Distribution of *G* concurrence of random pure states

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The average entanglement of random pure states of an $N\times N$ composite system is analyzed. We compute the average value of the determinant D of the reduced state, which forms an entanglement monotone. Calculating higher moments of the determinant, we characterize the probability distribution $P(D)$. Similar results are obtained for the rescaled Nth root of the determinant, called the G concurrence. We show that in the limit $N\to\infty$ this quantity becomes concentrated at a single point $G_\star=1/e$. The position of the concentration point changes if one consider an arbitrary $N\times K$ bipartite system, in the joint limit $N,K\to\infty$, with K/N fixed.

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I. INTRODUCTION

In designing protocols of quantum-information processing one usually deals with some particular initial states. One is then interested in describing the evolution of such a concrete quantum state and its properties in time. For instance, one studies the time dependence of the degree of quantum entanglement, which characterizes the non classical correlations between subsystems and is treated as a crucial resource in the theory of quantum information [1].

As a reference point one may compare the degree of entanglement of the analyzed state with analogous properties of a typical, random state. Such random states are also of direct physical interest since they arise under the action of a typical quantum chaotic system—see, e.g., [2]. In this work we investigate mean values of certain measures of quantum entanglement, averaged over the entire space of pure states of a Hilbert space of a given size.

There exist several measures of quantum entanglement that do not increase under local operations and satisfy the required properties listed in [3,4], but it is hardly possible to single out the "best" universal quantity. On the contrary, different entanglement measures are optimal for various tasks, so it is likely we will have to learn to live with quite a few of them [5,6].

The measures of quantum entanglement for a pure state of a bipartite system, $|\psi\rangle\in\mathcal{H}=\mathcal{H}_A\otimes\mathcal{H}_B$, rely on its Schmidt coefficients [7], equivalent to the spectrum $\vec{\Lambda}$ of the reduced system, $\rho=\mathrm{Tr}_B(|\psi\rangle\langle\psi|)$. By construction the sum of all Schmidt coefficients equals unity, $\sum_{i=1}^{N}\Lambda_i=1$, so just $(N-1)$ of them are independent. To quantify entanglement of a pure state one uses entanglement monotones [8], defined as quantities that do not increase under local operations and classical communication. Entanglement of a pure state of a $N\times N$ system is therefore completely described by a suitable set of $(N-1)$ independent entanglement monotones.

It is convenient to work with the ordered set of coefficients $\Lambda_1\geqslant\Lambda_2\geqslant\ldots\geqslant\Lambda_N\geqslant 0$. The first example of such a set of entanglement monotones found by Vidal consists of sums of the k largest coefficients, $E_k:=\sum_{i=1}^{k}\Lambda_i$ with $k=1,\ldots,N-1$ [8]. Alternatively, one can use Rényi entropies of $(N-1)$ different orders. Another set of monotones may be constructed out of symmetric polynomials of the Schmidt coefficients of order $k=2,\ldots,N$ [9],

$$\tau_2=\sum_{k=1}^{N}\sum_{l=k+1}^{N}\Lambda_k\Lambda_l,$$

$$\tau_3=\sum_{k=1}^{N}\sum_{l=k+1}^{N}\sum_{m=l+1}^{N}\Lambda_k\Lambda_l\Lambda_m,$$

$$\vdots$$

$$\tau_N=\prod_{k=1}^{N}\Lambda_k.$$

For large N these polynomials become small, so it is of advantage to consider cognate quantities $\tau_k'=(\tau_k)^{1/N}$. Gour noted that taking the Nth root of the polynomials does not spoil the monotonicity and proposed to used normalized quantities τ_k' as alternative measures of quantum entanglement [10]. In particular he found unique properties of the last polynomial τ_N, equal to the determinant of the reduced matrix $D=\det\rho$. Its rescaled Nth root

$$G:=ND^{1/N},\tag{1.1}$$

proportional to the geometric mean of all Schmidt coefficients, was called the G concurrence in [10], where its operational interpretation as a type of entanglement capacity was suggested. This quantity, extended by the convex roof construction for mixed states, played a crucial role in demonstration of asymmetry of quantum correlations [11] and was used to characterize the entanglement of assistance [12].

The aim of this work is to compute mean values and to describe probability distributions for the determinant D and its root G of random pure states of a bipartite system, generated with respect to the natural, unitary invariant measure on the space of pure states, also called the Fubini-Study

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measure. Our analysis is performed for a bipartite system of an arbitrary size N, and in particular we treat in detail the interesting limiting case $N\to\infty$. Although our study directly concerns bipartite systems, one may infer some statements valid also in the general case of multipartite systems.

The paper is organized as follows. In Sec. II we review the concept of a random pure state and describe certain probability measures in this set. Average values of the G concurrence are computed in Sec. III, while the subsequent section concerns the probability distribution of this measure of quantum entanglement. The paper is concluded with some final remarks while the discussion of the asymptotics of probability distributions is postponed to an Appendix.

II. RANDOM PURE STATES AND INDUCED MEASURES

Consider a pure state of a bipartite $N\times K$ system represented in a product basis

$$|\psi\rangle = \sum_{i=1}^{N}\sum_{j=1}^{K} A_{ij}|i\rangle \otimes |j\rangle .$$

