Entanglement induced in spin-1/2 particles by a spin chain near its critical points

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A relation between entanglement and criticality of spin chains is established. The entanglement we exploit is shared between auxiliary particles, which are isolated from each other, but are coupled to the same critical spin-1/2 chain. We analytically evaluate the reduced density matrix, and numerically show the entanglement of the auxiliary particles in the proximity of the critical points of the spin chain. We find that the entanglement induced by the spin chain may reach 1, and it can signal very well the critical points of the chain. A physical understanding and experimental realization with trapped ions are presented.

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Quantum entanglement lies at the heart of the difference between the quantum and classical multiparticle world, and can be treated as a useful resource in various tasks such as cryptography, quantum computation, and teleportation [1]. Quantum phase transitions [2] are transitions between quantitatively distinct phases of quantum many-body systems, driven solely by quantum fluctuations. In the past decade, a great effort has been devoted to understand the relations between entanglement and quantum phase transitions [3–9]. In fact, it is natural to associate the quantum phase transition and entanglement once correlations are behind both of them. By sharing this point of view, one anticipates that entanglement induced by a quantum critical many-body system will furnish a dramatic signature of quantum critical points for the many-body system.

On the other hand, we usually think of environment that surrounds quantum system as a source of decoherence. Recently researchers have started to investigate the positive effects [10–17] of environment, for example, environment assisted information processing and environment induced entanglement. These investigations pave a new way to engineer mechanisms of preventing, minimizing, or using the impact of environment in quantum information processing. In those works, however, the environment was modeled as a set of independent quantum systems, i.e., correlations among particles in the environment were ignored. An interesting open question is whether the correlation among environmental particles can affect the entanglement induced in a bipartite system that couples to it.

In this paper, we show how to exploit entanglement in auxiliary particles induced by a quantum critical many-body system as an essential tool to reveal quantum phenomena in the many-body quantum system. Indeed, quantum phase transitions are accompanied by a qualitative change in the nature of classical correlations, such drastic changes in the properties of ground states are often due to the collectiveness and/or randomness of interparticle couplings which are possibly reflected in entanglement between systems that couple to it. Here we adopt a spin-chain system described by the one-dimensional spin-$\frac{1}{2}$ XY model as the many-body system. Another pair of spin-$\frac{1}{2}$ systems that couple to the spin chain would act as the auxiliary particles. We observe that the entanglement of the auxiliary particles changes sharply in the proximity of quantum phase transition. This change can be traced down to the presence of collectiveness in the dominant couplings of the auxiliary particles to the chain, then it reflects the critical properties of the many-body system. This observation offers a new tool to study quantum critical phenomena, in particular for more general systems, where analytical solutions might not be available. A possible realization of this scheme is proposed with trapped ions. It utilizes off-resonant standing-waves driven ions in traps [18,19] to simulate the one-dimensional XY spin chain. The proposal also could be realized with ultracold atoms superposed by optical lattice [20]. This makes the study appealing for experimental test.

Consider a bipartite system consisting of two spin-$\frac{1}{2}$ particles a and b, and a quantum many-body system described by the one-dimensional spin-$\frac{1}{2}$ XY model. The system Hamiltonian H_S and the Hamiltonian H_B of the chain read

$$H_S = \frac{\omega_a}{2}\sigma_a^z + \frac{\omega_b}{2}\sigma_b^z,$$

$$H_B = -\sum_{l=1}^{N}\left(\frac{1+\gamma}{2}\sigma_l^x\sigma_{l+1}^x + \frac{1-\gamma}{2}\sigma_l^y\sigma_{l+1}^y + \frac{\lambda}{2}\sigma_l^z\right), \quad (1)$$

where N is the number of sites, $\sigma_i^\alpha(\alpha=x,y,z)$ are the Pauli matrices, and γ is the dimensionless anisotropy parameter. The periodic boundary condition $\sigma_{N+1}=\sigma_1$ is assumed for the spin chain. Suppose the coupling of the auxiliary particles to chain take the form

$$H_I = \sum_{l=1}^{N}(g\sigma_a^z\sigma_l^z + h\sigma_b^z\sigma_l^z), \quad (2)$$

