Hypothesis testing for an entangled state produced by spontaneous parametric down-conversion

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(Received 27 March 2006; revised manuscript received 2 August 2006; published 27 December 2006)

Generation and characterization of entanglement are crucial tasks in quantum information processing. A hypothesis testing scheme for entanglement has been formulated. Three designs were proposed to test the entangled photon states created by the spontaneous parametric down conversion. The time allocations between the measurement vectors were designed to consider the anisotropic deviation of the generated photon states from the maximally entangled states. The designs were evaluated in terms of the p value based on the observed data. It has been experimentally demonstrated that the optimal time allocation between the coincidence and anticoincidence measurement vectors improves the entanglement test. A further improvement is also experimentally demonstrated by optimizing the time allocation between the anticoincidence vectors. Analysis on the data obtained in the experiment verified the advantage of the entanglement test designed by the optimal time allocation.

DOI: 10.1103/PhysRevA.74.062321 PACS number(s): 03.67.Mn, 03.65.Wj, 42.50.−p, 03.65.Ud

I. INTRODUCTION

The concept of entanglement has been thought to be the heart of quantum mechanics. The seminal experiment by Aspect *et al.* [1] has proved the "spooky" nonlocal action of quantum mechanics by observing violation of Bell inequality [2] with entangled photon pairs. Recently, entanglement has been also recognized as an important resource for quantum information processing, explicitly or implicitly. For example, entanglement provides an exponential speedup in some computational tasks [3], and unconditional security in cryptographic communications [4]. A hidden entanglement between the legitimate parties guarantees the security in BB84 quantum cryptographic protocol [4,5]. Quantum communication between arbitrarily distant parties has been shown to be possible by a quantum repeater [6] based on quantum teleportation [7]. Practical realization of entangled states is therefore one of the most important issues in quantum information technology.

The practical implementation raises a problem to verify the amount of entanglement. A quantum information protocol requires a minimum entanglement. It is, however, not always satisfied in actual experiments. Unavoidable imperfections will limit the entanglement in the generation process. Moreover, decoherence and dissipation due to the coupling with the environment will degrade the entanglement during the processing. Therefore, it is a crucial issue to characterize the entanglement of the generated (or stored) states to guarantee the successful quantum information processing. For this purpose, quantum state estimation and quantum state tomography are known as a method of identifying the unknown state [8–10]. Quantum state tomography [11] has recently been applied to obtain full information of the 4×4 two-particle density matrix from the coincidence counts of 16 combinations of measurement [12]. However, characterization is not the goal of an experiment, but only a part of preparation. It is thus favorable to reduce the time for characterization and the number of consumed particles as possible. An entanglement test, which should be simpler than the full characterization, works well in most applications, because we only need to know whether the states are sufficiently entangled or not. We can reduce the resources for characterization with the entanglement test. Barbieri *et al.* [13] introduced an entanglement witness to test the entanglement of polarized entangled photon pairs. Hayashi *et al.* [14] studied the optimization problem on the entanglement tests with the mathematical statistics in the POVM framework. Hayashi *et al.* [15] treated this optimization problem in the framework of Poisson distribution, which describes the stochastic behavior of the measurement outcomes on the two-photon pairs generated by spontaneous parametric down conversion (SPDC).

In this paper, we will apply the experimental designs proposed in Ref. [15] to test the polarization entangled two-photon pairs generated by SPDC. The two-photon states can be characterized by the correlation of photon detection events in several measurement bases. In experiments, the correlation is measured by coincidence counts of photon detections on selected polarizations. The coincidence counts on a combination of the two from horizontal (H), vertical (V), 45° linear (X), 135° linear (D), clockwise circular (R), and anticlockwise circular (L) polarizations defines a measurement vector. We here use the same representations for the measurement vectors as the state vectors, such as $|HH\rangle$. If the state is close to $|\Phi^{(+)}\rangle=\frac{1}{\sqrt{2}}(|HH\rangle+|VV\rangle)$, the coincidence counts on the vectors $|HH\rangle$, $|VV\rangle$, $|DD\rangle$, $|XX\rangle$, $|RL\rangle$, and $|LR\rangle$ to yield the maximum values, whereas the counts on the vectors $|HV\rangle$, $|VH\rangle$, $|DX\rangle$, $|XD\rangle$, $|RR\rangle$, and $|LL\rangle$ take the minimum values. We will refer to the former vectors as the coincidence vectors, and to the latter as the anticoincidence vectors. The ratio of the minimum counts to the maximum counts measures the degree of entanglement. In the following sections, we formulate the hypothesis testing of entanglement in view of the statistics. We then improve the test by optimizing the allocation on the measurement time for each measurement, considering that the counts on the anticoincidence vectors are much smaller than those on the coincidence vectors.

The test can be further improved, if we utilize the knowledge on the tendency of the entanglement degradation. In general, the error from the maximally entangled states can be anisotropic, which reflects the generation process of the states. We can improve the sensitivity to the entanglement degradation by focusing the measurement on the expected error directions. In the present experiment, we generated polarization entangled photon pairs from a stack of two type-I phase matched nonlinear crystals [16,17], where one nonlinear crystal generates a photon pair polarized in the horizontal direction ($|HH\rangle$), and the other generates a photon pair polarized in the vertical direction ($|VV\rangle$). If the two pairs are indistinguishable, the generated photons are entangled in a two-photon state $\frac{1}{\sqrt{2}}(|HH\rangle+\exp[i\delta]|VV\rangle)$. Otherwise, the state will be a mixture of *HH* pairs and *VV* pairs. The quantum state tomography has shown that only the *HHHH*, *VVVV*, *VVHH*, *HHVV* elements are dominant [17], which implies that the density matrix can be approximated by a classical mixture of $|\Phi^{(+)}\rangle\langle\Phi^{(+)}|$ and $|\Phi^{(-)}\rangle\langle\Phi^{(-)}|$. We can improve the entanglement test on the basis of this property, as described in the following sections.

