_{12}\frac{\partial\tilde{M}_{12}}{\partial Q}\frac{\partial\tilde{M}_{12}}{\partial Q} - 4M_{22}M_{12}\frac{\partial\tilde{M}_{11}}{\partial Q}\frac{\partial\tilde{M}_{12}}{\partial Q} + M_{22}M_{22}\frac{\partial\tilde{M}_{11}}{\partial Q}\frac{\partial\tilde{M}_{11}}{\partial Q} \\ & - (M_{11}M_{22}-M_{12}M_{12})\left(M_{22}\frac{\partial^2\tilde{M}_{11}}{\partial Q^2} + M_{11}\frac{\partial^2\tilde{M}_{22}}{\partial Q^2} - 2M_{12}\frac{\partial^2\tilde{M}_{12}}{\partial Q^2}\right)\Bigg]_{\tilde{\mu},Q\to 0}. \end{aligned} \tag{B1}$$

Here, $M_{ij}=\tilde{M}_{ij}(\mathbf{Q}=\mathbf{0})$. To express Eq. (B1) in terms of current-free Green's functions, we use the identities

$$\left.\frac{\partial\tilde{G}_{0,11}(p)}{\partial Q}\right|_{\tilde{\mu},Q\to 0} = \left(\frac{\mathbf{p}\cdot\hat{\mathbf{Q}}}{2m}\right)[G_{0,11}(p)G_{0,11}(p)+G_{0,12}(p)G_{0,12}(p)], \tag{B2}$$

$$\left.\frac{\partial\tilde{G}_{0,12}(p)}{\partial Q}\right|_{\tilde{\mu},Q\to 0} = \left(\frac{\mathbf{p}\cdot\hat{\mathbf{Q}}}{2m}\right)[G_{0,12}(p)G_{0,11}(p)+G_{0,12}(p)G_{0,22}(p)], \tag{B3}$$

$$\left.\frac{\partial\tilde{G}_{0,22}(p)}{\partial Q}\right|_{\tilde{\mu},Q\to 0} = \left(\frac{\mathbf{p}\cdot\hat{\mathbf{Q}}}{2m}\right)[G_{0,22}(p)G_{0,22}(p)+G_{0,12}(p)G_{0,12}(p)], \tag{B4}$$

$$\left.\frac{\partial^2\tilde{G}_{0,11}(p)}{\partial Q^2}\right|_{\tilde{\mu},Q\to 0} = 2\left(\frac{\mathbf{p}\cdot\hat{\mathbf{Q}}}{2m}\right)^2[G_{0,11}(p)(G_{0,11}^2(p)+G_{0,12}^2(p))+G_{0,12}^2(p)(G_{0,11}(p)+G_{0,22}(p))], \tag{B5}$$

$$\left.\frac{\partial^2\tilde{G}_{0,12}(p)}{\partial Q^2}\right|_{\tilde{\mu},Q\to 0} = 2\left(\frac{\mathbf{p}\cdot\hat{\mathbf{Q}}}{2m}\right)^2 G_{0,12}(p)[G_{0,11}^2(p)+G_{0,22}^2(p)+G_{0,12}^2(p)+G_{0,11}(p)G_{0,22}(p)], \tag{B6}$$

and

$$\left.\frac{\partial^2\tilde{G}_{0,22}(p)}{\partial Q^2}\right|_{\tilde{\mu},Q\to 0} = 2\left(\frac{\mathbf{p}\cdot\hat{\mathbf{Q}}}{2m}\right)^2[G_{0,22}(p)(G_{0,22}^2(p)+G_{0,12}^2(p))+G_{0,12}^2(p)(G_{0,11}(p)+G_{0,22}(p))]. \tag{B7}$$

With these identities and substituting the BCS gap equation given by Eq. (52) into Eq. (30), we find

$$\left.\frac{\partial\tilde{M}_{11}(q)}{\partial Q}\right|_{\tilde{\mu},Q\to 0} = \left.\frac{\partial\tilde{M}^{22}(-q)}{\partial Q}\right|_{\tilde{\mu},Q\to 0} = \sum_k\left(\frac{\mathbf{k}\cdot\hat{\mathbf{Q}}+\mathbf{q}\cdot\hat{\mathbf{Q}}}{m}\right)[G_{0,11}(k+q)G_{0,11}(k+q)+G_{0,12}(k+q)G_{0,12}(k+q)]G_{0,22}(k), \tag{B8}$$