The Schmidt coefficients Λ_i coincide with the eigenvalues of a positive matrix $\rho_N = AA^\dagger$, equal to the density matrix obtained by a partial trace on the K-dimensional space. The matrix A need not be Hermitian; the only constraint is the trace condition $\mathrm{Tr}\, AA^\dagger = 1$. Furthermore, the natural unitarily invariant measure on the space of pure states corresponds to taking A as a matrix from the Ginibre ensemble [13]. Thus our problem consists in analyzing the distribution of determinants of random Wishart matrices $AA^\dagger$ normalized by fixing its trace. Schmidt coefficient distributions are given by [14]

$$P_{N,K}^{(\beta)}(\Lambda_1,\ldots,\Lambda_N) = B_{N,K}^{(\beta)}\delta\left(1-\sum_i \Lambda_i\right)\prod_i \Lambda_i^{[\beta(K-N)+\beta-2]/2} \times \theta(\Lambda_i)\prod_{i<j}|\Lambda_i-\Lambda_j|^\beta, \tag{2.1a}$$

in which the cases of real or complex A are distinguished by the *repulsion exponent* β [15] being 1 (2), and the normalization $B_{N,K}^{(\beta)}$ reads [13]

$$B_{N,K}^{(\beta)} := \frac{\Gamma(KN\beta/2)[\Gamma(1+\beta/2)]^N}{\prod_{j=0}^{N-1}\Gamma((K-j)\beta/2)\Gamma(1+(N-j)\beta/2)}. \tag{2.1b}$$

Formulas 2.1 describe a family of probability measures in the simplex of eigenvalues of a density matrix of size N. The integer number K, determining the size of the ancilla, can be treated as a free parameter.

Another important probability measure in the space of mixed quantum states is induced by the Euclidean geometry and the Hilbert-Schmidt (HS) distance. Assuming that each ball of a certain radius contains the same volume, one arrives at the HS measure [16]

$$P_{\mathrm{HS}}^{(\beta)}(\Lambda_1,\ldots,\Lambda_N) := H_N^{(\beta)}\delta\left(\sum_{i=1}^N \Lambda_i - 1\right)\prod_{i=1}^N \theta(\Lambda_i)\prod_{i<j}|\Lambda_i-\Lambda_j|^\beta, \tag{2.2a}$$

where the parameter β distinguishes as before between the real and the complex cases. The above normalization constant $H_N^{(\beta)}$ reads

$$\frac{1}{H_N^{(\beta)}} := \frac{1}{\Gamma(N+\beta N(N-1)/2)} \times \prod_{j=1}^N\left[\frac{\Gamma(1+j\beta/2)\Gamma[1+(j-1)\beta/2]}{\Gamma(1+\beta/2)}\right]. \tag{2.2b}$$

We observe that the distribution (2.2), normalization constants included, can be recasted into the form (2.1), provided that we choose $K=N-1+2/\beta$, that is,

$$K = \begin{cases} N & \text{for complex } \rho_N \text{ (with } \beta=2), \\ N+1 & \text{for real } \rho_N \text{ (with } \beta=1). \end{cases} \tag{2.3}$$

Using this observation, one can get a useful procedure for generating random density matrices distributed according to the HS measure taking normalized Wishart matrices $AA^\dagger$, with A belonging to the Ginibre ensemble of Hermitian matrices of appropriate dimension.

Aiming to derive the averaged moments needed in Sec. III, it is convenient to change variable in (2.1) by putting $K=2\alpha/\beta+N-1$ and obtaining

$$P_N^{(\alpha,\beta)}(\Lambda_1,\ldots,\Lambda_N) := C_N^{(\alpha,\beta)}\delta\left(\sum_{i=1}^N \Lambda_i - 1\right) \times \prod_{i=1}^N \Lambda_i^{\alpha-1}\theta(\Lambda_i)\prod_{i<j}|\Lambda_i-\Lambda_j|^\beta, \tag{2.4a}$$

with

$$\frac{1}{C_N^{(\alpha,\beta)}} := \frac{1}{\Gamma(\alpha N+\beta N(N-1)/2)} \times \prod_{j=1}^N\left[\frac{\Gamma(1+j\beta/2)\Gamma(\alpha+(j-1)\beta/2)}{\Gamma(1+\beta/2)}\right]. \tag{2.4b}$$

In the above formula the real variable α can be used as a free parameter instead of the integer K.

III. AVERAGE MOMENTS OF G CONCURRENCE

In this section we are going to compute averages over an ensemble of random density matrices distributed according to the HS measure, which is induced by the Euclidean geometry. This corresponds to fixing the size K of the ancilla according to (2.3), depending on whether the real or the complex case is concerned.

Denoting the eigenvalues of the density matrix ρ_N by $\{\Lambda_j\}$, the moments of the determinants $D(\Lambda_1,\ldots,\Lambda_N) = \prod_{j=1}^N \Lambda_j$ read

$$\langle D^M_{(\beta)}\rangle_N := \int_{-\infty}^{\infty} d\Lambda_1 \cdots \int_{-\infty}^{\infty} d\Lambda_N D^M(\Lambda_1,\ldots,\Lambda_N) \times P^{(\beta)}_{\mathrm{HS}}(\Lambda_1,\ldots,\Lambda_N). \quad (3.1)$$

The product of Heaviside step functions, present in the definition (2.2a) of $P^{(\beta)}_{\mathrm{HS}}$, allows us to extend the domain of integration on the entire axis. The integrand of (3.1) coincides with the factor present in the right hand side of Eq. (2.4a), provided that the parameter α is set there to $1+M$. Using this the integral (3.1) can be computed from (2.4b), and reads

$$\langle D^M_{\mathbb{C}}\rangle_N = \frac{C_N^{(1,2)}}{C_N^{(1+M,2)}} = \frac{\Gamma(N^2)}{\Gamma(MN+N^2)}\prod_{j=1}^N \frac{\Gamma(M+j)}{\Gamma(j)},$$

$$\langle D^M_{\mathbb{R}}\rangle_N = \frac{C_N^{(1,1)}}{C_N^{(1+M,1)}} = \frac{\Gamma\left(\frac{N^2+N}{2}\right)}{\Gamma\left(MN+\frac{N^2+N}{2}\right)}\prod_{j=1}^N \frac{\Gamma\left(M+\frac{j+1}{2}\right)}{\Gamma\left(\frac{j+1}{2}\right)}. \quad (3.2)$$

