Quantum-fluctuation properties of the ladder scheme of a three-level laser with incoherent pumping

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The dynamics and quantum statistical properties of the three-level laser ladder model are investigated. A comparison between this model to the other models, V and Λ ones, is presented. We found that the ladder-pumped lasers have both the highest intensity and the lowest linewidth, whereas the maximum noise reduction is more pronounced in the V and Λ models. We also show that at high pumping rate the ladder scheme behaves as the two-level one.

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I. INTRODUCTION

Interest in three-level laser models with incoherent pumping goes back to the beginning of the laser era [1]. Since then, two basic types of level schemes were suggested and deeply investigated [2–9]. In the Λ scheme, the ground state is not involved in the lasing transition, while in the V scheme, the pumping excites the atom from the ground state, which also serves as the lower lasing level. There exists a third type of laser scheme, the ladder or the cascade one, that has been studied in the literature on lasing without inversion physics [10]. Surprisingly, to the best of our knowledge, the cascade-type schemes have not been studied in the more simple case of *incoherent* pumping. In the present paper, we try to fill this gap by studying a simple cascade laser scheme. Both dynamical and statistical properties of such a laser are calculated analytically and discussed in detail. It is shown that such a scheme has attractive properties, such as high intensity and extremely narrow spectral line, which makes it a promising candidate for experimental realization.

II. THREE LEVEL LADDER MODEL

We study a simple cascade laser scheme shown in Fig. 1. The pumping process excites the atoms with the rate γ_{01}, from the ground state $|0\rangle$ to the first excited state $|1\rangle$. This level is the lower of the laser that acts on the $|2\rangle \to |1\rangle$ transition. Then the atoms are excited, with the rate γ_{12}, from the lower lasing level $|1\rangle$ to the second excited state $|2\rangle$, which is the upper lasing level in this case. The two lasing levels, i.e., levels $|1\rangle$ and $|2\rangle$, decay one level down with the rates γ_{10} and γ_{21}, respectively.

We start with the master equation for the atom-field density matrix R,

$$\frac{\partial R}{\partial t} = (\Lambda_f + \Lambda_a - iL)R, \qquad (1)$$

where

$$\Lambda_f R = \kappa([aR,a^\dagger] + [a,Ra^\dagger]), \qquad (2)$$

$$\Lambda_a R = \frac{1}{2}\sum_{k=1}^{N}\left\{ \sum_{i\neq j=0}^{2} \gamma_{ij}([\sigma_{ij,k}R,\sigma_{ji,k}] + [\sigma_{ij,k},R\sigma_{ji,k}]) + \sum_{i>j=0}^{2} \eta_{ij}([\sigma^3_{ij,k}R,\sigma^3_{ij,k}] + [\sigma^3_{ij,k},R\sigma^3_{ij,k}])\right\}, \qquad (3)$$

$$LR = \frac{1}{\hbar}[H,R],$$

$$H = \sum_{k=1}^{N} g\hbar(a^\dagger\sigma_{12,k} + \sigma^+_{21,k}a). \qquad (4)$$

Here, the interaction Hamiltonian H is taken in the rotating-wave approximation, κ is the cavity decay rate, $\sigma_{ij,k}=|i\rangle_k\langle j|_k$, $\sigma^3_{ij,k}=\frac{1}{2}(\sigma_{ij,k}\sigma_{ji,k}-\sigma_{ji,k}\sigma_{ij,k})$, γ_{ij} is either the rate of spontaneous emission into nonlasing modes or the pump rate, η_{ij} is the rate of collisional or/and reservoir-induced dephasing on $i\to j$ transition, g is a coupling constant, and N is the total number of atoms.

