Critical property of the geometric phase in the Dicke model

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We obtain the ground-state energy level and associated geometric phase in the Dicke model analytically by means of the Holstein-Primakoff transformation and the boson expansion approach in the thermodynamic limit. The nonadiabatic geometric phase induced by the photon field is derived with the time-dependent unitary transformation. It is shown that the quantum phase transition characterized by the nonanalyticity of the geometric phase is remarkably of the first order. We also investigate the scaling behavior of the geometric phase at the critical point, which can be measured in a practical experiment to detect the quantum phase transition.

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The Berry phase [1] with relaxation of its original restriction conditions [2–5] has been extensively generalized along various directions [6–9]. Recently, the geometric phase (GP) has been regarded as an essential way to implement the operation of a universal quantum logic gate in quantum computing [10–15] and as an important tool to detect a quantum phase transition (QPT) [16–20], which describes a structural change in the properties of the ground-state energy spectrum associated with variation of a controlling parameter [21] and originates from the singularity of the energy spectrum [22]. A single $1/2$ spin driven by a classical rotating magnetic field is a well-studied model to generate the Berry phase. Recently a quantized magnetic field has been considered and shown to induce the GP which reduces to the standard Berry phase in the semiclassical limit [23,24]. The generation of this framework of Berry phase to a many-body two-level-atom system interacting with a single bosonic mode known as the Dicke model [25,26] is certainly of interest, particularly in connection with the QPT. The Dicke model displays an interesting superradiant phenomenon describing the collective and coherent behaviors of atoms and exhibits the QPT from the normal to the superradiant phases [27–29] induced by variation of the coupling strength between the atom and field. It has been known that the atomic ensemble in the normal phase is collectively unexcited while it is macroscopically excited with coherent radiation in the so-called superradiant phase. The QPT has been related to the emergence of chaos in a corresponding classical Hamiltonian [30,31] and the logarithmic diverges of the von Neumann entropy at the critical transition point which describes the quantum entanglement between the atoms and field [32–34].

In this Brief Report we derive the nonadiabatic GP of a many-body atom-system driven by a quantized field. The ground state and corresponding GP are obtained with the Holstein-Primakoff transformation [35] and the boson expansion approach [36] in the thermodynamic limit. We demonstrate that the GP can serve as a critical criterion to characterize the QPT. The scaling behavior of the GP at the critical point is also studied.

The Dicke model of N two-level atoms in a single-mode light field is given in the rotating-wave approximation by

$$H = \omega a^\dagger a + \sum_{j=1}^{N}\left[\frac{\omega_0}{2}\sigma_z^j + \frac{\lambda}{\sqrt{N}}(\sigma_+^j a + \sigma_-^j a^\dagger)\right], \tag{1}$$

where a and $a^\dagger$ are the photon annihilation and creation operators, σ_+^j and σ_-^j are the pseudospin operators for the jth atom defined as $\sigma_\pm^j = \sigma_x^j \pm i\sigma_y^j$ with σ_x and σ_y being the Pauli matrices, λ denotes the coupling strength between the atom and the field, ω is the frequency of the electromagnetic wave, and ω_0 is the energy difference between two levels of the atom in the unit $\hbar = 1$. The prefactor $1/\sqrt{N}$ is inserted to have a finite free energy per atom in the thermodynamical limit $N \to \infty$. The Hamiltonian (1) is actually considered in a rotating frame along with the light field and becomes

$$H'(t) = R(t) H R^\dagger(t) - iR(t)\frac{dR^\dagger(t)}{dt} \tag{2}$$

in the laboratory frame with the time-dependent unitary transformation

$$R(t) = \exp[-i\varphi(t) a^\dagger a], \tag{3}$$

where $\varphi(t) = \omega t$ denotes the rotating angle. The nonadiabatic GP induced by the light field can be found by solving the time-dependent Schrödinger equation

$$i\frac{d|\Psi_n'(t)\rangle}{dt} = H'(t)|\Psi_n'(t)\rangle, \tag{4}$$

and the result is

$$\gamma_n = i\int_0^{2\pi}\left\langle \Psi_n'(\varphi(t))\left|\frac{d}{d\varphi}\right|\Psi_n'(\varphi(t))\right\rangle d\varphi = 2\pi\langle\Psi_n|a^\dagger a|\Psi_n\rangle, \tag{5}$$

where $|\Psi_n'(\varphi(t))\rangle = R(t)|\Psi_n\rangle$ with $|\Psi_n\rangle$ being the eigenstate of the Hamiltonian H such that $H|\Psi_n\rangle = E_n|\Psi_n\rangle$, which is equal to the Berry-phase formula in Refs. [23,24]. We are interested in the critical property of the GP associated with the ground state $|\Psi_0\rangle$ of the many-body system. Using collective giant-spin operators $J_z = \Sigma_{j=1}^N \sigma_z^j$ and $J_\pm = \Sigma_{j=1}^N \sigma_\pm^j$, which satisfy the usual SU(2) commutation relations $[J_z, J_\pm] = \pm J_\pm$ and $[J_+, J_-] = 2J_z$ with total spin quantum number $j = N/2$, the Hamiltonian (1) becomes

$$H = \omega a^\dagger a + \frac{\omega_0}{2} J_z + \frac{\lambda}{\sqrt{N}}(J_+ a + J_- a^\dagger). \tag{6}$$