For the sake of clarity, from now on the sub- and superscript $(\beta=2)$ and $(\beta=1)$ will be often replaced by $\mathbb{C}$ and $\mathbb{R}$. Making use of Eq. (1.1), one obtains the moments of the G concurrence by imposing $\alpha=1+M/N$ in the ratios $C_N^{(1,\beta)}/C_N^{(\alpha,\beta)}$, rescaled by a factor N^M. Thus we get now

$$\langle G^M_{\mathbb{C}}\rangle_N = N^M \frac{C_N^{(1,2)}}{C_N^{(1+M/N,2)}} = N^M \frac{\Gamma(N^2)}{\Gamma(M+N^2)}\prod_{j=1}^N \frac{\Gamma\left(\frac{M}{N}+j\right)}{\Gamma(j)},$$

$$\langle G^M_{\mathbb{R}}\rangle_N = N^M \frac{C_N^{(1,1)}}{C_N^{(1+M/N,1)}} = N^M \frac{\Gamma\left(\frac{N^2+N}{2}\right)}{\Gamma\left(M+\frac{N^2+N}{2}\right)}\prod_{j=1}^N \frac{\Gamma\left(\frac{M}{N}+\frac{j+1}{2}\right)}{\Gamma\left(\frac{j+1}{2}\right)}. \quad (3.3)$$

In Fig. 1 the mean values $\langle G_{(\beta)}\rangle_N$ and variance $\sigma_N^2 = \langle G^2_{(\beta)}\rangle_N - \langle G_{(\beta)}\rangle_N^2$ are represented as a function of N for both complex and real cases.

IV. PROBABILITY DISTRIBUTION $P_N^{(\beta)}(G)$

This section is devoted to a study of probability distributions. We shall start with the simplest problem of determining the distribution of the determinant D of a 2×2 density matrix ρ_2 distributed according to the HS measure. In this case an explicit solution is easily obtained by integrating the Dirac delta $\delta(D-\Lambda_1\Lambda_2)$ over the distribution $P^{(\beta)}_{\mathrm{HS}}(\Lambda_1,\Lambda_2)$ of (2.4), that is,

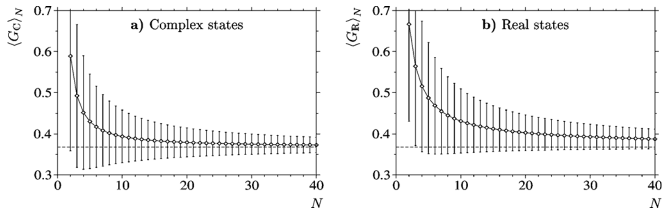

FIG. 1. Average of G concurrence for (a) complex and (b) real random mixed states of an $N\times(N+2-\beta)$ system distributed according to the HS measure. The average is computed by means of Eq. (3.3); error bars represent the variance of $P_N^{(\beta)}(G)$. Dashed line represents the asymptote $G_*=1/e$, whose explanation is given in Sec. V.

$$P_2^{(\beta)}(D) := C_2^{(1,\beta)}\int_{-\infty}^{\infty} d\Lambda_1 \int_{-\infty}^{\infty} d\Lambda_2 \delta(\Lambda_1+\Lambda_2-1)\theta(\Lambda_1)\theta(\Lambda_2) \times |\Lambda_1-\Lambda_2|^\beta \delta(D-\Lambda_1\Lambda_2).$$

It is a very simple distribution since $(\Lambda_1-\Lambda_2)^2 = (\Lambda_1+\Lambda_2)^2 - 4D = 1-4D$. Thus

$$P_2^{\mathbb{C}}(D) = 6\sqrt{1-4D}, \quad P_2^{\mathbb{R}}(D) = 4, \qquad D \in \left[0,\frac{1}{4}\right]. \quad (4.1)$$

The G-concurrence distribution $P_2^{(\beta)}(G)$ can be computed by either integrating $\delta(G-2\sqrt{\Lambda_1\Lambda_2})$ over $P^{(\beta)}_{\mathrm{HS}}(\Lambda_1,\Lambda_2)$, or simply using the latter result (4.1) together with $P_2^{(\beta)}(G)dG = P_2^{(\beta)}(D)dD$; in both cases (see Fig. 2)

$$P_2^{\mathbb{C}}(G) = 3G\sqrt{1-G^2}, \quad P_2^{\mathbb{R}}(G) = 2G, \qquad G\in[0,1], \quad (4.2)$$

Note that, due to $\Lambda_1+\Lambda_2=1$, (only) for the case $N=2$ the G-concurrence given by (1.1) reduces to the standard concurrence [17], $C=\sqrt{2(1-\mathrm{Tr}\rho_2^2)}$. Thus the formula (4.2) for the complex case coincides with the distribution of concurrence $P(C)$ obtained in [13].

For higher N we will construct the distribution $P_N^{(\beta)}(D)$ from all moments $\langle D^M_{(\beta)}\rangle_N$ given by Eq. (3.2); indeed

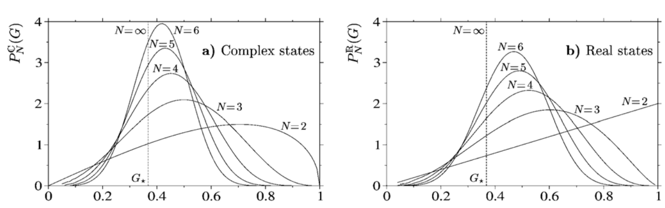

FIG. 2. G concurrence's distributions $P_N^{(\beta)}(G)$ are compared for different N in the case of (a) complex and (b) real random pure states. The distributions are obtained by performing numerically [18] the inverse Laplace transform of Eq. (4.3). Dashed vertical line centered in $G_*=1/e$ denotes the position of the Dirac delta corresponding to $P_{N\to\infty}^{(\beta)}(G)$, as it is shown in Sec. V.