From the master equation (1), one can derive, assuming the large number of photons $n \gg 1$, the following Fokker-Planck equation for the field density matrix $\rho = \mathrm{Tr}_A R$, where Tr_A means tracing out the atomic variables (see Refs. [5,11]):

$$\frac{1}{2\kappa}\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial n}\left[nf(n)\rho + D_n(n)\frac{\partial\rho}{\partial n}\right] + D_\varphi\frac{\partial^2\rho}{\partial\varphi^2}, \qquad (5)$$

where $n=a^\dagger a$ is the number of intracavity photons, the factor $f(n)$ in the drift term is given by

$$f(n) = 1 - \frac{\lambda[(n+1)\bar{r}_{22} - n\bar{r}_{11}]}{Sn(1+P_B+\alpha+\eta)}, \qquad (6)$$

the mean populations $\bar{r}_{11}$ and $\bar{r}_{22}$ of the two lasing levels, calculated using modified Maxwell-Bloch equations [7], are given by

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$$\bar{r}_{11} = \frac{P_A[1 + n + S\alpha(1 + P_B + \alpha + \eta)]}{1 + n + P_A(1 + 2n) + S[\alpha + P_A(P_B + \alpha)](1 + P_B + \alpha + \eta)}, \tag{7}$$

$$\bar{r}_{22} = \frac{P_A[n + SP_A(1 + P_B + \alpha + \eta)]}{1 + n + P_A(1 + 2n) + S[+ P_A(P_B + \alpha)](1 + P_B + \alpha + \eta)}, \tag{8}$$

and the dimensionless parameters λ, S, α, η, P_A, and P_B are defined as follows:

$$\lambda = \frac{N\gamma_{10}}{2\kappa}, \quad S = \frac{\gamma_{10}^2}{4g^2}, \quad \alpha = \frac{\gamma_{21}}{\gamma_{10}}, \quad \eta = \frac{\gamma_{col}}{\gamma_{10}}, \quad P_A = \frac{\gamma_{01}}{\gamma_{10}}, \quad P_B = \frac{\gamma_{12}}{\gamma_{10}}. \tag{9}$$

The diffusion coefficients D_n and D_φ in the Fokker-Planck equation (5) are given by

$$D_n(n) = \frac{\lambda A[n\alpha - P_B(n+1)] + B[n + 1 + P_A(2n+1)] + CS[\alpha + P_A(P_B + \alpha)]}{2\{1 + n + P_A(2n+1) + S[\alpha + P_A(P_B + \alpha)](1 + P_B + \alpha + \eta)\}}, \tag{10}$$

$$D_\varphi = \frac{\lambda}{4\bar{n}S(1 + P_B + \alpha + \eta)} \tag{11}$$

with

$$A = \frac{2\bar{r}_{00}[\bar{r}_{22} + n(\bar{r}_{22} - \bar{r}_{11})]}{1 + P_B + \alpha + \eta}, \quad B = \frac{2\bar{r}_{22}[\bar{r}_{22} + n(\bar{r}_{22} - \bar{r}_{11})]}{1 + P_B + \alpha + \eta}, \quad C = -2n\bar{r}_{22} + \frac{4[\bar{r}_{22} + n(\bar{r}_{22} - \bar{r}_{11})]^2}{S(1 + P_B + \alpha + \eta)}.$$

The Fokker-Planck equation (5) allows one to calculate both dynamical and statistical characteristics of the intracavity field. Below, we present analytical expressions for the mean photon number $\bar{n}=\langle n\rangle$, the Fano factor $F \equiv (\langle n^2\rangle - \langle n\rangle^2)/\langle n\rangle$, and the linewidth $\Delta\nu$. As for the second-order correlation function $G^{(2)} \equiv \langle a^\dagger a^\dagger aa\rangle/\langle a^\dagger a\rangle^2$, it can be expressed using the Fano factor as $G^{(2)} = 1 + (F-1)/\langle n\rangle$.

The mean photon number is given by

$$\langle n\rangle = \frac{1}{2}(-b + \sqrt{b^2 + 4c}), \tag{12}$$

where

$$b = -\frac{1 + S\alpha(1 + P_B + \alpha + \eta) + P_A[1 + S(P_B + \alpha)(1 + P_B + \alpha + \eta) + (\alpha - P_B)\lambda]}{1 + 2P_A}, \tag{13}$$

$$c = \frac{P_A P_B \lambda}{1 + 2P_A}. \tag{14}$$

The Fano factor F and the linewidth $\Delta\nu$ are given by

$$F = 1 + D_n(\langle n\rangle) \left.\frac{\partial f(n)}{\partial n}\right|_{n=\langle n\rangle}, \tag{15}$$