The construction of this article is following. Section II gives the mathematical formulation concerning statistical hypothesis testing. Section III defines the hypothesis scheme for the entanglement of the two-photon states generated by SPDC. Sections IV–VII describe testing methods as well as the design of the experiment. Section VIII examines the experimental aspects of the hypothesis testing on the entanglement. The designs on the time allocation are evaluated by the experimental data.

II. HYPOTHESIS TESTING FOR PROBABILITY DISTRIBUTIONS

A. Formulation

In this section, we review the fundamental knowledge of hypothesis testing for probability distributions [18]. Suppose that a random variable X is distributed according to a probability measure P_θ identified by an unknown parameter θ, and that the unknown parameter θ belongs to one of the mutually disjoint sets Θ_0 and Θ_1. When the task is to guarantee that the true parameter θ belongs to the set Θ_1 with a certain significance, we choose the null hypothesis H_0 and the alternative hypothesis H_1 as follows:

$$H_0:\theta \in \Theta_0 \quad \text{vs} \quad H_1:\theta \in \Theta_1. \tag{1}$$

The task is then described by a test, where we decide to accept the hypothesis H_1 by rejecting the null hypothesis H_0 with confidence. We make no decision when we cannot reject the null hypothesis. The test based is characterized as a function $\phi(x)$ taking a value in {0, 1}; we can reject H_0 if $\phi(x)=1$, but cannot reject it if $\phi(x)=0$. The test can also be described by the rejection region defined by $\{x|\phi(x)=1\}$, whether the data x falls into the rejection region or not. The function $\phi(x)$ should be designed properly according to the problem to provide an appropriate test.

For example, suppose that mutually independent data $x_1,\cdots,x_n$ obey a normal distribution with unknown mean μ and known variance σ_0^2,

$$P_\mu(x)=\frac{1}{\sqrt{(2\pi\sigma_0^2)^n}}\exp\left[-\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma_0^2}\right]. \tag{2}$$

We introduce a hypothesis testing

$$H_0:\mu \in \{\mu|\mu \le \mu_0\} \quad \text{vs} \quad H_1:\mu \in \{\mu|\mu > \mu_0\} \tag{3}$$

to decide if the mean is larger than a value μ_0 with a confidence level α. Then the function ϕ of the average on the data is defined by $\bar{x}=(1/n)\Sigma_{i=1}^{n}x_i$ as follows:

$$\phi(x)=\begin{cases}1, & \text{if } \sqrt{n}(\bar{x}-\mu_0)/\sigma_0 > z_\alpha,\\ 0, & \text{if } \sqrt{n}(\bar{x}-\mu_0)/\sigma_0 \le z_\alpha,\end{cases} \tag{4}$$

where z_α is given by

$$z_\alpha=\Phi^{-1}(\alpha) \tag{5}$$

$$\Phi(z) := \int_{-\infty}^{z}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy. \tag{6}$$

The rejection region is provided as follows:

$$\{x|\sqrt{n}(\bar{x}-\mu_0)/\sigma_0 > z_\alpha\} \tag{7}$$

in this test.

B. *p* values

In a usual hypothesis testing, we fix our test before applying it to the data. Sometimes, however, we compare tests in a class T by the minimum risk probability to reject the hypothesis H_0 with given data. This probability is called the p value, which depends on the observed data x as well as the subset Θ_0 to be rejected. For a given class T of tests, the p value is defined as follows:

$$\text{p-value} := \min_{\phi\in T:\phi(x)=1}\ \max_{\theta\in\Theta_0} P_\theta(\phi). \tag{8}$$

In the hypothesis test for a normal distribution defined previously [Eqs. (2)–(6)], the p value is given by

$$\text{p-value} = \Phi(\sqrt{n}(\bar{x}-\mu_0)/\sigma_0). \tag{9}$$

The concept of p value is useful for comparison of several designs of experiment.

C. Likelihood test

In mathematical statistics, the likelihood ratio test is often used as a class of standard tests [18]. Likelihood is a function of unknown parameter θ defined as the probability distribution $P_\theta(x)$ for given data x. When both Θ_0 and Θ_1 consist of single elements as $\Theta_0=\{\theta_0\}$ and $\Theta_1=\{\theta_1\}$, the rejection region of the likelihood ratio test is defined

$$\left\{ x \,\middle|\, \frac{P_{\theta_0}(x)}{P_{\theta_1}(x)} < r \right\},$$

where r is a constant, and the ratio $P_{\theta_0}(x)/P_{\theta_1}(x)$ is called the likelihood ratio. In the general case, i.e., the cases where Θ_0 or Θ_1 has at least two elements, the likelihood ratio test is given by its rejection region:

$$\left\{ x \,\middle|\, \frac{\sup_{\theta\in\Theta_0} P_\theta(x)}{\sup_{\theta\in\Theta_1} P_\theta(x)} < r \right\}.$$

In the hypothesis test for a normal distribution defined previously [Eqs. (2)–(6)], the logarithm of the likelihood ratio is given by

$$\log \frac{P_{\mu_0}(x)}{P_{\bar{\mu}=\bar{x}}(x)} = \frac{1}{2\sigma_0^2}\left\{\sum (x_i-\mu_0)^2 - \sum (x_i-\bar{\mu})^2\right\} = -\frac{n(\bar{x}-\mu_0)^2}{2\sigma_0^2}. \quad (10)$$

Therefore, the rejection region of the likelihood ratio test coincide with Eq. (7), if we set $r=\exp[-z_\alpha^2/2]$.