$$\langle D^M_{(\beta)}\rangle_N = \int_0^1 dD\, D^M P_N^{(\beta)}(D) = \int_0^\infty dx\, e^{-x(1+M)} P_N^{(\beta)}(e^{-x}),$$

with $D=e^{-x}$ and $dD=-dx\, e^{-x}$, and so we can obtain $P_N(D)$ by inverse Laplace transform or inverse Mellin transform as integral along the imaginary M axis:

$$P_N^{(\beta)}(D) = \int_{-i\infty}^{+i\infty} \frac{dM}{2\pi i} D^{-(1+M)} \langle D^M_{(\beta)}\rangle_N. \tag{4.3}$$

Although Eqs. (4.3) and (3.2) allow us to compute the $P_N^{(\beta)}(D)$ probabilities, the cognate quantities $P_N^{(\beta)}(G)$ can be determined as well by using

$$P_N^{(\beta)}(G)dG = P_N^{(\beta)}(D)dD;$$

by taking from (1.1) the explicit expression for dD/dG, one can indeed get the simple expression

$$GP_N^{(\beta)}(G) = NDP_N^{(\beta)}(D). \tag{4.4}$$

From now on formulas and figures will be given indifferently for both G and D distributions, their mutual relation being clear. In particular the D distribution is more indicated for showing details of calculation, for its simpler form, whereas the G distribution better shows features in the pictures, because its domain is independent of N. The asymptotic behavior of the Gamma function for large argument (Stirling's formula) is important:

$$\Gamma(z) = z^{(z-1/2)}\sqrt{2\pi}e^{-z}\left[1 + \frac{1}{12z} + O\left(\frac{1}{z}\right)^2\right]$$

for $|z|\to\infty$ and $\arg(z)<\pi$. This implies the asymptotic behavior of (3.2) for large $|M|$:

$$\langle D^M_{\mathbb{C}}\rangle_N \simeq D^{\mathrm{S}}_{\mathbb{C}}(M,N) := A_N^{\mathbb{C}} \frac{e^{-MN\ln N}}{M^{(N^2-1)/2}}$$

$$\langle D^M_{\mathbb{R}}\rangle_N \simeq D^{\mathrm{S}}_{\mathbb{R}}(M,N) := A_N^{\mathbb{R}} \frac{e^{-MN\ln N}}{M^{(N^2+N-2)/4}}$$

with

$$A_N^{\mathbb{C}} := \frac{(2\pi)^{(N-1)/2}\Gamma(N^2)}{N^{N^2-1/2}\prod_{j=1}^{N}\Gamma(j)},$$

$$A_N^{\mathbb{R}} := \frac{(2\pi)^{(N-1)/2}\Gamma((N^2+N)/2)}{N^{(N^2+N-1)/2}\prod_{j=1}^{N}\Gamma((j+1)/2)}. \tag{4.5}$$

As a consequence the integral (4.3) converges and moreover it vanishes if $x<N\ln N$ or $D>(1/N)^N$, because in that case we can close the contour in (4.3) in the right M half plane according to Jordan's lemma [19]. Physically this means that there are no density matrices with determinants greater than the one with maximal entropy.

In the rest of this section we will give the asymptotic behavior of distributions $P_N^{(\beta)}(D)$ for the two edges of the domain, that is, $D\to 0$ and $D\to(1/N)^N$. The details of calculation, together with the explicit N dependence of all coefficients listed here in the following, are collected in the Appendix.

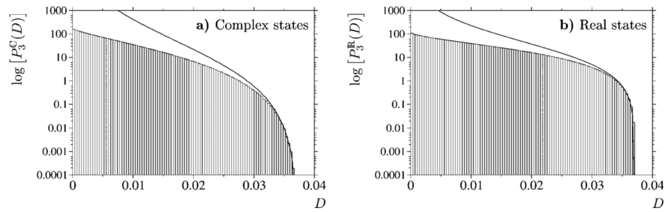

FIG. 3. In (a) a 100-bin histogram of 10^8 determinants of 3×3 complex density matrices distributed accordingly to the HS measure is compared with the right asymptote given by Eq. (4.6) (plotted as a solid line). Same analysis is depicted in (b), but for 3×3 real density matrices.

In particular, when very close to the completely mixed state, that is $D\simeq(1/N)^N$, we have the result (see Fig. 3)

$$P_N^{\mathbb{C}}(D) \simeq A_N^{\mathbb{C}} \frac{(-\ln D - N\ln N)^{(N^2-3)/2}}{D[(N^2-3)/2]!},$$

$$P_N^{\mathbb{R}}(D) \simeq A_N^{\mathbb{R}} \frac{(-\ln D - N\ln N)^{(N^2+N-6)/4}}{D[(N^2+N-6)/4]!}. \tag{4.6}$$

Moreover, using (4.4) together with

$$-\ln D - N\ln N \simeq 1 - DN^N = 1 - G^N,$$

we simply find

$$P_N^{\mathbb{C}}(G) \simeq \tilde{A}_N^{\mathbb{C}} \frac{(1-G^N)^{(N^2-3)/2}}{G},$$

$$P_N^{\mathbb{R}}(G) \simeq \tilde{A}_N^{\mathbb{R}} \frac{(1-G^N)^{(N^2+N-6)/4}}{G},$$

with

$$\tilde{A}_N^{\mathbb{C}} := A_N^{\mathbb{C}} \frac{N}{\Gamma((N^2-1)/2)},$$

$$\tilde{A}_N^{\mathbb{R}} := A_N^{\mathbb{R}} \frac{N}{\Gamma((N^2+N-2)/4)}.$$

For the other part of the spectrum, that is, for very small D, the probability $P_N^{\mathbb{C}}(D)$ can be expanded in a power series with some logarithmic corrections, as follows:

$$P_N^{\mathbb{C}}(D) \simeq Z_N^{\mathbb{C}} + X_N^{\mathbb{C}} D\ln D + \tilde{X}_N^{\mathbb{C}} D + V_N^{\mathbb{C}} D^2(\ln D)^2 + \tilde{V}_N^{\mathbb{C}} D^2(\ln D) + \tilde{\tilde{V}}_N^{\mathbb{C}} D^2 + O(D^3(\ln D)^3). \tag{4.7}$$

In particular, the coefficients $Z_N^{\mathbb{C}}, X_N^{\mathbb{C}}, \tilde{X}_N^{\mathbb{C}}$ are computed in the Appendix for all $N\geqslant 3$, whereas for $V_N^{\mathbb{C}}, \tilde{V}_N^{\mathbb{C}}$, and $\tilde{\tilde{V}}_N^{\mathbb{C}}$ we limit ourselves to explicitly solving the case $N=3$ [the case $N=2$ is simply given by formula (4.1)].