$$\Delta\nu = 2\kappa D_\varphi \tag{16}$$

In Fig. 2(a), we present the results for the mean number of photons and the Fano factor, and in Fig. 2(b) the second-order correlation function $G^{(2)}$ and the linewidth $\Delta\nu$ as functions of the pump parameter $P=P_A=P_B$. All quantities in Fig. 2 are calculated using Eqs. (12)–(16) with zero collisional dephasing rate $\eta=0$. As one can see in the figure, as the pump parameter increases, the photon number grows to some critical value at which it begins to decrease. The curve $n(P)$ has two kinks: the first kink corresponds to the threshold point, whereas the second one corresponds to the break point at which lasing ceases. This is similar to the known property of "self-quenching" or termination of lasing in V-type incoherently pumped lasers [4,7]. The Fano factor has two clear peaks. These peaks indicate a phase-transition-like

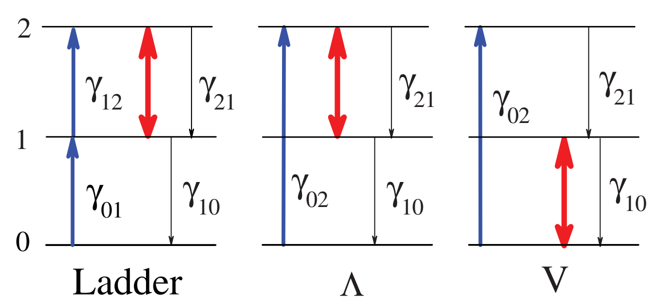

FIG. 1. (Color online) Ladder, Λ, and V schemes of a three-level laser.

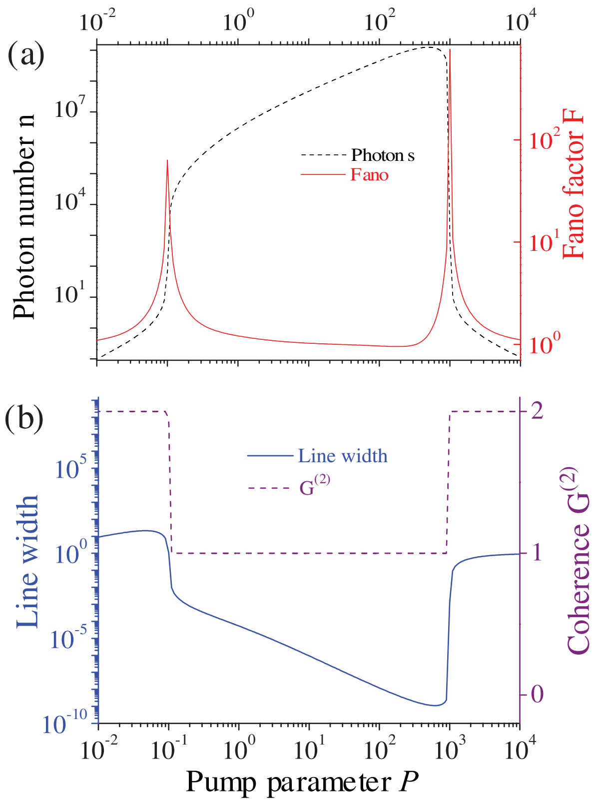

FIG. 2. (Color online) Statistical characteristics of a cascade-pumped three-level laser as functions of the pump parameter $P=P_A=P_B$. (a) The photon number and the Fano factor, (b) the linewidth in units of κ and the second-order coherence $G^{(2)}$. Parameters: $\lambda=10^7$, $\alpha=0.1$, $\eta=0$, $S=10^4$.

behavior at both the threshold and the termination point when lasing ceases. One can observe that the sharp peaks in the Fano factor allow us to estimate the width of the threshold phase transition. It also has a minimum of $F=0.95$ at very strong pump, which corresponds to maximal noise reduction. At that point, the photon statistics are found to be sub-Poisson and the laser light is squeezed. However, the degree of squeezing is low. The second-order correlation function demonstrates that the field turns from incoherent below threshold where $G^{(2)}=2$, into a coherent one above threshold where $G^{(2)}=1$. The reverse transition happens at the termination point where the field becomes incoherent again. The behavior of the linewidth is in some sense opposite to that of the photon number; it drops dramatically, about three orders of magnitude at the laser threshold, then it decreases down to extremely low values, and finally kinks up sharply, about eight orders of magnitude, at the ceasing point of the laser.

