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Multiparty correlation measure based on the cumulant

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We propose a genuine multiparty correlation measure for a multiparty quantum system as the trace norm of the cumulant of the state. The legitimacy of our multiparty correlation measure is explicitly demonstrated by proving it satisfies the five basic conditions required for a correlation measure. As an application we construct an efficient algorithm for the calculation of our measures for all stabilizer states.

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I. INTRODUCTION

Although a composite system can contain several parties, its physical properties do not necessarily equal the sum of its parts. Correlations among different parts in a composite system are usually invoked to describe the difference between the physical properties of a composite system and the sum of its parts. Roughly speaking, the physical properties for a composite system are reflected both in the sum of the physical properties of its parts and the correlations (among different parts) of the composite system. What makes a composite system more interesting in some sense can be largely attributed to the existence of correlations among the constituents of the system.

The rapid development of quantum information science in recent years has called for serious efforts to characterize correlations in a composite quantum system. Depending on whether quantum nonlocal resources are needed in the preparation of a quantum state, the correlations among different parties in a composite system can be further classified into classical and quantum correlation [1]. In the language of quantum-information science, quantum correlation is often called quantum entanglement to emphasize the inseparability of the quantum state of a composite system into those of its parties. Quantum entanglement is widely believed to be a useful resource in implementing quantum computation and information tasks [2].

The two-party correlation, especially the two-party quantum entanglement, has been extensively studied and by now is in many sense well understood [3–6]. However, very little is known about quantum entanglement properties for multiparty systems despite consorted efforts over the last decade. As we will show through our work, not only computational but also conceptual difficulties arise when one attempts to characterize genuine multiparty correlations [6]. For example, the two-party correlation of a quantum state $\rho^{(12)}$ is described by the mutual entropy of the composite state, i.e., $S(1:2)=S(\rho^{(1)})+S(\rho^{(2)})-S(\rho^{(12)})$, where $S(\rho)$ is the von Neaumann entropy of the quantum state ρ, which is defined as $S(\rho)=-\mathrm{Tr}(\rho\log_2\rho)$. The three-party mutual entropy for a three-state $\rho^{(123)}$, defined accordingly as

$$S(1{:}2{:}3)=S(\rho^{(123)})-S(\rho^{(12)})-S(\rho^{(23)})-S(\rho^{(13)})+S(\rho^{(1)})+S(\rho^{(2)})+S(\rho^{(3)}), \tag{1}$$

does not faithfully characterize genuine three-party correlation because it is known to take on negative values for some specific three-party quantum states. Vedral suggested an alternative three-party correlation measure using the relative entropy defined as [6]

$$S(\rho^{(123)}\|\rho^{(1)}\rho^{(2)}\rho^{(3)})=S(\rho^{(1)})+S(\rho^{(2)})+S(\rho^{(3)})-S(\rho^{(123)}). \tag{2}$$

Although semipositive definite for any three-party quantum state and easily generalizable to arbitrary multiparty cases, this relative entropy measure is not a proper correlation measure. For a product state $\rho^{(123)}=\rho^{(1)}\rho^{(23)}$ exhibiting only two-party correlation, the above relative entropy correlation measure $S(\rho^{(123)}\|\rho^{(1)}\rho^{(2)}\rho^{(3)})=S(2:3)$ simply measures two-party correlation, rather than the genuine three-party correlation. It turns out $S(\rho^{(123)}\|\rho^{(1)}\rho^{(2)}\rho^{(3)})$ measures the total correlation, the sum of both the two-party and three-party correlations in the three-party state.

The aim of this paper is to provide a genuine multiparty correlation measure for arbitrary quantum states. Our choice for the genuine multiparty correlation measure to be shown and justified below is intimately related to the cumulant (or the Ursell expansion) of a multiparty density matrix [7–10]. The research on the cumulant of a multiparty state has had a long history. Cumulants were first introduced by Thiele in 1899, who called them half-invariants [11]. The name of the cumulant is given by Fisher and Wishart in 1931 [12]. It was first introduced by Ursell into classical physics in 1927 [13], and Kahn and Ulenbeck gave the quantum mechanical treatment in 1938 [14].

The cumulant of a multiparty density matrix was known to be related to multiparty correlation, and was even called the correlation operator [15]. However, it does not provide a legitimate measure to compare the correlations in different quantum states.