The situation is similar when we do consider, in the same region of the domain, the probability $P_N^{\mathbb{R}}(D)$, corresponding to small determinants of reduced $N\times N$ real density matrices HS distributed. The expansion is still a power series (plus logarithmic corrections) but the exponents are now semi-integer, according to the mechanism described in the Appendix; thus the probability reads

$$P_N^{\mathbb{R}}(D) \simeq Z_N^{\mathbb{R}} + Y_N^{\mathbb{R}} D^{1/2} + X_N^{\mathbb{R}} D \ln D + \widetilde{X}_N^{\mathbb{R}} D + W_N^{\mathbb{R}} D^{3/2} \ln D + \widetilde{W}_N^{\mathbb{R}} D^{3/2} + O(D^2(\ln D)^2)$$

V. CONCENTRATION OF G CONCURRENCE FOR LARGE SYSTEM SIZE

Iterating the recursion relation for the Gamma function $\Gamma(n+1)=n\Gamma(n)$, we can recast expression (3.3) of the M moment of the G concurrence of a complex random pure state as

$$\langle G_C^M\rangle_N = \left\{\prod_{k=0}^{M-1}\frac{N}{N^2+k}\right\}\left\{\left[\frac{M}{N}\Gamma\left(\frac{M}{N}\right)\right]^N\right\}\left\{\prod_{k=1}^{N-1}\left(1+\frac{M}{kN}\right)^{N-k}\right\}$$

with the asymptotics characterized with help of the Euler constant $\gamma\approx 0.577\,215\,665\ldots$,

$$\prod_{k=0}^{M-1}\frac{N}{N^2+k} \underset{N\to\infty}{\sim} \frac{1}{N^M},$$

$$\left[\frac{M}{N}\Gamma\left(\frac{M}{N}\right)\right]^N \underset{N\to\infty}{\sim} e^{-\gamma M},$$

and

$$\prod_{k=1}^{N-1}\left(1+\frac{M}{kN}\right)^{N-k} \underset{N\to\infty}{\sim} N^M e^{M(\gamma-1)},$$

so that finally

$$\langle G_{\mathrm{C}}^M\rangle_N \underset{N\to\infty}{\sim} e^{-M}. \tag{5.1}$$

For the analogous moments of G concurrence of real random pure state, some technicality requires that the sequence of odd and even N has to be analyzed separately, although it is not hard to prove that the limit is the same. For that reason, we will simply illustrate the case $N=2p$, $p\in\mathbb{N}$, for which (3.3) gives

$$\langle G_{\mathbb{R}}^M\rangle_N = \left(\frac{2}{\sqrt{\pi}}\right)^p\left\{\prod_{k=0}^{M-1}\frac{2p}{2p^2+p+k}\right\}\left\{\left[\frac{M}{2p}\Gamma\left(\frac{M}{2p}\right)\right]^p\right\} \times\left\{\left[\left(\frac{M}{2p}+\frac{1}{2}\right)\Gamma\left(\frac{M}{2p}+\frac{1}{2}\right)\right]^p\right\} \times\left\{\prod_{k=1}^{p-1}\left(1+\frac{M}{2pk}\right)^{p-k}\right\}\left\{\prod_{k=3/2}^{p-1/2}\left(1+\frac{M}{2pk}\right)^{p-k-1/2}\right\},$$

with

$$\prod_{k=0}^{M-1}\frac{2p}{2p^2+p+k} \underset{p\to\infty}{\sim} \left(\frac{2}{2p+1}\right)^M,$$

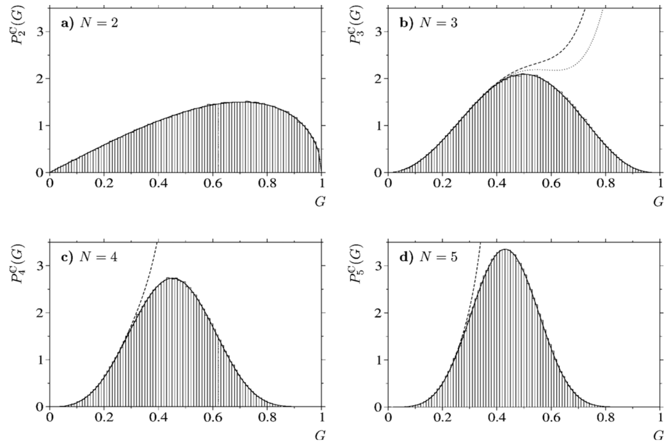

FIG. 4. In (a), formula (4.2) is compared with a 100-bin histogram of $10^6 G$ concurrence of 2×2 complex density matrices distributed according to the HS measure. (b), (c), and (d) show histograms (for different N) together with the distribution of G concurrence obtained by inverse Laplace transforming as in Eq. (4.3) (plotted as solid lines). The left asymptote given by Eq. (4.7), computed up to $O(D)$, is also plotted as a dashed line for comparison; in (b) we also add the contribution given by $V_N^{\mathrm{C}}, \widetilde{V}_N^{\mathrm{C}}, \widetilde{\widetilde{V}}_N^{\mathrm{C}}$ coefficients, using a dotted line.