The dephasing rate η has been set to zero in the above figure. Incorporating collisional or/and reservoir-induced dephasing processes gives rise to both increasing the threshold value and decreasing the pump rate at which the laser turns off. This results in narrowing the range of the laser operation. Varying other parameters such as saturation S results in similar behavior as in Fig. 2 with a lower degree of squeezing.

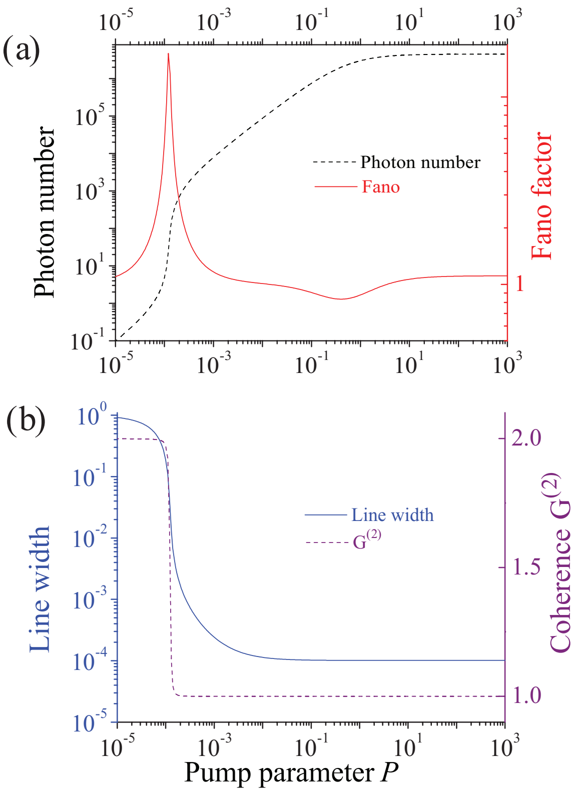

FIG. 3. (Color online) Statistical characteristics of the Λ scheme of a three-level laser as functions of pump parameter $P=\gamma_{02}/\gamma_{10}$. (a) The photon number and the Fano factor, (b) the linewidth in units of κ and the second-order coherence $G^{(2)}$.

III. COMPARISON BETWEEN LADDER, Λ AND V MODELS

In this Section, we compare the dynamic and the statistical properties of the ladder model with those of Λ and V models shown in Fig. 1. The comparison will focus on the following properties: (i) Lasing threshold (P_{on}), (ii) maximum number of photons, (iii) lasing cutoff threshold (P_{off}), (iv) maximum noise reduction, and (v) linewidth. First of all, we present in Figs. 3 and 4 the results for Λ and V models, respectively, in a fashion similar to Fig. 2.

Before making a comparison between the models, a short comment is needed concerning the meaning of the pumping rates in various models. In the Λ model, the external parameter is the pump parameter P or in the original terms (non-normalized), γ_{02}. The pumping excites the atoms in the lasing material from the ground level $|0\rangle$ to the second excited level $|2\rangle$. In the ladder model, the external parameters are the pump parameters P_A and P_B or in the original terms γ_{01} and γ_{12}, respectively. The pump P_A excites the atoms from the ground level $|0\rangle$ to the first excited level $|1\rangle$ and the pump P_B excites the atom from level $|1\rangle$ to level $|2\rangle$. Given equal energy gaps between levels $|0\rangle$ and $|2\rangle$ in both models, the external energy invested in both systems (models) is equal when $P=P_A=P_B$. For example, if the pumping rate in the Λ model $\gamma_{02}=1\ \mathrm{s}^{-1}$, i.e., pumping of one atom per second from level $|0\rangle$ to level $|2\rangle$, then equal investment of energy in the ladder model means that $\gamma_{01}=\gamma_{12}=1\ \mathrm{s}^{-1}$, i.e., pumping of