This paper is arranged as follows. In Sec. II, we outline three preliminaries: First, we will introduce the trace distance between two Hermitian operators of the same trace. Second, the cumulant of a multiparty density matrix is introduced in an instructive way. Third, the conditions for a legitimate

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multiparty correlation measure are discussed. In Sec. III, we define our multiparty correlation measure as the trace norm of the cumulant of the multiparty density matrix. We further prove that it is a legitimate multiparty correlation measure by explicitly demonstrating that it satisfies the five basic conditions for a multiparty correlation measure. We then apply our correlation measure to study three cases: simple two-party states, three-party states, and more general multiparty stabilizer states [16,17]. Finally, we summarize our results with a discussion.

II. PRELIMINARIES

This section contains the three preliminaries for a convenient introduction of our correlation measure. First, we generalize the trace distance between two quantum states to the trace distance between two Hermitian operators with the same trace. An important property of this distance is briefly reviewed, which constitutes the key element in proving the legitimacy of the proposed correlation measure. Second, we briefly review the concept of the cumulant for a multiparty density matrix in an instructive way. Third, we discuss the basic requirements for a legitimate multiparty correlation measure.

A. Trace distance

The trace distance between two Hermitian operators ρ and σ with the same trace is defined by

$$D(\rho,\sigma) \equiv \tfrac{1}{2}\mathrm{Tr}|\rho-\sigma|. \tag{3}$$

This definition is a simple extension of the trace distance between two quantum states [2]. Note that there is an alternative useful expression for the trace distance,

$$D(\rho,\sigma) = \max_P \mathrm{Tr}[P(\rho-\sigma)], \tag{4}$$

where the maximum is taken over all projectors P.

It is easy to check that it satisfies the three basic requirements for a distance, i.e.,

(1) $D(\rho,\sigma)=0 \Leftrightarrow \rho=\sigma$,

(2) $D(\rho,\sigma)=D(\sigma,\rho)$,

(3) $D(\rho,\tau) \leqslant D(\rho,\sigma)+D(\sigma,\tau)$.

The trace distance has the following important property, which is listed here as a theorem.

Theorem 1. (Trace-preserving quantum operations are contractive) Suppose $\mathcal{E}$ is a trace-preserving quantum operation. Let ρ and σ be Hermitian operators with the same trace. Then

$$D(\mathcal{E}(\rho),\mathcal{E}(\sigma)) \leqslant D(\rho,\sigma). \tag{5}$$

This theorem is a simple generalization of the well-known theorem where ρ and σ are density operators [2]. It plays an important role in proving our correlation measure is legitimate. The proof of this theorem is omitted here because its proof is almost the same to that of the original theorem, which can be found in several reference books.

B. Cumulants

We introduce an instructive way to understand the cumulant for an N-party density matrix $\rho^{(12\cdots N)}$. Suppose we know all the $(N-1)$-party reduced density matrices, we can construct a pseudo- N-party density matrix $\tilde{\rho}^{(12\cdots N)}$ such that all of its reduced density matrices are the same as those reduced from the density matrix $\rho^{(12\cdots N)}$. Then the cumulant of $\rho^{(12\cdots N)}$ is defined by

$$C(\rho^{(12\cdots N)}) \equiv \rho^{(12\cdots N)} - \tilde{\rho}^{(12\cdots N)}. \tag{6}$$

The pseudo-N-party density matrix $\tilde{\rho}^{(12\cdots N)}$ can be directly constructed by the following method. First, we find all the partitions of $T_N \equiv \{12\cdots N\}$. Any partition can be denoted by $\{S_1,S_2,\ldots,S_M\}$ where $M(\geqslant 2)$ is the number of partition, $\forall i,j \in T_M$, $S_i \neq \varnothing$, $S_i \cap S_j = \varnothing$, and $\Pi_{i=1}^{M} \cup S_i = T_N$. Second, the pseudo- N-party density matrix is expressible in the form

$$\tilde{\rho}^{(12\cdots N)} = \sum_{\{S_i\}} a_{\{S_i\}} \prod_i \rho^{(S_i)}, \tag{7}$$

where $a_{\{S_i\}}$ are constants dependent on the specific partition $\{S_i\}$. Third, all the constants can be determined by the conditions $\mathrm{Tr}_i\, \tilde{\rho}^{(12\cdots N)} = \mathrm{Tr}_i\, \rho^{(12\cdots N)}$, or

$$\mathrm{Tr}_i\, C(\rho^{(12\cdots N)}) = 0, \tag{8}$$

for any $i \in T_N$.

