Splitting quantum information via W states

Shi-Biao Zheng*

Department of Electronic Science and Applied Physics, Fuzhou University, Fuzhou 350002, People's Republic of China

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We describe a procedure for splitting quantum information into two or more parts so that if and only if the recipients cooperate, the original qubit can be reconstructed. Our scheme uses W-type entangled states as the quantum channel and thus the scheme is robust against decoherence. We illustrate the procedure in the ion-trap system, but the idea can also be realized in other systems.

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Entanglement, one of the most striking features of quantum mechanics, not only provides possibilities for testing quantum mechanics against local hidden-variable theories, but also has many practical applications in quantum information processing. One well-known example is teleportation [1]. In teleportation the sender (Alice) and the receiver (Bob) initially share a maximally entangled state of two particles. Alice then performs a joint measurement on her particle of the entangled pair and the particle whose state is to be teleported. With the outcome transmitted to Bob via a classical channel, he can recover the teleported state on his particle by an appropriate transformation.

On the other hand, Hillery *et al.* [2] have described a procedure for realizing quantum secret sharing by using multiparticle maximally entangled states, i.e., the Greenberger-Horne-Zeilinger (GHZ) states [3]. For classical information a shared key can be established between one party and two or more others, all of whom should work together to read the message. In the case of quantum information, Alice teleported a qubit in such a way that the qubit can be recovered if and only if two or more parties at the receiving end agree to collaborate. Karlsson *et al.* [4] and Cleve *et al.* [5] have also proposed different schemes for quantum secret sharing and quantum information splitting, which require the particle carrying the quantum information be first entangled with the other particles to share the information. Murao *et al.* [6] have shown that a telecloning scheme can also be applied to split quantum information into two or more parts. In this case the original state can be reconstructed if and only if the output clones and an ancilla is brought together. In order to broadcast quantum information from one sender to M recipients, one should exploit entanglement of $2M$ particles. In this paper we present an alternative scheme for splitting quantum information into M parts. Our scheme uses W-type entangled states of $M+1$ qubits as the quantum channel. The W state has some interesting properties [7]. For example, it retains bipartite entanglement when any one of the three qubits is traced out and thus it is much more robust than the GHZ states, as demonstrated in the recent experiment [8]. Thus the scheme is more robust against decoherence than the scheme of Ref. [2]. In comparison with the schemes of Refs. [4,5], our scheme does not require the particle carrying the quantum information be first entangled with the other particles. The number of required qubits is smaller than that in the scheme of Ref. [6].

We assume that Alice possesses a qubit, which is in an unknown state

$$|\phi_1\rangle = \alpha|0_1\rangle + \beta|1_1\rangle. \tag{1}$$

Alice, Bob, and Charie share an entangled state of the W type,

$$\frac{1}{2}|0_2 0_3 1_4\rangle + \frac{1}{2}|0_2 1_3 0_4\rangle + \frac{1}{\sqrt{2}}|1_2 0_3 0_4\rangle. \tag{2}$$

The state of the whole system is

$$\alpha\left(\frac{1}{2}|0_1 0_2 0_3 1_4\rangle + \frac{1}{2}|0_1 0_2 1_3 0_4\rangle + \frac{1}{\sqrt{2}}|0_1 1_2 0_3 0_4\rangle\right) + \beta\left(\frac{1}{2}|1_1 0_2 0_3 1_4\rangle + \frac{1}{2}|1_1 0_2 1_3 0_4\rangle + \frac{1}{\sqrt{2}}|1_1 1_2 0_3 0_4\rangle\right). \tag{3}$$

Then Alice performs a joint measurement on her two qubits with respect to Bell states [9]

$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|0_1 0_2\rangle \pm |1_1 1_2\rangle),$$

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|0_1 1_2\rangle \pm |1_1 0_2\rangle). \tag{4}$$

The particles held by Bob and Charie collapse onto one of the following entangled states:

$$\frac{1}{\sqrt{2}}\alpha(|0_3 1_4\rangle + |1_3 0_4\rangle) \pm \beta|0_3 0_4\rangle,$$

$$\alpha|0_3 0_4\rangle \pm \frac{1}{\sqrt{2}}\beta(|0_3 1_4\rangle + |1_3 0_4\rangle). \tag{5}$$