$$\left.\frac{\partial\tilde{M}_{12}(q)}{\partial Q}\right|_{\tilde{\mu},Q\to 0} = \sum_k\left(\frac{\mathbf{k}\cdot\hat{\mathbf{Q}}+\mathbf{q}\cdot\hat{\mathbf{Q}}}{m}\right)[G_{0,12}(k+q)G_{0,11}(k+q)+G_{0,12}(k+q)G_{0,22}(k+q)]G_{0,12}(k), \tag{B9}$$

$$\left.\frac{\partial^2\tilde{M}_{11}(q)}{\partial Q^2}\right|_{\tilde{\mu},Q\to0} = \left.\frac{\partial^2\tilde{M}_{22}(-q)}{\partial Q^2}\right|_{\tilde{\mu},Q\to0} = 2\sum_k\left(\frac{\mathbf{k}\cdot\hat{\mathbf{Q}}}{2m}\right)^2\frac{1}{\Delta}\{G_{0,11}(k)G_{0,12}(k)[G_{0,11}(k)+G_{0,22}(k)]+G_{0,12}(k)[G_{0,12}(k)G_{0,12}(k)+G_{0,22}(k)G_{0,22}(k)]\}+2\sum_k\left\{2\left(\frac{(\mathbf{k}\cdot\hat{\mathbf{Q}}+\mathbf{q}\cdot\hat{\mathbf{Q}})}{2m}\right)^2\Big[G_{0,22}(k)G_{0,11}(k+q)[G_{0,11}(k+q)G_{0,11}(k+q)+G_{0,12}(k+q)G_{0,12}(k+q)]+G_{0,22}(k)G_{0,12}(k+q)G_{0,12}(k+q)\times[G_{0,11}(k+q)+G_{0,22}(k+q)]\Big]+\frac{(\mathbf{k}\cdot\hat{\mathbf{Q}}+\mathbf{q}\cdot\hat{\mathbf{Q}})(\mathbf{k}\cdot\hat{\mathbf{Q}})}{2m^2}[G_{0,11}(k+q)G_{0,11}(k+q)+G_{0,12}(k+q)G_{0,12}(k+q)][G_{0,22}(k)G_{0,22}(k)+G_{0,12}(k)G_{0,12}(k)]\right\}, \tag{B10}$$

and

$$\left.\frac{\partial^2\tilde{M}_{12}(q)}{\partial Q^2}\right|_{\tilde{\mu},Q\to0} = +2\sum_k\left\{2\left(\frac{(\mathbf{k}\cdot\hat{\mathbf{Q}}+\mathbf{q}\cdot\hat{\mathbf{Q}})}{2m}\right)^2 G_{0,12}(k)G_{0,12}(k+q)[G_{0,11}(k+q)G_{0,11}(k+q)+G_{0,22}(k+q)G_{0,22}(k+q)+G_{0,12}(k+q)G_{0,12}(k+q)+G_{0,11}(k+q)G_{0,22}(k+q)]+\frac{(\mathbf{k}\cdot\hat{\mathbf{Q}}+\mathbf{q}\cdot\hat{\mathbf{Q}})(\mathbf{k}\cdot\hat{\mathbf{Q}})}{2m^2}G_{0,12}(k+q)[G_{0,11}(k+q)+G_{0,22}(k+q)]G_{0,12}(k)[G_{0,11}(k)+G_{0,22}(k)]\right\}. \tag{B11}$$

In deriving Eqs. (B8)–(B11), we have made use of the following identities for BCS Green's functions:

$$G_{0,11}(-k) = -G_{0,22}(k) \tag{B12}$$

and

$$G_{0,12}(-k) = G_{0,12}(k). \tag{B13}$$

Taken together, Eqs. (B1) and (B8)–(B11) give an explicit expression for the normal fluid density due to Bose fluctuations in terms of products of BCS Green's functions.

It is of interest to relate our result for ρ_n^B to that obtained recently by Andrenacci *et al.* [18], who used a direct diagrammatic evaluation of the response function definition of the normal fluid density (see Appendix A). Equations (14) and (25) in Ref. [18] give the following expression for the fluctuation contribution to the normal fluid density:

$$\rho_{n,AL}^B = -m\chi_{z,z}^{AL}(Q=0), \tag{B14}$$

where

$$\chi_{z,z}^{AL}(Q=0) = -\frac{1}{(2m)^2}\frac{1}{\beta^3}\sum_{k,k',q}\sum_{i,i',i^{me\prime}}\sum_{j,j',j''}(2k_z+2q_z)(2k_z'+2q_z)\Gamma_{j',i'}(q)\Gamma(q)_{i'',j'e}(q)\times G_{0,ii'}(k+q)G_{0,i''i}(k+q)G_{0,i''i'}(-k)G_{0,j'j}(k'+q)G_{0,jj''}(k'+q)G_{0,j'j''}(-k'). \tag{B15}$$