$$\left[\frac{M}{2p}\Gamma\left(\frac{M}{2p}\right)\right]^p \underset{p\to\infty}{\sim} e^{-\gamma M/2},$$

$$\left[\left(\frac{M}{2p}+\frac{1}{2}\right)\Gamma\left(\frac{M}{2p}+\frac{1}{2}\right)\right]^p \underset{p\to\infty}{\sim} \frac{\pi^{p/2}}{2^{(p+M)}}e^{M(1-\gamma/2)},$$

$$\prod_{k=1}^{p-1}\left(1+\frac{M}{2pk}\right)^{p-k} \underset{p\to\infty}{\sim} p^{M/2}e^{M(\gamma-1)/2},$$

and

$$\prod_{k=3/2}^{p-1/2}\left(1+\frac{M}{2pk}\right)^{p-k-1/2} \underset{p\to\infty}{\sim} e^{M(\gamma-1)/2}e^{-M}2^M\left(p+\frac{1}{2}\right)^{M/2}.$$

Putting all factors together we arrive at the general result [compare with (5.1)]

$$G(M) \coloneqq \lim_{N\to\infty}\langle G_{(\beta)}^M\rangle_N = e^{-M}. \tag{5.2}$$

The above expression, valid for both $\beta\in\{1,2\}$, is useful to derive the limiting distribution

$$P^{(\beta)}(G) \coloneqq \lim_{N\to\infty} P_N^{(\beta)}(G).$$

We see from (5.2) that its average is $1/e=0.367\,879\,441\ldots$ and its variance is 0; such behavior can be recognized in Fig. 4. Moreover, by fixing $G=e^{-x}$, one can see that $G(M)$ of (5.2) is nothing but the Laplace transform of the function

$$\eta(x) \coloneqq e^{-x}P^{(\beta)}(e^{-x})$$

so that, by inverse Laplace transforming, we obtain

$$e^{-x}P^{(\beta)}(e^{-x}) = GP^{(\beta)}(G) = \delta(-\ln(G)-1).$$

Rewriting the argument of the Dirac delta we finally arrive at

$$P^{(\beta)}(G) = \delta(G - e^{-1}). \tag{5.3}$$

In other words, we have shown that for large systems the G concurrence of random states is localized arbitrarily close to the averaged value.

A similar concentration effect has recently been quantified [20] for bipartite $N\times K$ systems. In particular the von Neumann entropy of the reduced density matrix of the first subsystem concentrates around the entropy of the maximally mixed state, $S(\mathbb{1}/N)=\ln N$, if we let the dimension K of the auxiliary subsystem go to infinity faster than N. When $K=N$, so that the induced distribution coincides with the Hilbert-Schmidt distribution, and $N\to\infty$, then the von Neumann entropy concentrates around $\ln N - 1/2$ [20,21]. Remarkably, the G concurrence displays a similar concentration effect; moreover, we are in position to prove the convergence of its distribution to a Dirac delta centered at a nontrivial value $1/e$.

The determinants and G concurrence may also be averaged in the general case of asymmetric induced measure (2.1). Consider an interesting case $K>N$. As for the HS distribution discussed in Sec. II the expectation value and the higher moments may be expressed as a ratio of normalization constants (2.1b) and (2.4b) For instance, the moments read

$$\langle G_{\mathbb{C}}^{M}\rangle_{N,K} = N^{M}\frac{B_{N,K}^{(2)}}{C_{N}^{(M/N+K-N+1,2)}} = N^{M}\frac{\Gamma(NK)}{\Gamma(NK+M)}\prod_{j=1}^{N}\frac{\Gamma(K-N+j+M/N)}{\Gamma(K-N+j)},$$

$$\langle G_{\mathbb{R}}^{M}\rangle_{N,K} = N^{M}\frac{B_{N,K}^{(1)}}{C_{N}^{(M/N+(K-N+1)/2,1)}} = N^{M}\frac{\Gamma(NK/2)}{\Gamma(NK/2+M)}\prod_{j=1}^{N}\frac{\Gamma((K-N+j)/2+M/N)}{\Gamma((K-N+j)/2)}. \tag{5.4}$$

Let us now study a particular case of the induced measure, for which we consider bipartite systems of arbitrarily large dimension, with the only constraint that the ratio between the size K of the ancilla and the size N of the principal subsystem is fixed and greater than 1. Let this ratio be expressed by the rational number $q=\ell_2/\ell_1$, with the ℓ_1 and ℓ_2 integers; this means that we are considering systems with $N=J\ell_1<K=J\ell_2$.

With the same tools used in computing (5.2), one can let J go to infinity and obtain

$$G(M) \coloneqq \lim_{J\to\infty}\langle G_{(\beta)}^{M}\rangle_{J\ell_1,J\ell_2} = X_q^{-M}, \quad \forall\, \beta\in\{1,2\}, \tag{5.5}$$

with

$$X_q \coloneqq \frac{1}{e}\left(\frac{q}{q-1}\right)^{q-1}, \quad q>1. \tag{5.6}$$

The limiting distribution $P_q(G)$, can be obtained as before and reads

$$P_q^{(\beta)}(G) \coloneqq \lim_{J\to\infty} P_{J\ell_1,J\ell_2}^{(\beta)}(G) = a(G - X_q),$$

for the complex as well as for the real case. Although the accumulation point X_q is not defined for the case $q=1$ (that is the case in which states in the principal system are HS distributed), we find, however, $\lim_{q\to 1} X_q = 1/e$, confirming our previous result (5.3). Moreover such values represent an infimum for X_q, whereas it attains the supremum on the other part of the domain, that is, for $q\to\infty$. This case corresponds an to extremely large environment, for which $X_\infty=1$, which is, in turn the G concurrence of the completely mixed state. Thus we find further evidence that a large environment concentrates reduced density matrices around the maximally mixed states [20].

