Dynamical trapping and transmission of matter-wave solitons in a collisionally inhomogeneous environment

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We investigate bright matter-wave solitons in the presence of a spatially varying scattering length. It is demonstrated that a soliton can be confined due to the inhomogeneous collisional interactions. Moreover, we observe the enhanced transmission of matter-wave solitons through potential barriers for suitably chosen spatial variations of the scattering length. The results indicate that the manipulation of atomic interactions can become a versatile tool to control matter-wave dynamics.

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I. INTRODUCTION

Recent developments in Bose-Einstein condensates (BECs) [1] have inspired, among others, many studies on their nonlinear excitations. Especially, dark [2], bright [3–5], and gap [6] matter-wave solitons have been observed experimentally. Devices, such as the atom chip [7], offer the possibility to control and manipulate matter-wave solitons. Their formal similarities with optical solitons indicate that they may be used in applications similarly to their optical counterparts [8].

Bright (dark) matter-wave solitons are formed in BECs with attractive (repulsive) interatomic interactions, i.e., for negative (positive) scattering length a. Employing magnetically induced Feshbach resonances the magnitude and sign of a can be changed by tuning the external magnetic field [9] (see also [3–5] where a Feshbach resonance was used for the formation of bright solitons). These studies paved the way for important experiments, such as the formation of molecular BECs [10] and the revelation of the BEC-BCS crossover [11]. On the other hand, it was predicted that a time-dependent modulation of a can be used to prevent collapse in attractive BECs [12], or to create matter-wave breathers [13]. Adding to a constant bias magnetic field a gradient in the vicinity of a Feshbach resonance allows for a spatial variation of a over the ensemble of cold atoms thereby yielding a collisionally inhomogeneous BEC. Due to the availability of magnetic and optical fields the external trapping potential and the spatial variation of a can be adjusted independently. Moreover, hyperfine species with the magnetic quantum number $M_F=0$ do not feel a potential due to the magnetic field but experience magnetically induced Feshbach resonances [14]. In this case the external potential is formed exclusively by an optical dipole potential and the magnetic-field configuration is responsible for the spatially dependent scattering length. Recently this has been exploited to study cold atomic gases in a collisionally inhomogeneous environment (CIE) [15,16].

Here we investigate the dynamics of bright matter-wave solitons of a quasi-one-dimensional (1D) ^{7}Li BEC [3,4] in a CIE. Our aim is to reveal phenomena which make the spatial manipulation of the scattering length a versatile tool. First we show that the inhomogeneity of a causes dynamical trapping of a bright matter-wave soliton. The proposed scheme leads to a *collision-induced breathing soliton*, which oscillates due to the effective confinement and is periodically compressed when passing through the region of large a. As a second situation for a CIE, we consider the transmission of a soliton through a potential barrier underneath which a suitably chosen scattering length $a(x)$ is present. This setup allows us to enhance the transmission of the soliton, i.e., the barrier becomes more transparent compared to the case of a spatially independent atom-atom interaction. In both cases we use the particular form $a(B)$ for a ^{7}Li BEC near a Feshbach resonance. This species has been used to prepare bright matter-wave solitons. The *collision-induced trapping and transmission* of matter-wave solitons reported here are two generic phenomena illustrating that collisionally inhomogeneous matter waves exhibit interesting features that could also be relevant for future applications.

II. COLLISION-INDUCED TRAPPING

To specify our setup we choose the magnetic-field dependence of the scattering length a of a ^{7}Li BEC [4]. It is emphasized that this case serves only as a typical example used for concreteness. We focus on the regime $0 \leqslant B \leqslant 590$ G, which is far to the left of the Feshbach resonance at 720 G, where the inelastic collisional loss of atoms is negligible. We have $a(B)<0$ for $150\text{ G}<B<520\text{ G}$ and $a(B)>0$ elsewhere. At $B\approx 352$ G, the scattering length reaches a minimum with $a\approx -0.23$ nm. Note that quasi-1D bright matter-wave solitons were observed experimentally [3,4] in the above-given regime of $a<0$.