III. HYPOTHESIS TESTING SCHEME FOR ENTANGLEMENT IN SPDC EXPERIMENTS

A. Formalization

This section introduces the hypothesis test for entanglement. We consider the entanglement of two-photon pairs generated by SPDC. The two-photon state is described by a density matrix σ. We assume the two-photon generation process to be identical but individual. Here we measure the entanglement by the fidelity between the generated state σ and the maximally entangled target state $|\Phi^{(+)}\rangle$:

$$F = \langle \Phi^{(+)}|\sigma|\Phi^{(+)}\rangle. \quad (11)$$

The purpose of the test is to guarantee, with a certain significance, that the state is sufficiently close to the maximally entangled state. For this purpose, the hypothesis that the fidelity F is less than a threshold F_0 should be disproved with a small error probability. In mathematical statistics, the above situation is formulated as hypothesis testing; we introduce the null hypothesis H_0 that entanglement is not enough and the alternative H_1 that the entanglement is enough:

$$H_0: F \le F_0 \quad \text{vs} \quad H_1: F > F_0, \quad (12)$$

with a threshold F_0.

The coincidence counts of photon pairs generated in the SPDC experiments can be assumed to be a random variable independently distributed according to a Poisson distribution. In the following, a symbol labeled by a pair (x,y) represents a random variable or parameter related to the measurement vectors $|x,y\rangle$. When the dark count is negligible, the number of detection events (i.e., coincidence counts) n_{xy} on the vectors $|x,y\rangle$ is a random variable according to the Poisson distribution $\mathcal{P}(\lambda\mu_{xy}t_{xy})$ of mean $\lambda\mu_{xy}t_{xy}$, where

(i) λ is a known constant related to the photon detection rate, determined from the averaged photon-pair-generation rate and the detection efficiency,

(ii) $\mu_{xy}=\langle x,y|\sigma|x,y\rangle$ is an unknown constant,

(iii) t_{xy} is a known constant of the time for detection.

The probability function of n_{xy} is

$$n_{xy} = \mathcal{P}(\lambda\mu_{xy}t_{xy}) = \exp(-\lambda\mu_{xy}t_{xy})\frac{(\lambda\mu_{xy}t_{xy})^{n_{xy}}}{n_{xy}!}.$$

Because the detections at different times are mutually independent, n_{xy} is independent of $n_{x'y'}$ ($x\neq x'$ or $y\neq y'$). In this paper, we discuss the quantum hypothesis testing under the above assumptions, whereas Usami *et al.* [12] discussed the state estimation under the same assumptions.

B. Modified visibility

Visibility of the two-photon interference is an indicator of entanglement commonly used in the experiments. The two-photon interference fringe is obtained by the measurement of coincidence counts on the vector $|x,y\rangle$, where the vector $|y\rangle$ is rotated along a great circle on the Poincaré sphere with a fixed vector $|x\rangle$. The visibility is calculated with the maximum and minimum number of coincidence counts, n_{max} and n_{min}, as the ratio $(n_{max}-n_{min})/(n_{max}+n_{min})$. We need to make the measurement with at least two fixed vectors $|x\rangle$ in order to exclude the possibility of the classical correlation. We may choose the two vectors $|H\rangle$ and $|D\rangle$ as $|x\rangle$, for example. However, our decision will contain a bias if we measure the coincidence counts only with two fixed vectors. The bias in the measurement emerges as the fact that the visibility reflects not only the fidelity but also the direction of the deviation of the given state from the maximally entangled target state. Hence, we cannot estimate the fidelity in a statistically proper way from the visibility.

In order to remove the bias based on such a direction, we propose to measure the counts on the coincidence vectors $|HH\rangle$, $|VV\rangle$, $|DD\rangle$, $|XX\rangle$, $|RL\rangle$, and $|LR\rangle$, and the counts of the anticoincidence vectors $|HV\rangle$, $|VH\rangle$, $|DX\rangle$, $|XD\rangle$, $|RR\rangle$, and $|LL\rangle$. The former corresponds to the maximum coincidence counts in the two-photon interference, and the latter does to the minimum. Since the equations

$$|HH\rangle\langle HH| + |VV\rangle\langle VV| + |DD\rangle\langle DD| + |XX\rangle\langle XX| + |RL\rangle\langle RL| + |LR\rangle\langle LR| = 2|\Phi^{(+)}\rangle\langle\Phi^{(+)}| + I \quad (13)$$

and

$$|HV\rangle\langle HV| + |VH\rangle\langle VH| + |XD\rangle\langle XD| + |DX\rangle\langle DX| + |RR\rangle\langle RR| + |LL\rangle\langle LL| = 2I - 2|\Phi^{(+)}\rangle\langle\Phi^{(+)}| \quad (14)$$

hold. In this paper, we call this proposed method the modified visibility method. Using this method, we can test the fidelity between the maximally entangled state $|\Phi^{(+)}\rangle\langle\Phi^{(+)}|$ and the given state σ, using the total number of counts of the coincidence events (the total count on coincidence event) n_1 and the total number of counts of the anticoincidence events (the total count on anticoincidence events) n_2 obtained by measuring on all the vectors with the time $\frac{t}{12}$. holds, we can estimate the fidelity by measuring the sum of the counts of the following vectors: $|HH\rangle$, $|VV\rangle$, $|DD\rangle$, $|XX\rangle$, $|RL\rangle$, and $|LR\rangle$, when λ is known [13,14]. This is because the sum $n_1 := n_{HH}+n_{VV}+n_{DD}+n_{XX}+n_{RL}+n_{LR}$ obeys the Poisson distribution with the expectation value $\lambda\frac{1+2F}{6}t_1$, where the measurement time for each vector is $\frac{t_1}{6}$. We call these vectors the coincidence vectors because these correspond to the coincidence events.