VI. CONCLUDING REMARKS

The generalized G concurrence is likely to be the first measure of pure state entanglement for which one not only could find the mean value over the set of random pure states, but also compute explicitly all moments and describe its probability distribution, deriving an analytic expression in the large-N limit. This offers various potential applications for our work. On one hand, by analyzing a concrete quantum state and its entanglement we may check to what extent its properties are nontypical. In practice this can be done by a comparison of its G concurrence G with the mean value $\langle G\rangle$, and by comparing its deviation from the average, $|G-\langle G\rangle|$, with the root of the variance of the distribution.

On the other hand, if one needs a quantum state of some particular properties, one may estimate how difficult it is to obtain such a state at random. For instance, looking for a state of a large degree of entanglement, with concurrence greater than a given value $\tilde{G}$, one can make use of the derived probability distribution by integrating it from $\tilde{G}$ to unity in order to evaluate the probability to generate the desired state by a fully uncontrolled, chaotic quantum evolution.

Although in this work we have concentrated our attention on pure states of bipartite systems, the averages obtained for the asymmetric induced measures (2.1) with $K>N$ may be easily applied for the more general, multipartite case. Consider a system containing n qudits (particles described in a d-dimensional Hilbert space). This system may be divided by an arbitrary bipartite splitting into m and $(n-m)$ particles, and one can study entanglement between the two subsystems—see, e.g., [22]. The partial trace over m qudits is equivalent to the partial trace performed over a single ancilla of size $K=d^m$, so setting the size of the system $N=d^{n-m}$ one may read out the average concurrence from Eq. (5.4). In particular, if n is even and we put $m=n/2+k$, then the ratio $q=K/N$ is equal to d^{2k} and in the asymptotic limit

$n\to\infty$ the concurrence concentrates around the mean (5.6) which depends only on the asymmetry k of the splitting.

Our research may also be considered as a contribution to the random matrix theory: we have found the distribution of the determinants of random Wishart matrices $AA^\dagger$, normalized by fixing their trace. Furthermore, the analysis of the distribution of G concurrence in the limit of large system sizes provides an illustrative example of the geometric concentration effect, since in high dimensions the distribution of the determinant is well localized around the mean value. This observation can also be related to the central limit theorem applied to logarithms of the eigenvalues of a density matrix, the sum of which is equal to the logarithm of the determinant.

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APPENDIX: COEFFICIENTS OF ASYMPTOTIC EXPANSIONS OF PROBABILITY

1. Right asymptote of $P_N^{(\beta)}(D)$: Proof of Eq. (4.6)

The starting point is integral (4.3). Since all the poles of the integrand are in the left half plane [see it in (3.2)], the contour integration along the imaginary axis can be modified into the one along the right asymptotic half plane, that is on a very large semicircle connecting $-i\infty$ to $+i\infty$; this allows us to use Stirling's formula for replacing $\langle D_{(\beta)}^M\rangle_N$ with $D_{(\beta)}^{\mathrm{S}}(M,N)$ [see formula (4.5)] in the integrand of (4.3). Of course we made an approximation, but we know that the formula we ended up with matches the correct result $[P_N^{(\beta)}(D)=0$ for $D>(1/N)^N]$ at the point $(1/N)^N$, so that this approximation will hold close to that point. Now we observe that $D_{(\beta)}^{\mathrm{S}}(M,N)$ has poles only at $M=0$, so that our contour of integration can be modified provided that we do not cross the origin, and we do so, obtaining

$$P_N^{(\beta)}(D)=\int_\gamma \frac{dM}{2\pi i}D^{-(1+M)}D_{(\beta)}^{\mathrm{S}}(M,N)$$

$$=\frac{A_N^{(\beta)}}{2\pi i D}\int_\gamma dM\, e^{M(-\ln D-N\ln N)}M^{-(N^2-1)/2},$$

where γ is now the contour that, starting from $-i\infty$, gets close to the negative real axis on the asymptotic left lower quarter plane, winds around $\mathbb{R}^-\cup\{0\}$ in the counterclockwise direction, and then approaches $+i\infty$ on the asymptotic left upper quarter plane. But now we apply once more Jordan's lemma and we remove the asymptotic semicircle from γ. After rescaling $M\to -M/\varepsilon$, with the latter defined by $\varepsilon=-\ln D-N\ln N$ and close to 1, we arrive at the well-known Hankel contour integral for the inverse of the Gamma function $(1/\Gamma)$ [23], which leads to (4.6) and gives the asymptotic behavior for $D\to(1/N)^N$.

2. Left asymptote of $P_N^{\mathbb{C}}(D)$ for complex random pure states

Now let us consider the behavior of $P_N^{\mathbb{C}}(D)$ at the lower edge of the spectrum $D\to 0$. In that case one can close the integral (4.3) in the left half plane obtaining contributions from all the poles of the Gamma functions in $\langle D_{\mathbb{R}}^M\rangle_N$ [see (3.2)]. Such poles are located at each of the negative integers $M=-1,-2,-3,\ldots$; fortunately there is the factor $D^{-(1+M)}$ such that we obtain a series in powers of D. Because of the multiple Gamma functions in (3.2), most of the poles are degenerate and the general feature (for an arbitrary large N) is that the pole in $-\ell$ is of order ℓ: due to this fact the D powers in the expansion get in general a logarithmic correction. The first pole at $M_1=-1$ is nondegenerate and yields

$$P_N^{\mathbb{C}}(0)=\frac{\Gamma(N^2)}{\Gamma(N^2-N)\Gamma(N)}=Z_N^{\mathbb{C}}.$$