This result is based on the Aslamazov-Larkin-type (AL) diagrammatic contributions to the transverse current correlation function $\chi_{z,z}$. Using our notation as defined in the text of this paper, the vertex functions in Eq. (B15) are

$$\Gamma_{11}(q) = \Gamma_{22}(-q) = \beta\frac{M_{22}(q)}{\det\mathbf{M}}, \tag{B16}$$

and

$$\Gamma_{12}(q) = \Gamma_{21}(q) = \beta\frac{M_{12}(q)}{\det\mathbf{M}}. \tag{B17}$$

To facilitate comparison with our results for ρ_n^B, we expand Eqs. (B14) and (B15) to give

$$\rho^B_{n,AL} = \frac{1}{m\beta^3}\sum_{k,k',q}(k_z+q_z)(k'_z+q_z)\{\Gamma_{11}(q)\Gamma_{11}(q)[G_{0,11}(k+q)G_{0,11}(k+q)+G_{0,12}(k+q)G_{0,12}(k+q)]G_{0,11}(-k)$$
$$\times[G_{0,11}(k'+q)G_{0,11}(k'+q)+G_{0,12}(k'+q)G_{0,12}(k'+q)]G_{0,11}(-k')+4\Gamma_{11}(q)\Gamma_{12}(q)[G_{0,11}(k+q)G_{0,11}(k+q)$$
$$+G_{0,12}(k+q)G_{0,12}(k+q)]G_{0,11}(-k)[G_{0,11}(k'+q)G_{0,12}(k'+q)+G_{0,22}(k'+q)G_{0,12}(k'+q)]G_{0,12}(-k')$$
$$+2\Gamma_{11}(q)\Gamma_{22}(q)[G_{0,11}(k+q)G_{0,12}(k+q)+G_{0,22}(k+q)G_{0,12}(k+q)]G_{0,12}(-k)[G_{0,11}(k'+q)G_{0,12}(k'+q)$$
$$+G_{0,22}(k'+q)G_{0,12}(k'+q)]G_{0,12}(-k')+2\Gamma_{12}(q)\Gamma_{12}(q)[G_{0,11}(k+q)G_{0,12}(k+q)$$
$$+G_{0,22}(k+q)G_{0,12}(k+q)]G_{0,12}(-k)[G_{0,11}(k'+q)G_{0,12}(k'+q)+G_{0,22}(k'+q)G_{0,12}(k'+q)]G_{0,12}(-k')$$
$$+2\Gamma_{12}(q)\Gamma_{12}(q)[G_{0,22}(k+q)G_{0,22}(k+q)+G_{0,12}(k+q)G_{0,12}(k+q)]G_{0,22}(-k)[G_{0,11}(k'+q)G_{0,11}(k'+q)$$
$$+G_{0,12}(k'+q)G_{0,12}(k'+q)]G_{0,11}(-k')+4\Gamma_{12}(q)\Gamma_{22}(q)[G_{0,22}(k+q)G_{0,22}(k+q)$$
$$+G_{0,12}(k+q)G_{0,12}(k+q)]G_{0,22}(-k)[G_{0,11}(k'+q)G_{0,12}(k'+q)+G_{0,22}(k'+q)G_{0,12}(k'+q)]G_{0,12}(-k')$$
$$+\Gamma_{22}(q)\Gamma_{22}(q)[G_{0,22}(k+q)G_{0,22}(k+q)+G_{0,12}(k+q)G_{0,12}(k+q)]G_{0,22}(-k)$$
$$\times[G_{0,22}(k'+q)G_{0,22}(k'+q)+G_{0,12}(k'+q)G_{0,12}(k'+q)]G_{0,22}(-k')\}. \tag{B18}$$