However, since the parameter λ is usually unknown, we need to perform another measurement on different vectors to obtain additional information that also holds. We can estimate the fidelity by measuring the sum of the counts of the following vectors: $|HV\rangle$, $|VH\rangle$, $|DX\rangle$, $|XD\rangle$, $|RR\rangle$, and $|LL\rangle$. The sum $n_2 := n_{HV}+n_{VH}+n_{DX}+n_{XD}+n_{RR}+n_{LL}$ obeys the Poisson distribution $\mathcal{P}(\lambda\frac{2-2F}{6}t_2)$, where the measurement time for each vector is $\frac{t_2}{6}$. Combining the two measurements, we can estimate the fidelity without the knowledge of λ. We call these vectors the anticoincidence vectors because these correspond to the anticoincidence events.

We can also consider a different type of measurement on λ. If we prepare our device to detect all photons, the detected number n_3 obeys the distribution $\mathcal{P}(\lambda t_3)$ with the measurement time t_3. We will refer to it as the total flux measurement. In the following, we consider the best time allocation for estimation and test on the fidelity, by applying methods of mathematical statistics. We will assume that λ is known or estimated from the detected number n_3.

IV. MODIFICATION OF VISIBILITY

The total count on coincidence events n_1 obeys $\mathcal{P}(\lambda\frac{2F+1}{12}t)$, and the count on total anticoincidence event s n_2 obeys the distribution $\mathcal{P}(\lambda\frac{2-2F}{12}t)$. These expectation values μ_1 and μ_2 are given as $\mu_1=\lambda\frac{2F+1}{12}t$ and $\mu_2=\lambda\frac{2-2F}{12}t$. Since the ratio $\frac{\mu_2}{\mu_1+\mu_2}$ is equal to $\frac{2}{3}(1-F)$, we can estimate the fidelity using the ratio $\frac{n_2}{n_1+n_2}$ as $\hat{F}(n_1,n_2)=1-\frac{3}{2}\frac{n_2}{n_1+n_2}$. Its variance is asymptotically equal to

$$\frac{1}{\lambda\left(\frac{t}{3(2F+1)}+\frac{t}{3(2-2F)}\right)}=\frac{(2F+1)(2-2F)}{\lambda t}. \quad (15)$$

Hence, similarly to the visibility, we can check the fidelity by using this ratio.

Indeed, when we consider the distribution under the condition that the total count n_1+n_2 is fixed to n, the random variable n_2 obeys the binomial distribution with the average value $\frac{2}{3}(1-F)n$. Hence, we can apply the likelihood test of the binomial distribution. In this case, by the approximation to the normal distribution, the likelihood test with the risk probability α is almost equal to the test with the rejection region concerning the null hypothesis $H_0:F\leqslant F_0$: $\left\{(n_1,n_2)\middle|\frac{n_2}{n_1+n_2}\leqslant\frac{2}{3}(1-F_0)+\Phi^{-1}(\alpha)\sqrt{\frac{(2-2F_0)(1+2F_0)}{9(n_1+n_2)}}\right\}$, where $\Phi(\alpha):=\int_{-\infty}^{\alpha}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx$. The p value of this kind of test is $\Phi\left(\frac{n_2(2F_0+1)-n_1(2-2F_0)}{\sqrt{(n_1+n_2)(2F_0+1)(2-2F_0)}}\right)$.

V. DESIGN I (λ: UNKNOWN, ONE-STAGE)

In this section, we consider the problem of testing the fidelity between the maximally entangled state $|\Phi^{(+)}\rangle\langle\Phi^{(+)}|$ and the given state σ by performing three kinds of measurement, coincidence, anticoincidence, and total flux, with the times t_1,t_2, and t_3, respectively. The data (n_1,n_2,n_3) obeys the multinomial Poisson distribution $\mathcal{P}(\lambda\frac{2F+1}{6}t_1,\lambda\frac{2-2F}{6}t_2,\lambda t_3)$ with the assumption that the parameter λ is unknown. In this problem, it is natural to assume that we can select the time allocation with the constraint for the total time $t_1+t_2+t_3=t$.

As is shown in Hayashi *et al.* [15], when $F_1 :=0.899\,519\leqslant F\leqslant 1$, the minimum variance of the estimator is asymptotically equal to $\frac{(1-F)(\sqrt{3}+\sqrt{1-F})^2}{\lambda t}$, which is attained by the optimal time allocation $t_1=0$, $t_2=\frac{\sqrt{3}}{(\sqrt{3}+\sqrt{1-F})}t$, $t_3=\frac{\sqrt{1-F}}{\sqrt{3}+\sqrt{1-F}}t$. Otherwise, it is asymptotically equal to $\frac{(2F+1)(1-F)(\sqrt{2F+1}+\sqrt{2-2F})^2}{3\lambda t}$, which is attained by the optimal time allocation $t_1=\frac{\sqrt{2-2F}}{\sqrt{2F+1}+\sqrt{2-2F}}t$, $t_2=\frac{\sqrt{2F+1}}{\sqrt{2F+1}+\sqrt{2-2F}}t$, $t_3=0$. This optimal asymptotic variance is much better than that obtained by the modified visibility method.

Since the optimal allocation depends only on the true parameter F, it is suitable to choose the optimal time allocation at the threshold F_0 for testing whether the fidelity is greater than the threshold F_0. In the both cases, when we perform the optimal allocation, the data obeys the binomial Poisson distribution. Hence, similar to the modified visibility, we can check the fidelity by applying the likelihood test of binomial distribution to the ratio between two kinds of counts.