where g and h denote the rescaled dimensionless coupling constants. Clearly, $[H_S, H_I]=0$, which implies that the energy of the auxiliary particles is conserved, but the coherence may not be conserved depending on the detail of the system-chain coupling. This leads to the following form of the time evolution operator $U(t)$, $U(t)=\sum_{i,j=0,1}U_{ij}(t)|ij\rangle\langle ij|$, where $|ij\rangle = |i\rangle_a\otimes|j\rangle_b$ and $|i\rangle_a(|i\rangle_b, i=0,1)$ represent the eigenstates of $\sigma_a^z(\sigma_b^z)$. It is easy to show that $U_{ij}(t)(i,j=0,1)$ satisfy $i\hbar\frac{\partial}{\partial t}U_{ij}(t)=H_{ij}U_{ij}(t)$ with $H_{ij}=-\sum_{l=1}^{N}(\frac{1+\gamma}{2}\sigma_l^x\sigma_{l+1}^x + \frac{1-\gamma}{2}\sigma_l^y\sigma_{l+1}^y + \frac{\Lambda_{ij}}{2}\sigma_l^z)$, where $\Lambda_{ij}=\lambda+(-1)^{i+1}2g+(-1)^{j+1}2h$, $i,j=0,1$. If the auxiliary particles are initially in state $|ij\rangle$, the dynamics and

statistical properties of the spin chain would be govern by H_{ij}, it takes the same form as H_B but with modified field strengths Λ_{ij}. The Hamiltonian H_{ij} can be diagonalized by a standard procedure [21] to be $H_{ij} = \Sigma_k \omega_{ij,k}(\eta^\dagger_{ij,k}\eta_{ij,k} - \frac{1}{2})$, where $\eta_{ij,k}(\eta^\dagger_{ij,k})$ are the annihilation (creation) operators of the fermionic modes with frequency $\omega_{ij,k} = \sqrt{\varepsilon^2_{ij,k} + \gamma^2 sin^2 \frac{2\pi k}{N}}$, $\varepsilon_{ij,k} = (\cos\frac{2\pi k}{N} - \Lambda_{ij})$, $k = -N/2, -N/2+1, \ldots, N/2-1$. The fermionic operator $\eta_{ij,k}$ was defined by the Bogoliubov transformation as $\eta_{ij,k} = d_k \cos\frac{\theta_{ij,k}}{2} - id^\dagger_{-k}\sin\frac{\theta_{ij,k}}{2}$, where $d_k = \frac{1}{\sqrt{N}}\Sigma_l a_l \exp(-i2\pi lk/N)$, and the mixing angle $\theta_{ij,k}$ was defined by $\cos\theta_{ij,k} = \varepsilon_{ij,k}/\omega_{ij,k}$. Fermionic operators a_l were connected with the spin operators by the Jordan-Wigner transformation via $a_l = (\Pi_{m<l}\sigma^z_m)(\sigma^x_l + i\sigma^y_l)/2$. The operators $\eta_{ij,k}$ parametrized by i and j clearly do not commute with each other, this will lead to entanglement in the auxiliary particles as shown later on. Before going on to calculate the reduced density matrix, we present a discussion on the diagonalization of H_{ij}. For a chain with periodic boundary condition, i.e., $\sigma_1 = \sigma_N$, boundary terms $H_{boun} \sim [(a^\dagger_N a_1 + \gamma a_N a_1)$ $+\text{H.c.}][\exp(i\pi M)+1]$ have to be taken into account [21,22]. In this paper, we would work with H_{boun} ignored [23], because we are interested in finding a link between the criticality of the chain and the entanglement in the auxiliary particles.

Having given an initial product (separable) state of the total system, $|\psi(0)\rangle = |\phi_a(0)\rangle \otimes |\phi_b(0)\rangle \otimes |\phi_B(0)\rangle$, we can obtain the reduced density matrix for the auxiliary particles as $\rho_{ab}(t) = \text{Tr}_B[U(t)|\psi(0)\rangle\langle\psi(0)|U^\dagger(t)]$, and it may be formally written in the form

$$\rho_{ab}(t) = \sum_{i,j,m,n} \rho_{ij;mn}(t)|ij\rangle\langle mn|. \tag{3}$$

A straightforward but somewhat tedious calculation shows that

$$\rho_{ij;mn}(t) = \rho_{ij;mn}(t=0)\Gamma_{ij;mn}(t),$$

$$\Gamma_{ij;mn}(t) = \prod_k e^{(i/2)(\omega_{ij,k} - \omega_{mn,k})t}\left[1 - (1 - e^{i\omega_{ij,k}t})\sin^2\frac{\theta_k - \theta_{ij,k}}{2}\right.$$