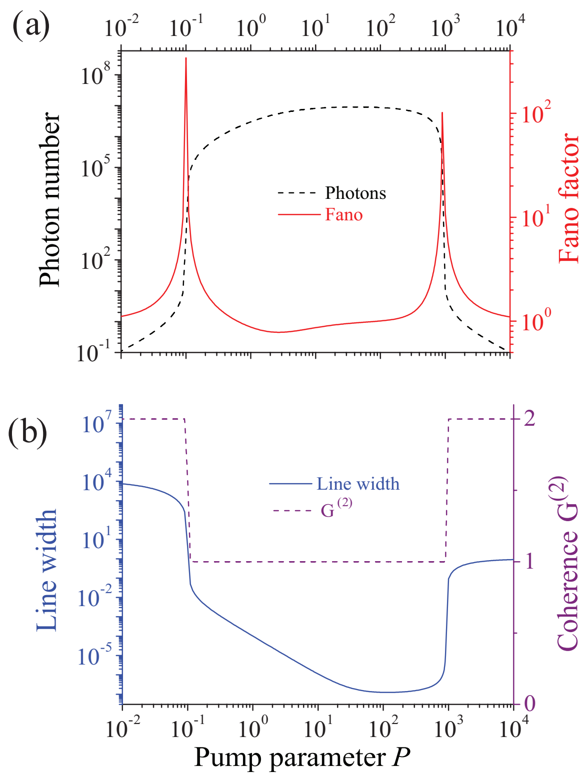

FIG. 4. (Color online) Statistical characteristics of the V scheme of a three-level laser as functions of pump parameter $P=\gamma_{02}/\gamma_{21}$. (a) The photon number and the Fano factor, (b) the linewidth in units of κ and the second-order coherence $G^{(2)}$.

one atom per second from level $|0\rangle$ to level $|1\rangle$ and one atom per second from level $|1\rangle$ to level $|2\rangle$. The last equality of the rates $\gamma_{01}=\gamma_{12}=1\ \mathrm{s}^{-1}$ stems from the physics of the absorption process, namely there is no spontaneous absorption process. In other words, the absorption of the two photons that leads from $|0\rangle$ to $|1\rangle$ and from $|1\rangle$ to $|2\rangle$ occurs simultaneously, without delay, i.e., without the need for population in level $|1\rangle$. Under these circumstances, the energy invested in both models can excite one atom from the ground state to the second excited state. The same line of reasoning applies to the relation between the ladder and V schemes, namely in both the ladder and V schemes the amount of energy needed to excite one atom from the ground state to the upper lasing level is the same.

Now, when one can recognize the situation in which the investment of external energy is equal in all three systems, let us make a comparison between Λ, V, and ladder models.

In order to demonstrate some of the differences between the models, we drew graphs of the photon number n as a function of the pump parameter P for each model on the same coordinates system (see Fig. 5). The pump parameter P is defined as $P=\gamma_{02}/\gamma_{10}$ for the Λ model and $P=\gamma_{02}/\gamma_{21}$ for the V model.

One can notice that although the graphs are drawn for a specific set of parameters, the qualitative relation between the models remains the same for a large range of parameters that enables lasing in a wide range of pump values.

In Fig. 5, the following significant differences can be noticed:

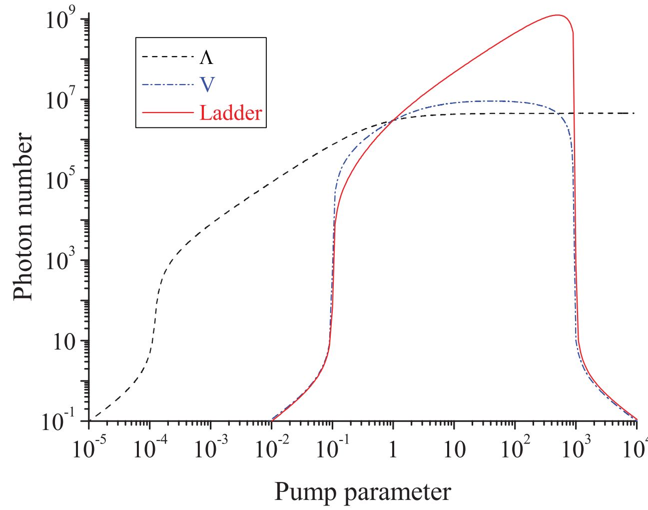

FIG. 5. (Color online) Comparison of photon number behavior for three different three-level laser models: Ladder, Λ, and V.