In the former case, the likelihood test with the risk probability α is almost equal to the test with the rejection region $H_0:F\leqslant F_0$: $\left\{(n_2,n_3)\middle|\frac{n_2}{n_2+n_3}\leqslant\frac{\sqrt{1-F_0}}{\sqrt{3}+\sqrt{1-F_0}}+\frac{\Phi^{-1}(\alpha)}{\sqrt{3}+\sqrt{1-F_0}}\sqrt{\frac{\sqrt{1-F_0}\sqrt{3}}{n_2+n_3}}\right\}$. The p value of this kind of test is $\Phi\left(\frac{n_2\sqrt{3}-n_3\sqrt{1-F_0}}{\sqrt{(n_2+n_3)\sqrt{1-F_0}\sqrt{3}}}\right)$. In the later case, the likelihood test with the risk probability α is almost equal to the test with the rejection region concerning the null hypothesis $H_0:F\leqslant F_0$: $\left\{(n_1,n_2)\middle|\frac{n_2}{n_2+n_1}\leqslant\frac{\sqrt{2-2F_0}}{\sqrt{2F_0+1}+\sqrt{2-2F_0}}+\frac{\Phi^{-1}(\alpha)}{\sqrt{2F_0+1}+\sqrt{2-2F_0}}\sqrt{\frac{\sqrt{2-2F_0}\sqrt{2F_0+1}}{n_1+n_2}}\right\}$. The p value of this kind of test is $\Phi\left(\frac{n_2\sqrt{2F_0+1}-n_1\sqrt{2-2F_0}}{\sqrt{(n_1+n_2)\sqrt{2F_0+1}\sqrt{2-2F_0}}}\right)$.

VI. DESIGN II (λ: KNOWN, ONE-STAGE)

In this section, we consider the case where λ is known. In this case, when $F\geqslant\frac{1}{4}$, the optimal time allocation is $t_2=t$, $t_1=t_3=0$. That is, the count on anticoincidence $(t_1=0;t_2=t)$ is better than the count on coincidence $(t_1=t;t_2=0)$. In fact, Barbieri *et al.* [13] measured the sum of the counts on the anticoincidence vectors $|HV\rangle$, $|VH\rangle$, $|DX\rangle$, $|XD\rangle$, $|RR\rangle$, $|LL\rangle$ to realize the entanglement witness in their experiment. In this case, the count on anticoincidence n_2 obeys the Poisson distribution with the average $\lambda\frac{1-F}{3}t$, the fidelity is estimated by $1-\frac{3n_1}{\lambda t}$. Its variance is $\frac{3(1-F)}{\lambda t}$. Then, the likelihood test with the risk probability α of the Poisson distribution is almost equal to the test with the rejection region: $\left\{n_2 \middle| n_2 \leqslant \frac{\lambda(1-F_0)}{3}t + \Phi^{-1}(\alpha)\sqrt{\frac{\lambda(1-F_0)}{3}t}\right\}$ concerning the null hypothesis $H_0:F \leqslant F_0$. The p value of likelihood tests is $\Phi\left(\frac{n_2-\lambda(1-F_0)t/3}{\sqrt{\lambda(1-F_0)t/3}}\right)$.

When $F<\frac{1}{4}$, the optimal time allocation is $t_1=t$, $t_2=t_3=0$. The fidelity is estimated by $\frac{3n_2}{\lambda t}-\frac{1}{2}$. Its variance is $\frac{3(1+2F)}{2\lambda t}$. Then, the likelihood test with the risk probability α of the Poisson distribution is almost equal to the test with the rejection region: $\left\{n_1 \middle| n_1 \geqslant \lambda\frac{1+2F_0}{6}t+\Phi^{-1}(1-\alpha)\sqrt{\lambda\frac{1+2F_0}{6}t}\right\}$ concerning the null hypothesis $H_0:F\leqslant F_0$. The p value of likelihood tests is $\Phi\left(\frac{-n_1+\lambda(1+2F_0)t/6}{\sqrt{\lambda(1+2F_0)t/6}}\right)$.

VII. DESIGN III (λ: KNOWN, TWO-STAGE)

Next, for a further improvement, we minimize the variance by optimizing the time allocation t_{HV}, t_{VH}, t_{DX}, t_{XD}, t_{RR}, and t_{LL} between the anticoincidence vectors $B=\{|HV\rangle,|VH\rangle,|DX\rangle,|XD\rangle,|RR\rangle,|LL\rangle\}$, respectively, under the restriction of the total measurement time: $t_{HV}+t_{VH}+t_{DX}+t_{XD}+t_{RR}+t_{LL}=t$. The number of the counts n_{xy} obeys Poisson distribution $\mathcal{P}(\lambda\mu_{xy}t_{xy})$ with the unknown parameter $\mu_{xy}=\langle x_A y_B|\sigma|x_A y_B\rangle$. The minimum value of estimation of the fidelity $F=1-\frac{1}{2}\sum_{(x,y)\in B}\mu_{xy}$ is

$$\frac{\left(\sum_{(x,y)\in B}\sqrt{\mu_{x,y}}\right)^2}{4\lambda t}, \tag{16}$$

which is attained by the optimal time allocation

$$t_{xy}=\frac{\sqrt{\mu_{xy}}t}{\sum_{(x,y)\in B}\sqrt{\mu_{x,y}}}, \tag{17}$$

which is called Neyman allocation and is used in sampling design [19].

However, this optimal time allocation is not applicable in the experiment, because it depends on the unknown parameters μ_{HV}, μ_{VH}, μ_{DX}, μ_{XD}, μ_{RR}, and μ_{LL}. In order to resolve this problem, we use a two-stage method, where the total measurement time t is divided into t_f for the first stage and t_s for the second stage under the condition of $t=t_f+t_s$. In the first stage, we measure the counts on the anticoincidence vectors for $t_f/6$ and estimate the expectation value for Neyman allocation on measurement time t_s. In the second stage, we measure the counts on anticoincidence vectors $|x_A y_B\rangle$ according to the estimated Neyman allocation. The two-stage method is formulated as follows:

(i) The measurement time for each vector in the first stage is given by $t_f/6$.