To determine the constants $a_{\{S_i\}}$ in Eq. (7), we will use the cumulants to expand the density matrix, which is the so-called Ursell expansion of the density matrix [9]. Let us denote $\rho^{(1)} = C^{(1)}$ and $C(\rho^{(12\cdots N)}) = C^{(12\cdots N)}$. From Eqs. (6) and (7), the density matrix

$$\rho^{S} = \sum_{\{S_i\}} b_{\{S_i\}} \prod_i C^{(S_i)} + C^{(S)}, \tag{9}$$

where the sum is taken over all the partitions $\{S_i\}$ of S with $S \subseteq T_N$. Using the condition (8) and the mathematical induction method, we can prove that the constants $b_{\{S_i\}} = 1$. Using Eq. (9), we obtain the unique solution of the constants in Eq. (7) satisfying the condition (8): $a_{\{S_i\}} = (-1)^M (M-1)!$ [7–9].

The cumulant of an N-party density matrix has the following important property.

Theorem 2. If an N-party density matrix is a product state, i.e., $\rho^{(12\cdots N)} = \rho^{(S_1)}\rho^{(S_2)}$, where $\{S_1,S_2\}$ is a partition of T_N, then $\tilde{\rho}^{(12\cdots N)} = \rho^{(12\cdots N)}$, i.e., the cumulant $C(\rho^{(12\cdots N)}) = 0$.

Proof. $N=2$, $C(\rho^{(12)})$ follows obviously from $\rho^{(12)} = \rho^{(1)}\rho^{(2)}$. Assume the theorem is valid for all $k \leqslant N-1$, we need to prove it is also valid for $k=N$. According to Eq. (9), we obtain

$$\rho^{(12\cdots N)} = \sum_{\{S_{1i}\}} \prod_i C^{(S_{1i})} \sum_{\{S_{2j}\}} \prod_j C^{(S_{2j})} + C^{(12\cdots N)} = \rho^{(S_1)}\rho^{(S_2)} + C^{(12\cdots N)},$$

where $\{S_{1i}\}$ and $\{S_{2i}\}$ are partitions of S_1 and S_2, respectively. The above equation implies $C^{(12\cdots N)} = 0$. This completes our proof.

C. Conditions of a genuine multiparty correlation measure

From a general physical consideration, a genuine N-party correlation measure $M_C(\rho^{(12\cdots N)})$ should satisfy the following five conditions [5].

(1) Negative correlation has no physical interpretation. $M_C(\rho^{(12\cdots N)}) \geqslant 0$.

(2) Any product state implies no genuine N-party correlation. If an N-party density matrix $\rho^{(12\cdots N)} = \rho^{(S_1)}\rho^{(S_2)}$, where $\{S_1, S_2\}$ is a partition of T_N, then $M_C(\rho^{(12\cdots N)}) = 0$.

(3) The correlation measure is invariant under local unitary transformations. $M_C(U_L\rho^{(12\cdots N)}U_L^{\dagger}) = M_C(\rho^{(12\cdots N)})$, where $U_L = \Pi_{i=1}^{N} U^{(i)}$.

(4) The correlation measure is invariant when the system is augmented by locally noncorrelated auxiliary subsystems. $M_C(\rho^{(12\cdots N)} \otimes \sigma_L^{(12\cdots N)}) = M_C(\rho^{(12\cdots N)})$, where $\sigma_L^{(12\cdots N)} = \Pi_{i=1}^{N}\sigma^{(i)}$.

(5) The correlation measure is nonincreasing under local operations. $C_M(\mathcal{E}_L(\rho^{(12\cdots N)})) \leqslant C_M(\rho^{(12\cdots N)})$, where $\mathcal{E}_L = \Pi_{i=1}^{N}\mathcal{E}^{(i)}$.

Here we emphasized that condition (2) is stronger than the following condition N-product version,

(2′) If an N-party density matrix $\rho^{(12\cdots N)} = \Pi_{i=1}^{N}\rho^{(i)}$, then $M_C(\rho^{(12\cdots N)}) = 0$.

As we have mentioned in the introduction, this condition can be used to define the total correlation, which includes different types of correlations in the state.

We include the extra condition (4) as a general requirement for a legitimate correlation measure because correlations in a system should not depend on the rest of the world or ancillary systems ($\sigma_L^{(12\cdots N)}$) if they are independent and uncorrelated ($\sigma_L^{(12\cdots N)} = \Pi_{i=1}^{N}\sigma^{(i)}$).