Let us assume a magnetic-field configuration $B=B_0+\varepsilon x$ (G), where ε is the field gradient and $B_0=450$ G, being far to the left of resonance. The gradient chosen is of the order of a few tens of G/μm which can be experimentally realized in atom chips [7]. Other resonances and species might require much smaller values of the gradient to implement a significant change of a on the BEC scale. To extract the spatial

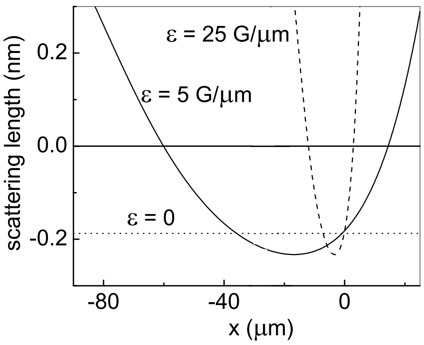

FIG. 1. The spatial variation of the scattering length for $B=450+\varepsilon x$ (G) and magnetic field gradients $\varepsilon=0$ (dotted line), $\varepsilon=5$ G/μm (solid line) and $\varepsilon=25$ G/μm (dashed line).

dependence of the scattering length we apply a fifth-order polynomial fit to the data for $a(B)$ given in Fig. 2 of Ref. [4] in the interval $0\leqslant B\leqslant 590$ G, i.e., we approximate the scattering length as $a(B)\approx\Sigma_{j=0}^{5}A_jB^j$ (A_j are constants). Then, substituting $B=B_0+\varepsilon x$ (G) in this expression, we obtain $a(x)$ as a fifth-order polynomial of x (its form is cumbersome and not shown here). In Fig. 1 we show the function $a(x)$ (for two values of ε), which possesses a single minimum in the considered interval. The dependence of a on x is, as expected, much more pronounced for a larger value of ε. For $\varepsilon=0$ we have $a=-0.182$ nm.

The evolution of an *untrapped*, quasi-1D bright matter-wave soliton in a CIE is described by the following normalized Gross-Pitaevskii (GP) equation:

$$i\partial_t\psi=-\frac{1}{2}\partial_x^2\psi-g(x)|\psi|^2\psi. \tag{1}$$

Here, ψ is the mean-field wave function (with the density $|\psi|^2$ measured in units of the peak 1D density n_0), while x and t are, respectively, normalized to $\xi=\hbar/\sqrt{n_0|g_0|m}$ and ξ/c, where $c=\sqrt{n_0|g_0|/m}$; here, $g_0=2\hbar\omega_\perp a(B_0)$, with $\omega_\perp$ being the confining frequency in the transverse direction. Finally, the spatially dependent nonlinearity is $g(x)=a(x)/a(B_0)$, where $a(x)\equiv a(B_0+\varepsilon\xi x)$ [note that $g(x=0)=g(\varepsilon=0)=1$]. Typically, for a quasi-1D ^{7}Li BEC with $\omega_\perp=2\pi\times 1$ kHz and $n_0=10^9$ m^{-1}, the scales ξ and c take the values 2 μm and 4.6 mm/s, respectively.

Introducing the transformation $\psi=u/\sqrt{g}$ in the region $g(x)>0$ we reduce Eq. (1) to the following perturbed nonlinear Schrödinger (NLS) equation:

$$i\partial_t u+\frac{1}{2}\partial_x^2 u+|u|^2u=R(u), \tag{2}$$

with a perturbation $R(u)$ given by

$$R(u)\equiv\frac{d}{dx}\ln(\sqrt{g})\partial_x u+\frac{1}{2}\left[\frac{d^2}{dx^2}\ln(\sqrt{g})-\left(\frac{d}{dx}\ln(\sqrt{g})\right)^2\right]u. \tag{3}$$

In the case $R=0$ (i.e., for $\varepsilon=0$), Eq. (2) has a bright soliton solution of the form [17]

$$u(x,t)=\eta\,\mathrm{sech}[\eta(x-x_0)]\exp\{i[kx-\phi(t)]\}, \tag{4}$$

where η is the amplitude and inverse width of the soliton, and x_0 is the soliton center; $k=dx_0/dt$ defines both the soliton velocity and wave number, and $\phi(t)=(1/2)(k^2-\eta^2)t$ is its phase. For $\varepsilon\neq 0$, there are two characteristic spatial scales in the problem, the one of the soliton, $L_S=1/\eta$, and the one of the inhomogeneity, $L_B=B_0/(\varepsilon\xi)$. The first term on the right-hand side of Eq. (3) is of order $O(L_S/L_B)$, while the last two terms are of order $O(L_S^2/L_B^2)$. For the above-mentioned typical values of the parameters, and for $\eta=1$ (a soliton with 10^3 atoms and width 4 μm), it is clear that $L_S/L_B=\xi\varepsilon/B_0$. Thus for sufficiently small values of ε, e.g., for $\varepsilon=5$ G/μm, the perturbation R is of order $O(10^{-2})$. In such a case, we may employ the adiabatic perturbation theory for solitons [18]. According to this approach the soliton parameters η, k, and x_0 become unknown, slowly varying functions of time t, but the functional form of the soliton [see Eq. (4)] remains unchanged. This way, we obtain for the soliton center

$$\frac{d^2x_0}{dt^2}=-\frac{\partial V_{\mathrm{eff}}}{\partial x_0},\qquad V_{\mathrm{eff}}(x_0)\equiv-\frac{1}{6}\eta^2(0)g^2(x_0). \tag{5}$$