After the Bell-state measurement the quantum information is encoded in the coefficients of the two components $\frac{1}{\sqrt{2}}(|0_3 1_4\rangle + |1_3 0_4\rangle)$ and $|0_3 0_4\rangle$. The two components are not symmetric, with one component being a Bell state and the other being a product state. Due to this fact, the initial W-type state used as the quantum channel should also be asymmetric, i.e., a nonmaximally entangled W-type state, as shown in Eq. (2). The nonclone theorem allows only one particle to be in the original state of particle 1 [10]. Neither

*Email address: sbzheng@pub5.fz.fj.cn

Bob nor Charie can recover the original qubit only by local operations on their own particles. In order to reconstruct the original qubit the two particles should be brought together to the same location. We here assume that Bob and Charie agree to let Charie possess the final qubit and thus Bob sends his particle to Charie. Then a unitary transformation is performed on the two particles

$$\frac{1}{\sqrt{2}}(|0_3 1_4\rangle + |1_3 0_4\rangle) \rightarrow |0_3 1_4\rangle,$$

$$|0_3 0_4\rangle \rightarrow |0_3 0_4\rangle). \tag{6}$$

In this case the fourth particle evolves into

$$(\alpha|1_4\rangle \pm \beta|0_4\rangle),$$

$$(\alpha|0_4\rangle \pm \beta|1_4\rangle), \tag{7}$$

with the third particle left in the state $|0_3\rangle$. With the outcome of the joint measurement transmitted to Charie he can perform a rotation on his particle to reconstruct the original qubit. In the following we illustrate the idea in the ion-trap system, but it should be applicable to other systems.

We consider four two-level ions confined in a linear trap. Assume that the first ion is initially in the state of Eq. (1). We simultaneously drive the other three ions with a laser beam, tuned to $\omega_0 - \nu - \delta$, where ω_0 is the frequency of the electronic transition $|1\rangle \rightarrow |0\rangle$ and ν is the frequency of the center-of-mass mode of the collective motion of the ions. We here assume that $\delta \ll \nu$. In this case, the excitation of the stretch modes is far off-resonant and thus can be discarded. We consider the resolved sideband regime, where the vibrational frequency ν is much larger than other characteristic frequencies. In the Lamb-Dicke regime, i.e., $\eta\sqrt{\bar{n}+1} \ll 1$, with η being the Lamb-Dicke parameter and $\bar{n}$ being the mean phonon number of the center-of-mass mode, the interaction Hamiltonian can be approximated by

$$H_{i,1} = i\eta\Omega e^{-i\phi}\sum_{j=2}^{4} \sigma_j^+ a e^{i\delta t} + \text{H.c.}, \tag{8}$$

where $\sum$ represents summation, $\sigma_j^+ = |1_j\rangle\langle 0_j|$ is the spin flip operator, and a is the annihilation operators for the collective vibrational mode. The Hamiltonian is mathematically identical to that describing the interaction of two-level atoms with a single-mode cavity mode, with the cavity mode replaced by the vibrational mode [11]. In the case $\delta \gg \eta\Omega\sqrt{\bar{n}+1}$, there is no energy exchange between the external and internal degrees of freedom. The energy-conserving transitions are between $|1_j 0_k n\rangle$ and $|0_j 1_k n\rangle$. Then the effective Hamiltonian is

$$H_{e,1} = \lambda\left[\sum_{j=2}^{4}(|1_j\rangle\langle 1_j|aa^\dagger - |0_j\rangle\langle 0_j|a^\dagger a) + \sum_{j,k=2}^{4}\sigma_j^+\sigma_k^-\right], \quad j \neq k. \tag{9}$$

The first and second terms describe the phonon-number-dependent Stark shifts, and the third term describes the coupling between the jth and kth ions induced by the virtual vibrational excitation. The effective coupling strength is independent of the phonon-number because there are two transition paths that interfere destructively. Assume that the vibrational mode is initially in the vacuum state. It will remain in this state throughout the procedure since the vibrational quantum number conserves. Then H_e reduces to

$$H_{e,1} = \lambda\left(\sum_{j=2}^{4}|1_j\rangle\langle 1_j| + \sum_{j,k=2}^{4}\sigma_j^+\sigma_k^-\right), \quad j \neq k, \tag{10}$$

where

$$\lambda = (\eta\Omega)^2/\delta. \tag{11}$$

Assume that ions 2, 3, and 4 are initially in the state $|1_2\rangle$, $|0_3\rangle$, and $|0_4\rangle$, respectively. Thus the initial state of the ionic system is $|1_2 0_3 0_4\rangle$. After an interaction time τ_1 the state of the system is

$$|\psi(\tau_1)\rangle = \frac{2+e^{-i3\lambda\tau_1}}{3}|1_2 0_3 0_4\rangle + \frac{e^{-i3\lambda\tau_1}-1}{3}(|0_2 1_3 0_4\rangle + |0_2 0_3 1_4\rangle). \tag{12}$$