$$- (1 - e^{-i\omega_{mn,k}t})\sin^2\frac{\theta_k - \theta_{mn,k}}{2}$$

$$+ (1 - e^{i\omega_{ij,k}t})(1 - e^{-i\omega_{mn,k}t})$$

$$\left.\times\left(\sin\frac{\theta_k - \theta_{ij,k}}{2}\sin\frac{\theta_k - \theta_{mn,k}}{2}\cos\frac{\theta_{ij,k} - \theta_{mn,k}}{2}\right)\right],$$

$$\tag{4}$$

where $\cos\theta_k = [\cos(2\pi k/N) - \lambda]/$ $\sqrt{[\cos(2\pi k/N) - \lambda]^2 + \gamma^2 sin^2(2\pi k/N)}$. To derive this result, the spin chain was assumed to be initially in the ground state of H_B. Discussions on Eq. (4) are in order. For $g = h$, one has $\Lambda_{ij} = \lambda$ when $i=0$ and $j=1$, or $i=1$ and $j=0$. Hence, $|01\rangle$ and $|10\rangle$ expand a decoherence free subspace. Subsequently, the entanglement of state in this subspace remains unchanged due to $\Gamma_{ij;mn}(t) = 1$. The situation changes if $h \neq g$, where the decoherence free subspace does not exist. The entanglement
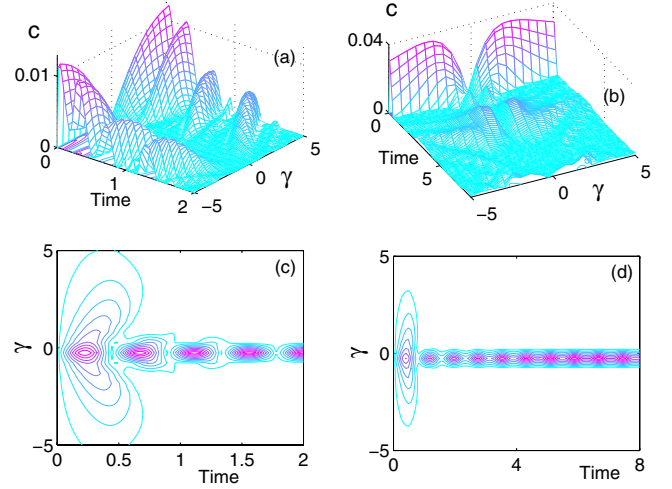

FIG. 1. (Color online) Wootters concurrence of auxiliary particles as a function of time and the anisotropy parameter γ. The figure was plotted for $N=1502$ sites, and $g=h=0.05$. (a), (b), (c), and (d) correspond to $\lambda=5$, $\lambda=2$, $\lambda=1$, and $\lambda=0.8$, respectively.

shared between the auxiliary particles evolves with time according to Eq. (4) in this situation.

With these expressions, we now turn to study entanglement shared between the auxiliary particles a and b in state (3). To be specific, we choose $|\phi_a(0)\rangle \otimes |\phi_b(0)\rangle = 1/\sqrt{2}(|0\rangle_a + |1\rangle_a \otimes 1/\sqrt{2}(|0\rangle_b + |1\rangle_b$ as the initial state of the auxiliary particles, while the spin chain is assumed to be in the ground state of Hamiltonian H_B. The entanglement measured by the Wootters concurrence can be calculated and the numerical result was shown in Figs. 1–4. The time t was rescaled dimensionless according to the coupling constants. The regions of criticality appear when the ground and first excited states become degenerate. We first focus on the criticality in the XX model. The XX model, which corresponds to $\gamma=0$, has a criticality region along the line between $\lambda=1$ and $\lambda=-1$. The criticality is reflected in the entanglement of the auxiliary particles, which appear in Fig. 1 and 2. Figure 1 shows the Wootters concurrence as a function of time and the anisotropy parameter γ, it is clear that the Wootters concurrence has a sharp change in the limit $\gamma \to 0$. This result can be understood by considering the value of $\theta_{ij,k}$, which take 0 or π depending on the sign of $\cos(2\pi k/N) - \Lambda_{ij}$ in this limit. In
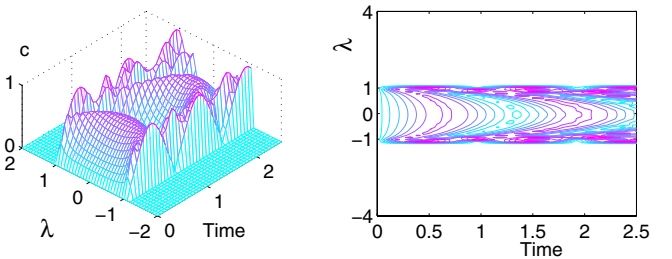