Including the next order-2 pole $(M_2=-2)$ contribution we find the asymptotic expansion for $D\to 0$,

$$P_N^{\mathbb{C}}(D)\simeq Z_N^{\mathbb{C}}+X_N^{\mathbb{C}}D\ln D+\tilde{X}_N^{\mathbb{C}}D$$

with

$$X_N^{\mathbb{C}}=\frac{\Gamma(N^2)}{\Gamma(N^2-2N)\Gamma(N)\Gamma(N-1)},$$

$$\tilde{X}_N^{\mathbb{C}}=X_N^{\mathbb{C}}[N+N\psi(N^2-2N)-4-2\psi(1)-(N-2)\psi(N-2)]. \tag{A1}$$

Here $\psi(x)$ is the Digamma function [24], or polygamma function of order 0, with

$$\psi(1)=-\gamma,\quad \psi(n)=-\gamma+\sum_{k=1}^{n-1}\frac{1}{k}\quad \text{for } n>1. \tag{A2}$$

Note that the Euler constant γ cancels everywhere. By adding the next order-3 pole $(M_3=-3)$ contribution one gets in general the terms in (4.7) corresponding to the $V_N^{\mathbb{C}}$, $\tilde{V}_N^{\mathbb{C}}$, and $\tilde{\tilde{V}}_N^{\mathbb{C}}$ coefficients, although the latter are in general rather complicated, involving polygamma function of order higher than 0. This is not the case when $N=3$, for which a cancelation makes $M_3=-3$ a pole of order 2, and the coefficients read:

$$V_3^{\mathbb{C}}=0,\quad \tilde{V}_3^{\mathbb{C}}=6\times 7!=30\,240,\quad \tilde{\tilde{V}}_3^{\mathbb{C}}=9\times 7!=45\,360.$$

3. Left asymptote of $P_N^{\mathbb{R}}(D)$ for real random pure states

We will apply the same reasoning as in the previous case, just now differing in the fact that, when $\beta=1$, the ℓth pole M_ℓ of the integrand of (4.3) is $-(\ell+1)/2$; in general, for arbitrarily large N, its corresponding order is given by $\lfloor(\ell+1)/2\rfloor$, where $\lfloor x\rfloor$ means the largest integer not exceeding x. In particular, the first two poles $M_1=-1$ and $M_2=-\frac{3}{2}$ are nondegenerate and yield [25]

$$Z_N^{\mathbb{R}} = \frac{2^{N-1}\Gamma\left(\frac{N^2+N}{2}\right)}{\Gamma\left(\frac{N^2-N}{2}\right)\Gamma(N)} \quad \text{and}$$

$$Y_N^{\mathbb{R}} = -\sqrt{\pi}\frac{2^{N-1}\Gamma\left(\frac{N^2+N}{2}\right)}{\Gamma\left(\frac{N^2-2N}{2}\right)\Gamma\left(\frac{N+1}{2}\right)\Gamma(N-1)}.$$

Including the next two order-2 pole contributions ($M_3=-2$ and $M_4=-\frac{5}{2}$) we determine, for $N>3$,

$$X_N^{\mathbb{R}} = -\frac{2^{2N-3}\Gamma\left(\frac{N^2+N}{2}\right)}{\Gamma\left(\frac{N^2-3N}{2}\right)\Gamma(N)\Gamma(N-2)},$$

$$\widetilde{X}_N^{\mathbb{R}} = X_N^{\mathbb{R}}\left\{N + N\psi\left(\frac{N^2-3N}{2}\right) - 8 - \frac{3}{2}\psi\left(\frac{1}{2}\right) - 2\psi(1) - \frac{N-3}{2}\psi\left(\frac{N-3}{2}\right) - \frac{N-4}{2}\psi\left(\frac{N-4}{2}\right)\right\}, \tag{A3}$$

and for $N>4$

$$W_N^{\mathbb{R}} = -\frac{\sqrt{\pi}}{3}\frac{2^{2N-3}\Gamma\left(\frac{N^2+N}{2}\right)}{\Gamma\left(\frac{N^2-4N}{2}\right)\Gamma\left(\frac{N+1}{2}\right)\Gamma(N-1)\Gamma(N-3)},$$

$$\widetilde{W}_N^{\mathbb{R}} = W_N^{\mathbb{R}}\left\{N + N\psi\left(\frac{N^2-4N}{2}\right) - \frac{35}{3} - \frac{5}{2}\psi\left(\frac{1}{2}\right) - 2\psi(1) - \frac{N-4}{2}\psi\left(\frac{N-4}{2}\right) - \frac{N-5}{2}\psi\left(\frac{N-5}{2}\right)\right\}, \tag{A4}$$

where we made use once more of the ψ Digamma function [26] of (A2). The case $N=3$ constitutes an exception for $X_3^{\mathbb{R}}$ and $W_3^{\mathbb{R}}$ coefficients, because of the lowering of the order of M_3 and M_4 poles; moreover, for the latter pole, the same happens also for $N=4$. All these coefficients need separate calculations and read

$$\widetilde{X}_3^{\mathbb{R}} = 12 \times 5!, \quad \widetilde{W}_3^{\mathbb{R}} = 4 \times 5!,$$

$$\widetilde{W}_4^{\mathbb{R}} = 2^8 \times 8!, \quad X_3^{\mathbb{R}} = W_3^{\mathbb{R}} = W_4^{\mathbb{R}} = 0.$$

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[24] In Eq. (A1) we have used $\sum_{k=1}^{n}\psi(k)=n\psi(n)-n+1$.
[25] From now on we will often make use of the identity $\Gamma(n/2)\Gamma((n+1)/2)=\sqrt{\pi}\Gamma(n)/2^{n-1}$, $n\in\mathbb{N}^+$.
[26] In Eqs. (A3) and (A4) we have used $\sum_{k=1}^{n}\psi(k/2) =(n/2)\psi(n/2)+[(n-1)/2]\psi((n-1)/2)+(1/2)\psi(1/2)-n+2$; moreover, the notation $0\psi(0)=\lim_{\varepsilon\to 0}\varepsilon\psi(\varepsilon)=-1$ is understood.