We choose $\lambda\tau_1 = [\pi - \arccos(1/8)]/3$. Then we obtain the three-particle W state

$$|\psi(\tau_1)\rangle = \frac{1}{\sqrt{2}}e^{i\theta}|1_2 0_3 0_4\rangle + \frac{1}{2}(|0_2 1_3 0_4\rangle + |0_2 0_3 1_4\rangle), \tag{13}$$

where

$$\theta = -\arctan\frac{\sin(3\lambda\tau_1)}{1+\cos(3\lambda\tau_1)} - \arctan\frac{\sin(3\lambda\tau_1)}{2+\cos(3\lambda\tau_1)}. \tag{14}$$

We here have discarded the common phase $\arctan\frac{\sin(3\lambda\tau_1)}{1+\cos(3\lambda\tau_1)}$. Performing the single-qubit rotation $|1\rangle_1 \rightarrow e^{-i\theta}|1\rangle_1$, we obtain the state of Eq. (2).

Now we perform the transformation $|0_2\rangle \rightarrow i|0_2\rangle$, leading to

$$|\Psi^\pm\rangle \rightarrow \frac{1}{\sqrt{2}}(i|0_1 0_2\rangle \pm |1_1 1_2\rangle),$$

$$|\Phi^\pm\rangle \rightarrow \frac{1}{\sqrt{2}}(|0_1 1_2\rangle \pm i|1_1 0_2\rangle). \tag{15}$$

Then we excite the first two ions with two lasers of frequencies $\omega_0 + \nu + \delta$ and $\omega_0 - \nu - \delta$. Under the conditions $\delta \ll \nu$ and $\eta\sqrt{\bar{n}+1} \ll 1$ the Hamiltonian in the interaction picture is

$$H_i = i\eta\Omega e^{-i\phi}\sum_{j=1}^{2}\sigma_j^+(a^\dagger e^{-i\delta t} + a e^{i\delta t}) + \text{H.c.} \tag{16}$$

When $\delta \gg \eta\Omega$ and $\phi = \pi/2$ the effective Hamiltonian is [12]

$$H_{e,2}=\lambda\left[\sum_{j=1}^{2}(|1_j\rangle\langle 1_j|+|0_j\rangle\langle 0_j|)+2(\sigma_1^{\dagger}\sigma_2^{\dagger}+\sigma_1^{+}\sigma_2^{-}+\text{H.c.})\right], \tag{17}$$

with λ given by Eq. (11). The appearance of the factor 2 is due to the fact that there are two pairs of paths for each transition [12]. The Hamiltonian leads to the transition

$$\begin{aligned}
|1_1 1_2\rangle &\to e^{-2i\lambda\tau_2}[\cos(2\lambda\tau_2)|1_1 1_2\rangle - i\sin(2\lambda\tau_2)|0_1 0_2\rangle],\\
|0_1 0_2\rangle &\to e^{-2i\lambda\tau_2}[\cos(2\lambda\tau_2)|0_1 0_2\rangle - i\sin(2\lambda\tau_2)|1_1 1_2\rangle],\\
|1_1 0_2\rangle &\to e^{-2i\lambda\tau_2}[\cos(2\lambda\tau_2)|1_1 0_2\rangle - i\sin(2\lambda\tau_2)|0_1 1_2\rangle],\\
|0_1 1_2\rangle &\to e^{-2i\lambda\tau_2}[\cos(2\lambda\tau_2)|0_1 1_2\rangle - i\sin(2\lambda\tau_2)|1_1 0_2\rangle].
\end{aligned} \tag{18}$$