FIG. 2. (Color online) Wootters concurrence versus time and λ with fixed $\gamma=10^{-4}$ (close to zero). This corresponds to the XX model, and the critical points with $\lambda=\pm1$ are clearly shown. The figure on the left is a contour plot for the right. The other parameters chosen are $N=1502$ and $g=h=0.05$.
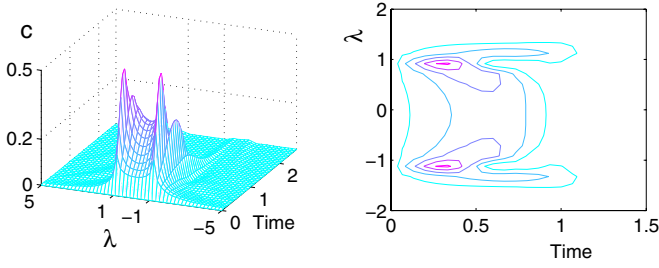
FIG. 3. (Color online) Entanglement (measured by Wootters concurrence) induced by the transverse Ising spin-$\frac{1}{2}$ chain, which may be obtained from the *XY* model by setting $\gamma=1$. The numerical calculation was performed with $N=1502$ sites, and $g=h=0.05$.

either case $|\Gamma_{ij;mn}(t)|=1$, which indicates that the norm of any element of the reduced density matrix remains unchanged. Figure 2 shows the entanglement of the auxiliary particles near the criticality region with $\gamma=0$ and $\lambda=\pm1$. The entanglement changes sharply along the line of $\lambda=\pm1$ in the time-λ plane. This is a reflection of the critical phenomena in the entanglement of the auxiliary particles.

Next we turn to study the criticality in the transverse Ising model. The structure of the transverse model ground state changes dramatically as the parameter λ is varied. The dependence of the entanglement on λ is quite complicated [2]. Here we present analyses for the $\lambda\rightarrow\infty$, $\lambda=1$, and $\lambda=0$ limits. In the $\lambda\rightarrow\infty$ limit the ground state of the chain approaches a product of spins pointing the positive z direction. The mixing angle $\theta_{ij,k}$ and θ_k tend to $-\pi$ in this limit. This leads to $\Gamma_{ij;mn}(t)=1$, which indicates that the initial state does not evolve with time, i.e., the auxiliary particles remain in separable states. The $\lambda=0$ limit is fundamentally different from the $\lambda\rightarrow\infty$ limit because the corresponding ground state is doubly degenerate under the global spin flip by $\Pi_{l=1}^{N}\sigma_l^z$. This symmetry breaks at $|\lambda|=1$ and the chain develops a nonzero magnetization $\langle\sigma^x\rangle\neq0$ which grows as λ is decreased. This dramatic change in the ground state of the spin chain can be found in the entanglement of auxiliary particles in Fig. 3. In fact, the ground state of the *XY* models is very complicated with many different regimes of behavior [24,25]. With whatever γ, there is a sharp change in the entanglement across the line $|\lambda|=1$ (Fig. 4). This signals the change in the ground state of the spin chain from paramagnetic phase to the others. Although the entanglement in both cases with $\gamma=0$ and $\gamma=1$ shares the same properties along
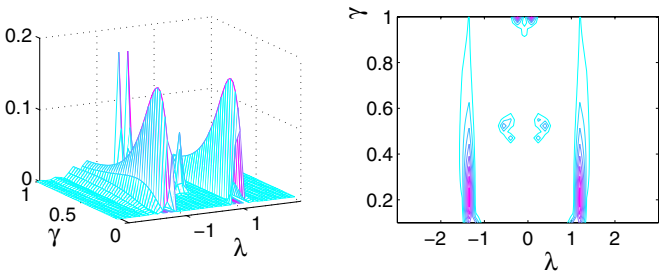

FIG. 4. (Color online) Wootters concurrence at time $t=0.75$. The locations of peaks on the λ axis do not change with time. Various parameters are $N=1502$, $g=h=0.05$.

the line of $|\lambda|=1$, i.e., it changes dramatically across this critical region, the two case are quite different in its nature. The two auxiliary particles never evolve in the case of $\gamma=0$, while the two will eventually evolve into the pointer states in another case (except for $\lambda\rightarrow\infty$), hence no entanglement share between the auxiliary particles in the later case in the $t\rightarrow\infty$ and $N\rightarrow\infty$ limit.