(ii) In the second stage, we measure the counts on a vector $|x_A y_B\rangle$ the measurement time $\tilde{t}_{xy}$ defined

$$\tilde{t}_{xy}=\frac{m_{xy}}{\sum_{(x,y)\in B}\sqrt{m_{xy}}}(t-t_f),$$

where m_{xy} is the observed count in the first stage.

(iii) Define $\hat{\mu}_{xy}$ and $\hat{F}$,

$$\hat{\mu}_{xy}=\frac{n_{xy}}{\lambda\tilde{t}_{xy}}, \quad \hat{F}=1-\frac{1}{2}\sum_{(x,y)\in B}\hat{\mu}_{x,y},$$

where $n_{x,y}$ is the number of the counts on $|x_A y_B\rangle$ for $\tilde{t}_{xy}$. Then, we can estimate the fidelity by $\hat{F}$.

We can test our hypothesis by using the likelihood test, which has the rejection region:

$$\left\{\vec{n}\middle| \sup_{\vec{\mu}\cdot\vec{w}\geqslant c_0}\mathcal{P}(\vec{\mu})(\vec{n}) \leqslant r \sup_{\vec{\mu}\cdot\vec{w}<c_0}\mathcal{P}(\vec{\mu})(\vec{n})\right\},$$

where $w_i := \frac{1}{2\lambda\tilde{t}_i}$ and $c_0 := 1-F_0$. However, it is very difficult to choose r such that the likelihood test has a given risk probability α. In order to resolve this problem, Hayashi *et al.* [15] proposed a better method. For treating this method, we number the anticoincidence vectors by $i=1,\ldots,6$. Then, the rejection region with the risk probability α is given by $\mathcal{B}_{R_\alpha}$, where the set $\mathcal{B}_R$ is defined by

$$\mathcal{B}_R := \left\{\vec{n}'\middle| \sum_{i=1}^{6}\frac{n_i'}{\tilde{\mu}_i(R)} \leqslant 1\right\}, \tag{18}$$

and $\tilde{\mu}_i(R)$ are defined as follows:

$$\frac{c_0}{w_i}-\tilde{\mu}_i(R)+\tilde{\mu}_i(R)\log\frac{\tilde{\mu}_i(R)w_i}{c_0}=R \quad \text{if } R\leqslant R_{0,i}, \tag{19}$$

$$\frac{c_0}{w_M}+\tilde{\mu}_i(R)\log\frac{w_M-w_i}{w_M}=R \quad \text{if } R>R_{0,i}, \tag{20}$$

where $w_M := \max_i w_i$ and $R_{0,i} := \frac{c_0}{w_M}+\frac{c_0(w_M-w_i)}{w_i w_M}\log\frac{w_M-w_i}{w_M}$. The value R_α is defined by

$$-\min_{i\neq j} z_{i,j}(R_\alpha)=\Phi^{-1}(\alpha), \tag{21}$$

where $z_{i,j}(R)$ is given by

$$z_{i,j}(R) := \begin{cases} \dfrac{x_i(R)}{\sqrt{y_i(R)}} & \text{if } \dfrac{2x_j(R)y_i(R)}{x_i(R)y_i(R)+x_i(R)y_j(R)} \geqslant 1, \\ \dfrac{x_j(R)}{\sqrt{y_j(R)}} & \text{if } \dfrac{2x_i(R)y_j(R)}{x_j(R)y_j(R)+x_j(R)y_i(R)} \geqslant 1. \end{cases} \tag{22}$$

Otherwise,

$$z_{i,j}(R) := \frac{2\{x_i(R)x_j(R)[y_i(R)+y_j(R)] - x_i(R)^2 y_j(R) - x_j(R)^2 y_i(R)\}}{\sqrt{[x_i(R)-x_j(R)][y_i(R)-y_j(R)]}\sqrt{x_i(R)y_j(R)^2 + x_j(R)y_i(R)^2 - y_i(R)y_j(R)[x_i(R)+x_j(R)]}}, \tag{23}$$

where $x_i(R) := \frac{c_0}{w_i\tilde{\mu}_i(R)} - 1$ and $y_i(R) := \frac{c_0}{w_i\tilde{\mu}_i(R)^2}$.

Concerning these tests, the p value is calculated to

$$\Phi[-\min_{i\neq j} z_{i,j}(R'(\vec{n}))], \tag{24}$$

where $R'(\vec{n})$ is defined

$$\sum_{i=1}^{6} \frac{k_i}{\tilde{\mu}_i(R'(\vec{n}))} = 1. \tag{25}$$

VIII. ANALYSIS OF EXPERIMENTAL DATA

The experimental setup for the hypothesis testing is shown in Fig. 1. The nonlinear crystals (BBO), the optical axis of which were set to orthogonal to one another, were pumped by a pulsed UV light polarized in 45° direction to the optical axis of the crystals. One nonlinear crystal generates two photons polarized in the horizontal direction ($|HH\rangle$) from the vertical component of the pump light, and the other generates ones polarized in the vertical direction ($|VV\rangle$) from the horizontal component of the pump. The second harmonic of the mode-locked Ti:S laser light of about 100 fs duration and 150 mW average power was used to pump the nonlinear crystal. The wavelength of SPDC photons was thus 800 nm. The group velocity dispersion and birefringence in the crystal may differ the space-time position of the generated photons and make the two processes to be distinguished [17]. Fortunately, this timing information can be erased by compensation; the horizontal component of the pump pulse should arrive at the nonlinear crystals earlier than the vertical component. The compensation can be done by putting a set of birefringence plates (quartz) and a variable wave plate before the crystals. We could control the two photon state from highly entangled states to separable states by shifting the compensation from the optimal setting.