This corresponds to

$$\begin{aligned}
\frac{1}{\sqrt{2}}(i|0_1 0_2\rangle \pm |1_1 1_2\rangle) \to \frac{1}{\sqrt{2}}e^{-2i\lambda\tau_2}\{&i[(\cos 2\lambda\tau_2)\mp\sin(2\lambda\tau_2)]\\
&\times[|0_1 0_2\rangle]+[\sin(2\lambda\tau_2)\pm(\cos 2\lambda\tau_2)]\\
&\times|1_1 1_2\rangle\},\\
\frac{1}{\sqrt{2}}(|0_1 1_2\rangle \pm i|1_1 0_2\rangle) \to \frac{1}{\sqrt{2}}e^{-2i\lambda\tau_2}\{&[(\cos 2\lambda\tau_2)\pm\sin(2\lambda\tau_2)]\\
&\times[|0_1 1_2\rangle]+i[-\sin(2\lambda\tau_2)\\
&\pm(\cos 2\lambda\tau_2)]|1_1 0_2\rangle\}.
\end{aligned} \tag{19}$$

Choosing $2\lambda\tau_2=\pi/4$, we obtain

$$|\Psi^{\pm}\rangle \to \begin{cases}|1_1 1_2\rangle\\ i|0_1 0_2\rangle,\end{cases}$$

$$|\Phi^{\pm}\rangle \to \begin{cases}|0_1 1_2\rangle\\ -i|1_1 0_2\rangle.\end{cases} \tag{20}$$

The common phase factor $e^{-i\pi/4}$ is discarded. Hence, the joint measurement can be achieved by detecting ions 1 and 2 separately. After the Bell-state measurement, the third and fourth ions collapse to one of the states of Eq. (5).

Then we perform the rotation $|1_3\rangle\to i|1_3\rangle$, leading to

$$\frac{1}{\sqrt{2}}(|0_3 1_4\rangle+|1_3 0_4\rangle)\to\frac{1}{\sqrt{2}}(|0_3 1_4\rangle+i|1_3 0_4\rangle). \tag{21}$$

We simultaneously drive the third and fourth ions with a laser beam, tuned to $\omega_0-\nu-\delta$. Under the above-mentioned conditions the effective Hamiltonian is

$$H_{e,3}=\lambda\left(\sum_{j=3}^{4}|1_j\rangle\langle 1_j|+\sigma_3^{+}\sigma_4^{-}+\sigma_3^{-}\sigma_4^{+}\right). \tag{22}$$

Thus we obtain the evolution

$$|0_3 1_4\rangle \to e^{-i\lambda\tau_3}[\cos(\lambda\tau_3)|0_3 1_4\rangle - i\sin(\lambda\tau_3)|1_3 0_4\rangle],$$

$$|1_3 0_4\rangle \to e^{-i\lambda\tau_3}[\cos(\lambda\tau_3)|1_3 0_4\rangle - i\sin(\lambda\tau_3)|0_3 1_4\rangle]. \tag{23}$$

Choosing $\lambda\tau_3=\pi/4$ we have

$$\frac{1}{\sqrt{2}}(|0_3 1_4\rangle+i|1_3 0_4\rangle)\to e^{-i\pi/4}|0_3 1_4\rangle. \tag{24}$$

On the other hand, $|0_3 0_4\rangle$ undergoes no change. Then we perform the transformation $|1_4\rangle\to e^{i\pi/4}|1_4\rangle$, leading to the transformation of Eq. (6). By this way, the fourth ion is in one of the states given by Eq. (7). According to the outcome of the Bell-state measurement, one can perform an appropriate rotation to reconstruct the initial state of particle 1.

We now consider the error induced by decoherence. According to the experiment of the Innsbruck group [8], one Zeeman level of the $S_{1/2}$ ground state of $^{40}Ca^{+}$ ions can act as the state $|0\rangle$, while one Zeeman level of the metastable $D_{5/2}$ state can act as the state $|1\rangle$. The lifetime of the metastable state is very long and thus the spontaneous emission is negligible. For the setup of the NIST group, one can use the Raman transition between a pair of the hyperfine $S_{1/2}$ ground states of $^{9}Be^{+}$ ions through a virtual excited state to suppress the spontaneous emission [13]. Setting $\Omega=0.1\nu$, $\delta=0.2\nu$, and $\eta=0.1$, we have $\lambda\simeq 0.5\times 10^{-3}\nu$. The total time required to complete the procedure is about $T=0.54\pi/\lambda\simeq 3.393\times 10^{3}/\nu$. The probability of the vibrational mode being excited via the laser excitation is on the order of $P\sim(\eta\Omega)^2/\delta^2=0.0025$. Set $\Gamma=0.005\nu$ [14], with Γ being the decoherence rate of the vibrational mode. Then the error caused by the decoherence of the vibrational mode is on the order of $P\Gamma T\simeq 0.0424$.