As $g(x)$ is proportional to $a(x)$, it is clear (see Fig. 1) that the soliton "feels" a collision-induced effective confinement potential V_{eff} *although there is no external trapping potential* in the axial direction [see Eq. (1)]. Notice that this is a *dynamic* potential, given its dependence on the soliton characteristics [i.e., $\eta(t=0)$]. Hence, under certain conditions (see below), the soliton will perform oscillations. The above predictions have been verified by direct numerical integration of the GP Eq. (1) with $\varepsilon=5$ G/μm and for a soliton initially placed at $x_0(0)=0$ (where $B=B_0$) and with zero initial velocity. The result is shown in the right panel of Fig. 2, where the spatiotemporal contour plot of the soliton density is directly compared to the analytical prediction of Eq. (5) (dashed line); the agreement between the two is excellent. We observe that the matter wave is periodically compressed whenever it reaches the region of large scattering lengths ($x\approx-18$ μm with $a\approx-0.23$ nm), exhibiting a robust *breathing behavior* in the CIE. An approximation of the relevant potential with a parabolic one [for $\eta(0)=1$] indicates that it is tantamount to the presence of a magnetic trapping of $\omega_x=2\pi\times 18$ Hz. It is important, however, to note that such dynamics occur if the soliton moves in regions with $g(x)>0$, or $a(x)<0$, since it cannot exist in the case of repulsive interactions. If the initial soliton velocity $k(0)\equiv dx_0(0)/dt=0$, Eq. (5) shows that $k^2(t)=(1/3)\{g^2[x_0(t)]-1\}$, yielding the simple condition $g(x)\geqslant 1$, i.e., $a(x)\leqslant a(B_0)$. The latter guarantees that the soliton is moving in regions with $a<0$, which is always satisfied for $150<B_0<520$ G. On the other hand, if $k(0)\neq 0$ then there exists a velocity threshold $k_{\mathrm{cr}}(0)$, above which the soliton departs from the region with $a(x)<0$. For the above initial conditions this critical value has been found numerically to be $k_{\mathrm{cr}}(0)\approx 0.4$ (or 1.84 mm/s in physical units).

For significantly larger field gradients, e.g., $\varepsilon=25$ G/μm, the soliton width becomes comparable to the length scale L_B. Thus the perturbation R in Eq. (2) is of the order

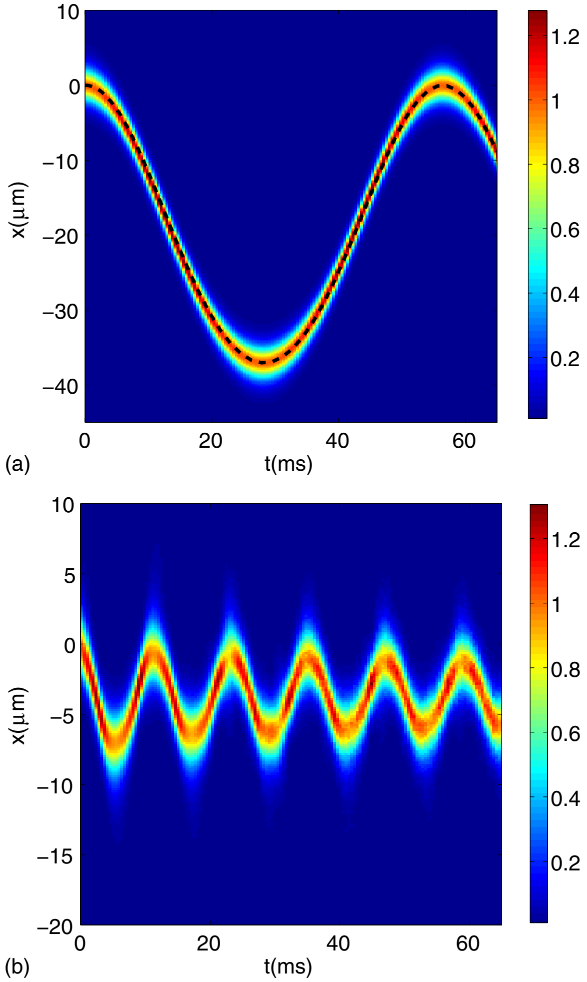

FIG. 2. (Color online) Spatiotemporal contour plot of the density of a bright matter-wave soliton for $B_0=450$ G and magnetic field gradients $\varepsilon=5$ G/μm (top panel) and $\varepsilon=25$ G/μm (bottom panel). The dashed line in the left panel corresponds to the analytical prediction according to Eq. (5).