The connection between the entanglement in auxiliary particles and the criticality of spin chain may be understood as singularity in one or more $\rho_{ij;mn}(t)$ (the elements of the reduced density matrix) at the critical points. To show this point, we recall that

$$\rho_{ij;mn}(t)=\frac{1}{4}\langle\phi_B(0)|e^{-iH_{ij}t}e^{iH_{kl}t}|\phi_B(0)\rangle, \quad (5)$$

with the same notations and initial states given before Eq. (3). Notice that $|\phi_B(0)\rangle$ was taken as the ground state $|0\rangle_B$ of H_B, but it is not an eigenstates of H_{ij} with $\Lambda_{ij}\neq\lambda$. Expanding $|0\rangle_B$ in terms of eigenstate $|\alpha\rangle_B^{ij}$ of H_{ij}, $(\alpha=0,1,\ldots,)$

$$|0\rangle_B=c_0^{ij}|0\rangle_B^{ij}+\sum_{\alpha\neq0}c_\alpha^{ij}|\alpha\rangle_B^{ij}, \quad (6)$$

one can easily prove that $|c_{\alpha\neq0}^{ij}|\sim h,g$ when $g\ll1$ and $h\ll1$. This is exactly the case under our consideration. Where $|0\rangle_B^{ij}$ denotes the ground state of H_{ij}. Therefore,

$$\rho_{ij;mn}(t)\sim\frac{1}{4}c_0^{ij}[c_0^{kl}]^{*kl}_B\langle0|0\rangle_B^{ij}e^{-i(\Omega_{ij}-\Omega_{kl})t}, \quad (7)$$

up to first order in g and h. Here Ω_{ij} stands for the ground-state energy of H_{ij}. As observed in [26], a sudden drop in $^{kl}_B\langle0|0\rangle_B^{ij}$ can signal the regions of criticality in the spin chain. Therefore the entanglement as a function of $\rho_{ij;mn}(t)$ can signal the criticality of the chain.

This discussion, apart from its theoretical interests, offers a possible experimental method to study critical phenomena without the need to identify the state of the system, in particular in the presence of degeneracy. The *XY* model can be realized with trapped ions under the action of off-resonant standing waves [19] as follows. Consider a system of trapped ions, whose physical implementation corresponds to Coulomb chains in Penning traps or an array of ion microtraps. The Coulomb repulsion together with the trapped potential and ions' motion yield a set of collective vibrational modes, these collective modes can be coupled to the internal states of ions by conditional forces. The directions of the conditional forces [19] would determine the effective couplings between the effective spins. To simulate the *XY* model, we apply two conditional forces in both directions x and y. The effective coupling is proportional to $1/r^3$, where r represents the relative distance between ions. Although the couplings are distance dependent, the critical properties are shown to be similar to these ideal models [19,27] considered in this paper. The coupling of the auxiliary particles to the chain, which share another trapping with the ions in the chain, can be simulated in the same way, where a conditional force in the z direction is applied. The particles in the chain may circle the auxiliary particles, such that the couplings of the auxiliary particles to the particles in the chain are equal. The

proposal could also be realized with ultracold atoms in optical lattice by the method represented in [28]. We would like to note that preventing any energy exchange between the chain and the auxiliary particles might be a challenging task in experiments. The requirement of uniform coupling also is experimentally very challenging. Nevertheless, this paper established a link between the entanglement and criticality from the other aspect, which might shed light on the understanding of criticality from the auxiliary particles, and it may open up a way to characterize and experimentally detect quantum phase transition from the viewpoint of quantum information theory [29], where two ancillas coupled to two particles at different sites in the chain are considered. By this model, the relation between the state transfer quality and the spectral gap of the chain was established.

To conclude, we have proposed a method to study critical phenomena in many-body systems. The criticality was found to have a reflection in the entanglement of auxiliary particles that couple to it. Indeed, we have found that the entanglement change dramatically along the line of critical points. This dramatic change has been explained in terms of reflection of quantum phase transitions, which lead to the collectiveness in the couplings of the spin chain to the auxiliary particles. Quantum critical many-body system can also make a reflection for its criticality in decoherence of a quantum system which couples to it [30]. In addition to shedding light on the role of environment in quantum information processing, these pave a way to study critical phenomena in many-body systems. The generalization of these results to a wide variety of critical phenomena and their relation to the critical exponents is a promising and challenging question which deserves extensive future investigation.

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