We suppose the fluctuation of the Rabi frequency of the laser field to be δ_Ω. Then the infidelity caused by the fluctuation is about $\varepsilon_r=\frac{4}{9}(\pi\delta_\Omega/6\Omega)^2+4(\pi\delta_\Omega/4\Omega)^2$. Setting $\delta_\Omega=0.05\Omega$ we have $\varepsilon_r=0.00647$. We note that the effective Hamiltonians of Eqs. (10) and (22) is independent of the phase of the laser field. Thus the state evolutions governed by these two Hamiltonians are not affected by the fluctuation of the phase of the laser field. On the other hand, in order to obtain the effective Hamiltonian of Eq. (17) we have assumed that both lasers have the same phase $\pi/2$. Suppose the fluctuations of the phases of the two laser fields to be δ_{ϕ_1} and δ_{ϕ_2}. The error caused by the fluctuations is about $\varepsilon_p=\frac{1}{4}(\delta_{\phi_1}+\delta_{\phi_2})^2$. Setting $\delta_{\phi_1}=\delta_{\phi_2}=0.1$ we obtain $\varepsilon_p=0.01$. With all of the above-mentioned nonideal situations being considered, the fidelity is about 0.941.

The scheme can easily be generalized to split quantum information among more than two parties. In order to do so, the sender and the M parties should initially share an M-particle entangled state

$$\frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{M}}(|0_2 0_3\cdots 0_M 1_{M+1}\rangle+|0_2 0_3\cdots 0_{M-1}1_M 0_{M+1}\rangle+\cdots+|0_2 1_3 0_4\cdots 0_M 0_{M+1}\rangle)+|1_2 0_3\cdots 0_M 0_{M+1}\rangle\right]. \tag{25}$$

After Alice performs a joint measurement on her two qubits the $M-1$ particles at the receiving ends collapse onto one of the following states:

$$\frac{1}{\sqrt{M}}\alpha(|0_3 0_4 \cdots 0_M 1_{M+1}\rangle + |0_3 0_4 \cdots 0_{M-1} 1_M 0_{M+1}\rangle + \cdots + |0_3 1_4 0_5 \cdots 0_M 0_{M+1}\rangle) \pm \beta|0_3 \cdots 0_M 0_{M+1}\rangle,$$

$$\alpha|0_3 \cdots 0_M 0_{M+1}\rangle \pm \frac{1}{\sqrt{M}}\beta(|0_3 0_4 \cdots 0_M 1_{M+1}\rangle + |0_3 0_4 \cdots 0_{M-1} 1_M 0_{M+1}\rangle + \cdots + |1_3 0_4 0_5 \cdots 0_M 0_{M+1}\rangle). \tag{26}$$

Then the $M-1$ particles are brought together to the same location and a transformation is performed on them;

$$\frac{1}{\sqrt{M}}(|0_3 0_4 \cdots 0_M 1_{M+1}\rangle + |0_3 0_4 \cdots 0_{M-1} 1_M 0_{M+1}\rangle + \cdots + |1_3 0_4 0_5 \cdots 0_M 0_{M+1}\rangle) \rightarrow |0_1 \cdots 0_M 1_{M+1}\rangle,$$

$$|0_3 \cdots 0_M 0_{M+1}\rangle \rightarrow |0_3 \cdots 0_M 0_{M+1}\rangle. \tag{27}$$

By this way, the $(M+1)$th particle evolves into

$$\alpha|1_{M+1}\rangle \pm \beta|0_{M+1}\rangle,$$

$$\alpha|0_{M+1}\rangle \pm \beta|1_{M+1}\rangle, \tag{28}$$

with all of the other particles left in the state $|0\rangle$. The original qubit can be reconstructed by performing a proper unitary transformation.

In summary, we have described a procedure for splitting quantum information into two or more parts using multiparticle W-type entangled states. In order to recover the qubit the particles at the receiving sides should be brought together. The key point of our scheme is that a nonmaximally entangled state can be used for quantum information processing with a probability of success being 1. The scheme is much more robust than the previous scheme using GHZ states. We illustrate the idea in the ion-trap system. However, it can also be realized in other systems.

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