$L_S/L_B=O(10^{-1})$ and nonadiabatic effects are expected. The numerical integration of the GPE confirms this expectation and reveals that, although the soliton is still confined and performs oscillations, its evolution is nonadiabatic, i.e., emission of radiation is observed. The corresponding results are shown in the bottom panel of Fig. 2: Larger field gradients lead to oscillations with a higher frequency and smaller amplitude.

The above discussed dynamical soliton confinement is experimentally feasible, as there exist already relevant quasi-1D BEC experiments [3,4]. Also, it has a chance to be observed in a higher-dimensional setting, as in the 3D experiment reported in [5]: In this case, the remarkably robust bright solitons observed may follow the dynamics of their 1D counterparts, for sufficiently short time scales (i.e., before higher-dimensional effects, such as collapse, come into play).

III. COLLISION-INDUCED TRANSMISSION

In our second setup we consider the scattering of a bright matter-wave soliton off an external potential barrier thereby comparing the results for homogeneous and inhomogeneous atomic interactions. In particular, we consider a GP equation in the form $i\partial_t\psi=-(1/2)\partial_x^2\psi-g(x)|\psi|^2\psi+V_b(x)\psi$, with the initial condition of a soliton in Eq. (4) for $t=0$ and $\eta=1$. Here, $V_b(x)=V_0\,\mathrm{sech}^2[\alpha(x-x_B)]$ is the barrier, characterized by an amplitude V_0, width α^{-1}, and location x_B. In the following we assume $V_0=1$ and $\alpha^{-1}=1/2$, i.e., the barrier's width is half the soliton's width, for reasons similar to those discussed in [19]. Our aim is to compare the transmission coefficient T for $g=1$ and $g=g(x)$.

In order to compare the transmission in these cases, the incoming and outgoing scattering environments should be identical, i.e., the scattering lengths should asymptotically (outside the range of the barrier) take the same value. Regarding the CIE case with $g=g(x)$, this is possible in the case of ^{7}Li, due to the convenient form of the function $a(B)$ (see Fig. 2 of [4]) in the considered range $150<B<520$ G with $a<0$: It is readily observed that one could choose two specific field values, e.g., B_1 and B_2, so that the respective scattering lengths (sufficiently far from the barrier) are equal, i.e., $a(B_1)=a(B_2)$. The same value of a should also be used for the homogeneous case. To implement the above, we assume a localized inhomogeneous magnetic field,

$$B(x)=\frac{1}{2}\{(B_1+B_2)+(B_1-B_2)\tanh[w(x-x_B)]\}, \qquad (6)$$

where the parameters w and x_B characterize, respectively, the inverse width and location of the region of inhomogeneity; note that the inhomogeneity is centered at the same position where the barrier is located (at $x=x_B$). We also assume that $B_1=450$ G and $B_2=265$ G for which $a=-0.182$ nm (other choices are, of course, equally possible and lead to similar results). We finally note that the above-mentioned field configuration can be realized by a multiwire setup or a current density flowing in a half plane augmented by a homogeneous bias field.

The setup is illustrated in Fig. 3: The initial ($t=0$) form of the soliton, as well as its transmitted and reflected parts (for the inhomogeneous case at $t=6$ ms) are depicted and labeled, respectively. At the location of the barrier (shaded region), and for the collisionally inhomogeneous case, there exists a local spatial change of the scattering length. In particular, the form of $a(x)$ is obtained by inserting the magnetic field $B(x)$ [Eq. (6)] into the fifth-order polynomial fit $a(B)\approx\Sigma_{j=0}^{5}A_jB^j$ approximating the data of $a(B)$ [4] in the interval $0\leqslant B\leqslant 590$ G (the same fit was also used in the previous section). The resulting spatial dependence of the scattering length is also shown in Fig. 3; as noted above, the scattering length takes equal values far from the barrier in both the homogeneous and inhomogeneous cases. In dimensionless units (used in the numerics—see below), we have assumed that $g(x)=a[B(x)]/a(B_1)$. Notice that in the homogeneous case the field is $B=B_1$, and thus $g=1$; in the inhomogeneous case, the field is $B=B_1$ and $B=B_2$ for $x\rightarrow\pm\infty$, respectively, and thus $g=1$ as well.