The lasing threshold is much lower (in two orders of magnitudes) in the Λ model compared with the other two schemes, the V and ladder. In both of the latter cases, the lasing threshold is practically the same. The physical reason for that is that population inversion can be achieved much easier in the Λ scheme than in the two other schemes. This is due to the fact that the pumping excites the atoms from the level outside the lasing transition (level $|0\rangle$ in Λ model) directly to the upper level of the lasing transition, level $|2\rangle$.

The calculation indicates that the maximal number of photons is significantly higher (about two orders of magnitude) in the ladder scheme compared with the other two models, Λ and V. The reason is that in the ladder scheme, there is a direct pumping between the lasing levels $|1\rangle$ and $|2\rangle$ that contributes directly to population inversion. Thanks to this pump, there is no bottleneck in the transition $|1\rangle\to|0\rangle$, like that observed in the Λ model. Unlike the ladder scheme, the finite relaxation rate γ_{21} in the transition $|2\rangle\to|1\rangle$ limits the number of photons in the V model.

Because of the incoherent pumping from the lower lasing level that exists in both the V and the ladder models, the lasing action ceases when the pumping rate is fast enough. As one can see in the figure, the cutoff point is almost the same for both models.

It is interesting to note that when the pumping rate (γ_{01} in the ladder scheme and γ_{02} in both Λ and V schemes) is lower than the depletion rate (γ_{10} in the Λ and ladder schemes and γ_{21} in the V scheme) in the nonlasing transition, i.e., at $P<1$, the highest number of photons has the Λ model. When $P=1$, i.e., when the pumping rates are equal to the spontaneous relaxation rate in the nonlasing transition (transition $|1\rangle\to|0\rangle$ in the Λ and ladder models and $|2\rangle\to|1\rangle$ in the V model), the number of photons is equal in all three models ($n\approx 3\times 10^{6}$). At high pumping rates, i.e., at $P>1$, the nonlasing transition becomes a bottleneck in both Λ and V schemes, which is not the case in the ladder scheme. This is why the maximal number of photons in the ladder scheme is a few orders of magnitude larger than in Λ and V schemes.

Another important property of the laser performance is the ability to produce "squeezed light," i.e., the ability to

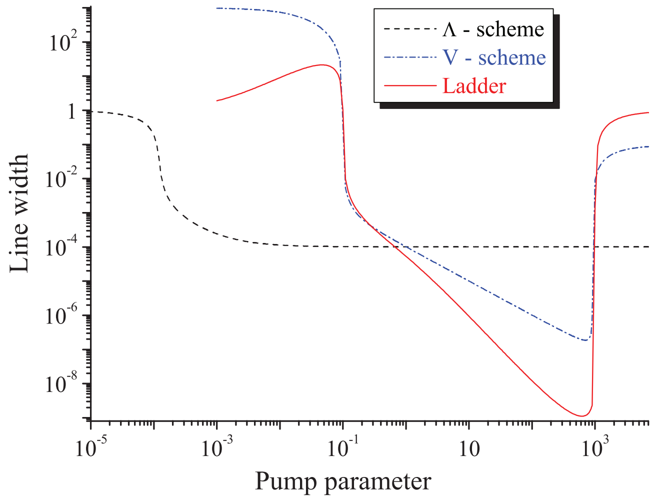

FIG. 6. (Color online) Comparison of linewidth behavior for three different three-level laser models: Ladder, Λ, and V.

suppress the quantum noise. In other words, the ability to get a narrow, as much as possible, distribution of the number of photons in steady state. From this point of view, the Λ and V models are much better than the ladder model. In the first two, one can get up to 25% noise reduction, whereas in the latter one (the ladder model) one can get a maximum of 5% noise reduction.