The system undergoes a QPT at a critical value of the atom-field coupling strength λ_c (to be determined below) from the normal phase when $\lambda < \lambda_c$ to the superradiant phase when $\lambda > \lambda_c$ in the thermodynamical limit. By means of the boson expansion approach and the Holstein-Primakoff transformation of the collective angular momentum operators defined as $J_+ = b^\dagger\sqrt{N - b^\dagger b}$, $J_- = \sqrt{N - b^\dagger b}b$, and $J_z = (b^\dagger b - N/2)$, where the new boson operators satisfy the commutation relation $[b, b^\dagger] = 1$ [35]. With this transformation the Hamiltonian becomes

$$H = \omega a^\dagger a + \frac{\omega_0}{2}(b^\dagger b - N/2) + \frac{\lambda}{\sqrt{N}}(b^\dagger\sqrt{N - b^\dagger b}a + \sqrt{N - b^\dagger b}ba^\dagger). \tag{7}$$

Following the procedure of Refs. [31,36], we introduce the shifting boson operators $c^\dagger$ and $d^\dagger$ with properly scaled auxiliary parameters α and β such that

$$c^\dagger = a^\dagger + \sqrt{N}\alpha, \quad d^\dagger = b^\dagger - \sqrt{N}\beta \tag{8}$$

to evaluate the critical transition point λ_c and the ground-state energies of both phases. It should be noticed that the displacements $c^\dagger = a^\dagger - \sqrt{N}\alpha$ and $d^\dagger = b^\dagger + \sqrt{N}\beta$ also lead to the same result except for the change of sign. Expanding Hamiltonian (7) with the displacement boson operators $c^\dagger$ and $d^\dagger$ as power series of $1/\sqrt{N}$ we can obtain

$$H = NH_0 + N^{1/2}H_1 + N^0 H_2 + \cdots, \tag{9}$$

with

$$H_0 = \omega\alpha^2 + \frac{\omega_0}{2}\left(\beta^2 - \frac{1}{2}\right) - 2\lambda\alpha\beta\sqrt{k},$$

$$H_1 = (-\omega\alpha + \lambda\beta\sqrt{k})(c^\dagger + c) + \left[\frac{\omega_0\beta}{2} - \lambda\alpha\sqrt{k}(1 - 2\beta^2)\right](d^\dagger + d),$$

$$H_2 = \omega c^\dagger c + \frac{\omega_0}{2}d^\dagger d + \lambda\left[\sqrt{k}d^\dagger - \frac{\beta^2}{2\sqrt{k}}(d^\dagger + d)\right]c + \lambda\left[\sqrt{k}d - \frac{\beta^2}{2\sqrt{k}}(d^\dagger + d)\right]c^\dagger + \frac{\lambda\alpha\beta}{2\sqrt{k}}\left[4d^\dagger d + (d^\dagger)^2 + d^2 + \frac{\beta^2}{2k}(d^\dagger + d)^2\right],$$

where $k = 1 - \beta^2$. The first term of H gives rise to the Hartree-Bogoliubov ground-state energy [37]

$$E_0 = \langle\Psi_0|NH_0|\Psi_0\rangle = \begin{cases} -N\omega_0/4, & \lambda < \lambda_c, \\ -N\left[\frac{\lambda^2}{4\omega}(1 - \delta^2) + \frac{\omega_0\delta}{4}\right], & \lambda > \lambda_c, \end{cases} \tag{10}$$

and the critical value

$$\lambda_c = \sqrt{\omega_0\omega/2}, \tag{11}$$

where $|\Psi_0\rangle = |0\rangle_{at}|0\rangle_{ph}$ is the vacuum of the quasiboson operators such that $c|0\rangle_{ph} = 0$ and $d|0\rangle_{at} = 0$ (correspondingly, $a|0\rangle_{ph} = \sqrt{N}\alpha|0\rangle_{ph}$ and $b|0\rangle_{at} = -\sqrt{N}\beta|0\rangle_{at}$) with $\delta = \omega\omega_0/2\lambda^2$. The auxiliary parameters

$$\alpha = \begin{cases} 0, & \lambda < \lambda_c, \\ \lambda\sqrt{1 - \delta^2}/2\omega, & \lambda > \lambda_c, \end{cases} \tag{12}$$

$$\beta = \begin{cases} 0, & \lambda < \lambda_c, \\ \sqrt{(1 - \delta)/2}, & \lambda > \lambda_c, \end{cases} \tag{13}$$

are determined from minimizing the ground-state energy (10). In the normal phase ($\lambda < \lambda_c$) the system is essentially in the lower-energy state and is only microscopically excited, whereas above the transition point both the field and atomic ensemble acquire macroscopic excitations.