The count on the vector $|x_A y_B\rangle$ was measured by adjusting the half wave plates (HWPs) and the quarter wave plates (QWPs) in Fig. 1. We accumulated the counts for one second, and recorded the counts every one second. Therefore, the time allocation of the measurement time on a vector must be an integral multiple of one second. Figure 2 shows the histogram of the counts in one second on the vector

$$B = \{|VH\rangle, |HV\rangle, |XD\rangle, |DX\rangle, |RR\rangle, |LL\rangle\},$$

when the visibility of the two-photon states was estimated to be 0.92. The measurement time was 40 sec on each vector. The distribution of the coincidence events obeys the Poisson distribution. Only small numbers of counts were observed on the vectors $|HV\rangle$ and $|VH\rangle$. Those observations agree with the prediction, therefore, we expect that the hypothesis testing in the previous sections can be applied.

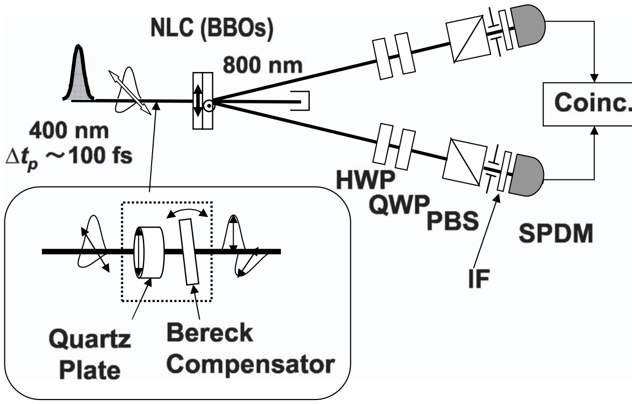

FIG. 1. Schematic of the entangled photon pair generation by spontaneous parametric down conversion. Cascade of the nonlinear crystals (NLC) generate the photon pairs. Group velocity dispersion and birefringence in the NLCs are pre-compensated with quartz plates and a Bereck compensator. Two-photon states are analyzed with half wave plates (HWP), quarter wave plates (QWP), and polarization beam splitters (PBS). Interference filters (IF) are placed before the single photon counting modules (SPCM).

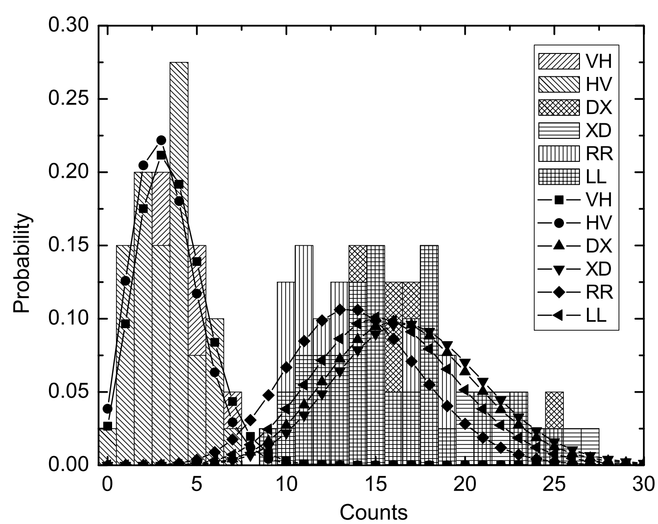

FIG. 2. Distribution of the counts obtained in one second on the vectors $|VH\rangle$, $|HV\rangle$, $|XD\rangle$, $|DX\rangle$, $|RR\rangle$, and $|LL\rangle$. Bars present the histograms of the measured numbers, and lines show the Poisson distribution with the mean values estimated from the experiment. Measurement time was 40 sec for each vector.

In the following, we compare four testing methods on experimental data with the fixed total time t. The testing method employs the different time allocations $\{t_{HH}, t_{VV}, t_{DD}, t_{XX}, t_{RL}, t_{LR}, t_{HV}, t_{VH}, t_{DX}, t_{XD}, t_{RR}, t_{LL}\}$ between the measurement vectors:

(i) *Modified visibility method*: λ is unknown. The coincidence and the anticoincidence are measured with the equal time allocation;

$$t_{HH}=t_{VV}=t_{DD}=t_{XX}=t_{RL}=t_{LR}=t_{HV}=t_{VH}=t_{DX}=t_{XD}=t_{RR}=t_{LL}=\frac{t}{12}. \qquad (26)$$

(ii) *Design I*: λ is unknown. The counts on coincidence and anticoincidence are measured with the optimal time allocation at the target threshold $F_0 \leqslant 0.899\,519$;

$$t_{HH}=t_{VV}=t_{DD}=t_{XX}=t_{RL}=t_{LR}=\frac{t_1}{6},$$

$$t_{HV}=t_{VH}=t_{DX}=t_{XD}=t_{RR}=t_{LL}=\frac{t_2}{6}, \qquad (27)$$

where

$$t_1=\frac{t\sqrt{2-2F_0}}{\sqrt{2F_0+1}+\sqrt{2-2F_0}},$$

$$t_2=\frac{t\sqrt{2F_0+1}}{\sqrt{2F_0+1}+\sqrt{2-2F_0}}. \qquad (28)$$

(iii) *Design II*: λ is known. Only the counts on anticoincidence are measured with the equal time allocation at the target threshold $F_0 \geqslant 1/4$;

$$t_{HV}=t_{VH}=t_{DX}=t_{XD}=t_{RR}=t_{LL}=\frac{t}{6}. \qquad (29)$$

(iv) *Design III*: λ is known. Only the counts on anticoincidence are measured. The time allocation is given by the two-stage method:

$$t_{HV}=t_{VH}=t_{DX}=t_{XD}=t_{RR}=t_{LL}=\frac{t_f}{6} \qquad (30)$$

in the first stage, and

$$\tilde{t}_{xy}=\frac{m_{xy}}{\sum_{(x,y)\in B}\sqrt{m_{xy}}}(t-t_f) \qquad (31)$$

in the second stage. The observed count m_{xy} in the first stage determines the time allocation in the second stage.