In Fig. 6, the linewidths in the three models as a function of pump are depicted together. Here one can see a strong difference between the models: the linewidth in the V scheme is about three orders of magnitude smaller than that in the Λ scheme. Moreover, the linewidth in the ladder scheme is about two orders of magnitude smaller than that in the V scheme, so that the ladder scheme has an extremely narrow spectral line, five orders of magnitude narrower than the Λ scheme.

Another interesting comparison can be made with a two-level laser scheme. The point is that the two-level model was the first model deeply studied in laser theory. On the one hand, it is relatively simple and has some attractive properties; on the other hand, it is a rather abstract model since it cannot be realized experimentally. In Fig. 7, the properties of the two-level model are presented in order to compare it with the ladder scheme. Comparing the two models (see Figs. 2 and 7, respectively), one may notice that both models demonstrate similar behavior in the range of laser operation. The reason is that at high enough pumping rate, when $P \gg 1$, the population of the ground level in the ladder model is negligible, so that it remains empty and does not contribute to lasing. This is seen in the calculations. Thus one can conclude that the ladder scheme can be thought of as a feasible experimental realization of an otherwise abstract two-level laser model.

As a potential real-life example, we mention a cascade pumping scheme $1s_3 \to 2p_2 \to 2s_2$ in an electric discharge He-Ne laser. A similar scheme was elaborated upon in Ref.

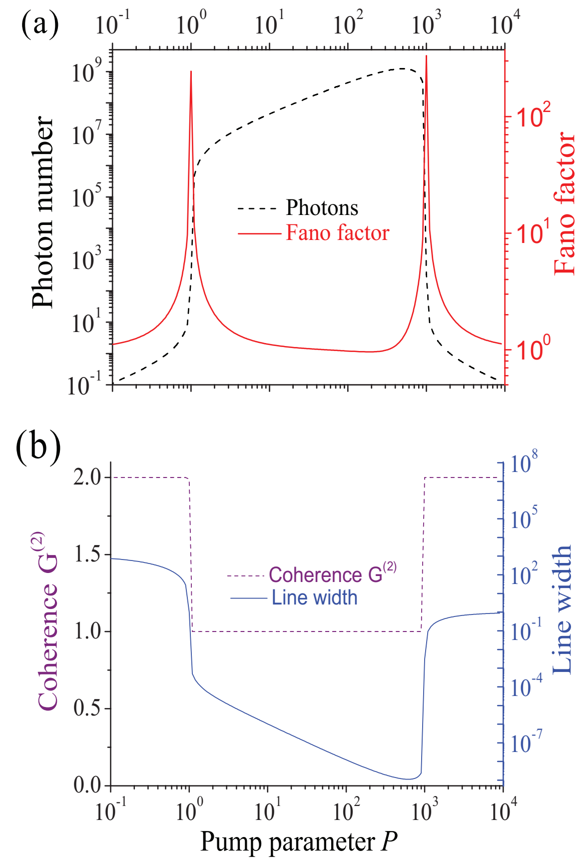

FIG. 7. (Color online) Statistical characteristics of a two-level laser as functions of pump parameter $P = \gamma_{01}/\gamma_{10}$. (a) The photon number and the Fano factor, (b) the linewidth in units of κ. Comparing this model with the ladder one (see Fig. 2), one may notice that both models demonstrate similar behavior in the range of laser operation.

[12] in a different context, namely lasing without inversion in which the coherent pumping is essential. The present model, however, does not require a coherent pump. The dynamic behavior in the case of incoherent pump is entirely different, as discussed above.

IV. SUMMARY

A three-level ladder model has been analyzed and compared with two other three-level models, Λ and V, as well as with a two-level model. We have shown that the ladder scheme has two apparent advantages: first, it can produce much more intensive light than both the Λ and V schemes, and second, it has an extremely narrow spectral line (the linewidth is a few orders of magnitude smaller than that in the Λ and V schemes). A deficiency of the ladder scheme is that the maximum noise reduction that can be achieved in the intensity fluctuations is 5% below the shot noise limit, whereas both the Λ and V models allow up to 25% noise reduction. It has also been shown that at a high enough pumping rate, the ladder scheme effectively behaves as a two-level one.

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