The GP of the many-body system associated with the ground state $|\Psi_0\rangle$ is obtained in terms of the quasiboson operator (8) as

$$\gamma_0 = 2\pi\langle\Psi_0|a^\dagger a|\Psi_0\rangle = \begin{cases} 0, & \lambda < \lambda_c, \\ \frac{\pi N}{2\omega^2}\left(\lambda^2 - \frac{\lambda_c^4}{\lambda^2}\right), & \lambda > \lambda_c. \end{cases} \tag{14}$$

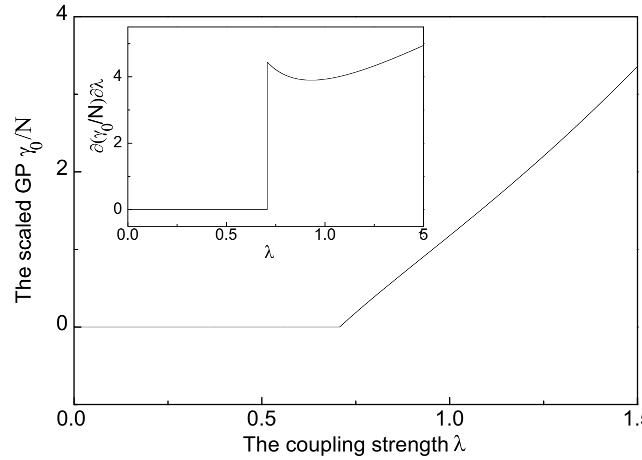

FIG. 1. The scaled ground-state geometric phase γ_0/N (in the units of ω_0^{-2}) as a function of the coupling parameter λ (in units of ω_0) in the resonant condition $\omega = \omega_0$. Inset: the first-order derivative of γ_0/N with respect to the coupling parameter λ.

The scaled GP γ_0/N and its first-order derivative with respect to the coupling parameter λ are shown in Fig. 1 with the resonant condition $\omega = \omega_0 = 1$. It can be seen that the GP vanishes when $\lambda < \lambda_c$ and increases abruptly with λ when $\lambda > \lambda_c$, indicating a first-order phase transition at the critical point λ_c. It is apparent that the GP can be used to detect the quantum criticality in systems described by the Dicke model. The quantum criticality is also shown to relate the quantum entanglement governed by the von Neumann entropy which has a cusplike behavior at the critical transition point [32,33].

The scaling behavior of the GP at the critical point can be found in the thermodynamical limit as

$$\frac{\gamma_0}{N}(\lambda \to \lambda_c) = \frac{2\pi\lambda_c}{\omega^2}|\lambda - \lambda_c|. \quad (15)$$

On the other hand, the first-order derivative of the GP diverges linearly with the atom number N at the transition point λ_c as

$$\lim_{N\to\infty} \frac{d\gamma_0}{d\lambda}\bigg|_{\lambda=\lambda_c} = \frac{2\pi\lambda_c}{\omega^2}N, \quad (16)$$

which is different from the logarithmic divergence of the first-order derivative of the Berry phase in the *XY* model [16–18].

In recent years it has been shown that some nanosystems—for example, semiconductor quantum dots—coupled with a high-quality single-cavity mode can be a promising candidate for implementing the Hamiltonian of the Dicke model and realistic parameter values are given by $\lambda \sim 0.1-1.5$ [38–40] in units of ω_0. While ω_0 is of the order of the underlying bulk band gap with a value range from 1.5 eV in GaAs-like semiconductors down to 0.1 eV in narrow-gap semiconductors. The value of λ is also variable in a wide range from weak- to strong-coupling regimes based on current nanotechnology. The GP can be observed directly by measuring the mean photon number out of the cavity.

In conclusion, we have investigated the critical property of the nonadiabatic GP in terms of a time-dependent unitary transformation in the single-mode super-radiant Dicke model, which displays a first-order QPT. The GP is shown to be proportional to the mean photon number and therefore is calculated analytically for the ground state of the Dicke model in the thermodynamical limit. We also provide the scaling behavior of the GP as a probe to test the QPT.

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