We have numerically integrated the above-mentioned GPE (with the potential barrier term $V_b\psi$) to determine the transmission coefficient T. The results, shown in Fig. 4,

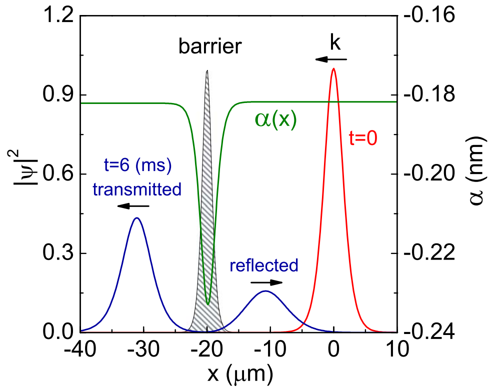

FIG. 3. (Color online) Scattering of a soliton off a barrier. The left (right) vertical axis shows the soliton density $|\psi|^2$ (scattering length a). The incident soliton is shown at its initial location at $x=0$. The transmitted and reflected parts of the soliton are shown at $t=6$ ms (inhomogeneous case); the former is located at $x\approx-31$ μm and the latter at $x\approx-11$ μm. The barrier is located at $x_B=-20$ μm (shaded area). The spatial dependence of the scattering length $a(x)$, forming a dip at the location of the barrier, is also shown.

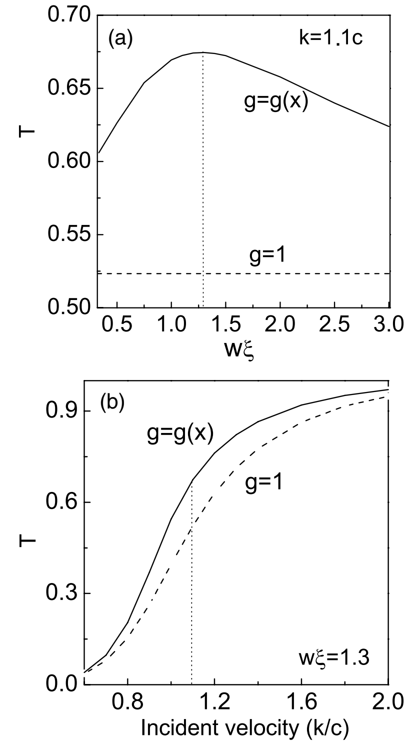

FIG. 4. The transmission T as a function of the inverse width w of the inhomogeneity, in units of ξ^{-1} (top panel), or the soliton's incident velocity k, in units of c (bottom panel). Dashed and solid lines correspond to the collisionally homogeneous and inhomogeneous cases, respectively.

present T as a function of the width of the inhomogeneity (top panel) and the soliton's incident velocity (bottom panel). Generally, for a fixed soliton velocity, or width of the inhomogeneity, the transmission T in the inhomogeneous case is always larger than the one in the homogeneous case. Particularly, for $k=1.1c$ and $w\xi=1.3$, the relative difference of the transmission T for the two cases becomes maximal, being $\approx 15\%$. This result clearly demonstrates that the transmission of a matter-wave soliton may be enhanced in the presence of a spatially dependent collisional interaction. This is due to the fact that, within the barrier, the inhomogeneity causes the local scattering length to be more negative, resulting in a compression of the soliton. The latter raises the soliton's chemical potential, and increases its amplitude and its speed; this, in turn, leads to facilitated transmission, compared to the homogeneous case [see also [19] and particularly their Eq. (2)].

IV. CONCLUSIONS

We have explored the dynamics of bright matter-wave solitons subject to a spatially varying nonlinearity, which can be realized by means of an external inhomogeneous magnetic field on top of a bias field. It was demonstrated that a dynamical trapping of the matter wave can be achieved solely on the basis of the spatial change of the collisional interaction, thereby creating a breathing matter-wave state in the collisionally inhomogeneous environment. In an adiabatic regime, such a state could be well described within the realm of soliton perturbation theory, while for abrupt variations of the scattering length, radiative emissions are nontrivial resulting in energy losses and hence shorter period oscillations. Using a localized spatial variation of the scattering length, we have shown that the transmission of matter-wave solitons through a barrier can be enhanced by suitably manipulating the collisional properties of the condensate in the vicinity of the potential barrier. Collisionally inhomogeneous environments therefore hold considerable promise in the effort to control and manipulate matter waves.

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