We have compared the p values at the fixed threshold $F_0=7/8=0.875$ with the total measurement time $t=240$ sec. As shown in Sec. II B, the p value measures the minimum risk probability to reject the hypothesis H_0, i.e., the probability to make an erroneous decision to accept insufficiently entangled states with the fidelity less than the threshold. The results of the experiment and the analysis of obtained data are described in the following.

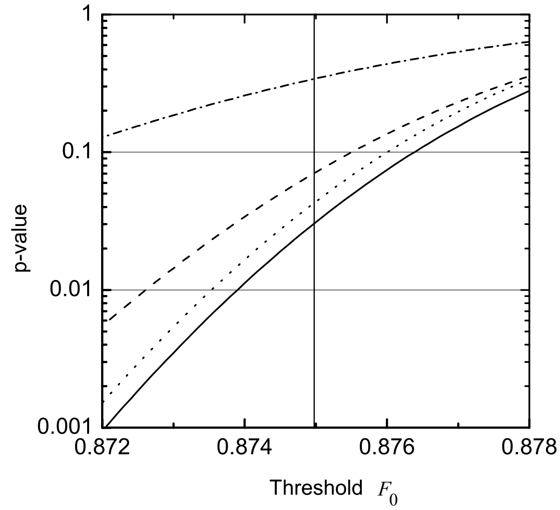

FIG. 3. Calculated p value as a function of the threshold. Dash-dotted line: (i) the modified visibility, dash line: (ii) design I, dotted line: (iii) design II, solid line: (iv) design III.

In the method (i), we measured the counts on each vector for 20 sec. We obtained $n_1=9686$ and $n_2=868$ in the experiment, which yielded the p value 0.343.

In the method (ii), the optimal time allocation was calculated with (28) to be $t_1=55.6$ sec and $t_2=184.4$ sec. However, since the time allocation should be the integral multiple of second in our experiment, we used the time allocation $t_1=54$ and $t_2=186$. That is, we measure the count on each coincidence vector for 9 sec and on each anticoincidence vector for 31 sec. We obtained $n_1=7239$ and $n_2=2188$ in the experiment, which yielded the p value 0.0715.

In method (iii), we measured the count on each anticoincidence vector for 40 sec. We used $\lambda=290$ estimated from another experiment. We obtained $n=2808$ in the experiment, which yielded the p value 0.0438.

In the method (iv), the calculation is rather complicated. Similarly to (iii), λ was estimated to be 290 from another experiment. In the first stage, we measured the count on each anticoincidence vector for $t_f/6=1$ sec. We obtained the counts 6, 3, 13, 20, 11, and 23 on the vectors $|HV\rangle$, $|VH\rangle$, $|DX\rangle$, $|XD\rangle$, $|RR\rangle$, and $|LL\rangle$, respectively. We made the time allocation of remaining 234 sec for the second stage according to (31), and obtained $t_{HV}=28.14$, $t_{VH}=19.90$, $t_{DX}=41.42$, $t_{XD}=51.37$, $t_{RR}=38.10$, and $t_{LL}=55.09$. Since the time allocation should be the integral multiple of second in our experiment, we used the time allocation $\{t_{HV}, t_{VH}, t_{DX}, t_{XD}, t_{RR}, t_{LL}\}=\{28,20,42,51,38,55\}$. We obtained the counts on anticoincidence $n_{HV}=99$, $n_{VH}=66$, $n_{DX}=703$, $n_{XD}=863$, $n_{RR}=531$, and $n_{LL}=853$. Applying the counts and the time allocation to the formula (24), we obtained the p values to be 0.0310.

The p values obtained in the four methods are summarized in the table. We also calculated the p values at different values of the threshold F_0 as shown in Fig. 3. We fixed time allocation for design I at $t_1=54$ s and $t_2=186$ s. As clearly seen, the optimal time allocation between the coincidence vectors measurement and the anticoincidence vectors measurement reduces the risk of a wrong decision on the fidelity (the p value) in analyzing the experimental data. The counts on the anticoincidence vectors is much more sensitive to the degradation of the entanglement. This matches our intuition that the deviation from zero provides a more efficient measure than that from the maximum does. The comparison between (iii) and (iv) shows that the risk can be reduced further by the time allocation between the anticoincidence vectors, as shown in Fig. 3. The optimal (Neyman) allocation implies that the measurement time should be allocated preferably to the vectors that yield more counts. Under the present experimental conditions, the optimal allocation reduces the risk probability to about 75%. The improvement should increased as the fidelity. However, the experiment showed almost no gain when the visibility was larger than 0.95. In such high visibility, errors from the maximally entangled state are covered by dark counts, which are independent of the setting of the measurement apparatus.

	(i)	(ii)	(iii)	(iv)
p value at 0.875	0.343	0.0715	0.0438	0.0310

IX. CONCLUSION

We have applied the formulation of the hypothesis testing scheme and the design of experiment for the hypothesis testing of entanglement to the two-photon state generated by SPDC. Using this scheme, we have handled the fluctuation in the experimental data properly. It has been experimentally demonstrated that the optimal time allocation improves the test in the terms of p values: the measurement time should be allocated preferably to the anticoincidence vectors in order to reduce the minimum risk probability. This design is particularly useful for the experimental test, because the optimal time allocation depends only on the threshold of the test. We do not need any further information of the probability distribution and the tested state. We have also experimentally demonstrated that the test can be further improved by optimizing time allocation among the anticoincidence vectors by using the two-stage method, when the error from the maximally entangled state is anisotropic.

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