Comparing with our results in Eqs. (B1), (B8), (B9), and (B9)–(B11), it is apparent that the expression for $\rho^B_{n,AL}$ in Eq. (B18) does not contain terms analogous to those in Eq. (B1) arising from second-order derivatives of $\tilde{M}_{ij}$ with respect to Q. Only the terms in Eq. (B1) that involve products of first-order derivatives of $\tilde{M}_{ij}$ correspond to the AL terms in Eq. (B18). Separating the contributions from first- and second-order derivatives with respect to Q in Eq. (B1), we use Eqs. (B16) and (B17) to rewrite Eq. (B1) as follows:

$$\rho^B_n = \frac{2m}{\beta^3}\sum_q\Bigg[\Gamma_{11}(q)\Gamma_{11}(q)\frac{\partial\tilde{M}_{11}}{\partial Q}\frac{\partial\tilde{M}_{11}}{\partial Q}-4\Gamma_{11}(q)\Gamma_{12}(q)\frac{\partial\tilde{M}_{11}}{\partial Q}\frac{\partial\tilde{M}_{12}}{\partial Q}+2\Gamma_{11}(q)\Gamma_{22}(q)\frac{\partial\tilde{M}_{12}}{\partial Q}\frac{\partial\tilde{M}_{12}}{\partial Q}+2\Gamma_{12}(q)\Gamma_{12}(q)\frac{\partial\tilde{M}_{11}}{\partial Q}\frac{\partial\tilde{M}_{22}}{\partial Q}$$
$$+2\Gamma_{12}(q)\Gamma_{12}(q)\frac{\partial\tilde{M}_{12}}{\partial Q}\frac{\partial\tilde{M}_{12}}{\partial Q}-4\Gamma_{12}(q)\Gamma_{22}(q)\frac{\partial\tilde{M}_{22}}{\partial Q}\frac{\partial\tilde{M}_{12}}{\partial Q}+\Gamma_{22}(q)\Gamma_{22}(q)\frac{\partial\tilde{M}_{22}}{\partial Q}\frac{\partial\tilde{M}_{22}}{\partial Q}\Bigg]$$
$$-\frac{2m^2}{\beta}\sum_q\left[\frac{1}{(M_{11}M_{22}-M_{12}M_{12})}\left(M_{22}\frac{\partial^2\tilde{M}_{11}}{\partial Q^2}+M_{11}\frac{\partial^2\tilde{M}_{22}}{\partial Q^2}-2M_{12}\frac{\partial^2\tilde{M}_{12}}{\partial Q^2}\right)\right]_{\tilde{\mu},Q\to 0}. \tag{B19}$$

Using Eqs. (B8) and (B9), as well as Eqs. (B12) and (B13), and taking $\hat{\mathbf{Q}}=\hat{\mathbf{z}}$, one can show that the terms in Eq. (B19) that involve a product of first-order Q derivatives of the matrix elements $\tilde{M}_{ij}$ are precisely equal to twice the AL expression for $\rho^B_{n,AL}$ given by Eq. (B18). To summarize, we have shown that the expression in Eq. (B1) reduces to Eq. (B19). This is equivalent to

$$\rho^B_n = 2\rho^B_{n,AL}-\frac{2m}{\beta}\sum_q\left[\frac{1}{(M_{11}M_{22}-M_{12}M_{12})}\left(M_{22}\frac{\partial^2\tilde{M}_{11}}{\partial Q^2}+M_{11}\frac{\partial^2\tilde{M}_{22}}{\partial Q^2}-2M_{12}\frac{\partial^2\tilde{M}_{12}}{\partial Q^2}\right)\right]_{\tilde{\mu},Q\to 0}, \tag{B20}$$

where $\rho^B_{n,AL}$ is given by Eq. (B18).

In Ref. [18], it is shown that the normal fluid density is indeed twice the AL contribution given by Eq. (B14), owing to an additional AL-type diagram that is topologically non-equivalent to the diagram that gives rise to Eq. (B15) (see Fig. 2 in Ref. [18]). This extra diagram gives rise to a contribution to the transverse current correlation function that is equal to Eq. (B15) for the case of a contact interaction potential. Hence, the final result obtained in Ref. [18] for the Bose contribution to the transverse current correlation function in the BCS-BEC crossover is given by $\rho^B_n=2\rho^B_{n,AL}$.

However, we see from Eq. (B20) that in addition to the AL-type contribution, our expression for the normal fluid density includes terms which arise from second-order derivatives of the matrix elements of the inverse Gaussian fluctuation propagator with respect to the superfluid velocity $Q=Mv_s$. In the BEC limit, one finds that these contributions vanish, which explains why Ref. [18] also obtains the Landau formula in Eq. (75) in the BEC limit. However, the extra terms in Eq. (B20) are important in the unitarity and BCS regions. Our numerical results for ρ_n which are discussed in Ref. [14] include the contributions from all terms in Eq. (B20). These extra terms are given explicitly by Eqs. (B10) and (B11). It would be interesting to understand which diagrams give rise to these extra contributions.

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