An optional requirement for a legitimate correlation measure is related to the so-called additivity, i.e., requiring $M_C(\rho^{(12\cdots N)} \otimes \sigma_L^{(12\cdots N)}) = M_C(\rho^{(12\cdots N)}) + M_C(\sigma^{(12\cdots N)})$ for an absolute correlation scale. This additivity requirement is clearly stronger than our proposed condition (4). We feel such a strong condition is not needed as argued previously in the basic requirements of two-party entanglement measure [4].

III. A GENUINE MULTIPARTY CORRELATION MEASURE

In this section, we present our central result of a genuine multiparty correlation measure. It is first proposed and further proved to satisfy the aforementioned five basic requirements for a multiparty correlation measure, thus it constitutes a legitimate multiparty correlation measure. We end this section by demonstrating the applications of our proposed measures to several important classes of examples.

A. General formalism

Definition. An N-party correlation measure of the state $\rho^{(12\cdots N)}$ is proposed as

$$M_C(\rho^{(12\cdots N)}) \equiv D(\rho^{(12\cdots N)}, \tilde{\rho}^{(12\cdots N)}) = \tfrac{1}{2}\mathrm{Tr}|C(\rho^{(12\cdots N)})|, \tag{10}$$

which constitutes a legitimate genuine multiparty correlation measure because of the following theorem, which is the main result of our paper.

Main theorem. $M_C(\rho^{(12\cdots N)})$ is a legitimate N-party correlation measure, i.e., it satisfies the five basic conditions for an N-party correlation measure.

Proof. Let us prove the five conditions, respectively,

(1) $M_C(\rho^{(12\cdots N)}) = \frac{1}{2}\mathrm{Tr}|C(\rho^{(12\cdots N)})| \geqslant 0$.

(2) $\rho^{(12\cdots N)} = \rho^{(S_1)}\rho^{(S_2)} \Rightarrow C(\rho^{(12\cdots N)}) = 0 \Rightarrow M_C(\rho^{(12\cdots N)}) = 0$. Note that we have used Theorem 2 in the first step of this proof.

(3) Under the action of U_L,

$$\rho^{(S_i)} \mapsto \mathrm{Tr}_{T_N - S_i}(U_L\rho^{(12\cdots N)}U_L^{\dagger}) = U_{S_i}\rho^{(S_i)}U_{S_i}^{\dagger},$$

where $U_{S_i} = \Pi_{j\in S_i}U^{j}$. Using the expression (7) for the pseudodensity matrix, we obtain

$$\tilde{\rho}^{(12\cdots N)} \mapsto U_L\tilde{\rho}^{(12\cdots N)}U_L^{\dagger}.$$

Thus, the cumulant

$$C(U_L\rho^{(12\cdots N)}U_L^{\dagger}) = U_L C(\rho^{(12\cdots N)})U_L^{\dagger}.$$

Therefore,

$$M_C(U_L\rho^{(12\cdots N)}U_L^{\dagger}) = \tfrac{1}{2}\mathrm{Tr}|U_L C(\rho^{(12\cdots N)})U_L^{\dagger}| = M_C(\rho^{(12\cdots N)}). \tag{11}$$

(4) It is easy to prove that

$$C(\rho^{(12\cdots N)} \otimes \sigma_L^{(12\cdots N)}) = C(\rho^{(12\cdots N)}) \otimes \sigma_L^{(12\cdots N)}.$$

Thus

$$\begin{aligned} M_C(\rho^{(12\cdots N)} \otimes \sigma_L^{(12\cdots N)}) &= \tfrac{1}{2}\mathrm{Tr}|C(\rho^{(12\cdots N)}) \otimes \sigma_L^{(12\cdots N)}| \\ &= M_C(\rho^{(12\cdots N)}). \end{aligned} \tag{12}$$

(5) Under the action of $\mathcal{E}_L$,

$$\rho^{(S_i)} \mapsto \mathrm{Tr}_{T_N - S_i}[\mathcal{E}_L(\rho^{(12\cdots N)})] = \mathcal{E}_{S_i}(\rho^{(S_i)}),$$

where $\mathcal{E}_{S_i} = \Pi_{j\in S_i}\mathcal{E}^{(j)}$. Using the expression (7) for the pseudodensity matrix, we obtain

$$\tilde{\rho}^{(12\cdots N)} \mapsto \mathcal{E}_L(\tilde{\rho}^{(12\cdots N)}).$$

Therefore,

$$\begin{aligned} M_C[\mathcal{E}_L(\rho^{(12\cdots N)})] &= D(\mathcal{E}_L(\rho^{(12\cdots N)}), \mathcal{E}_L(\tilde{\rho}^{(12\cdots N)})) \\ &\leqslant M_C(\rho^{(12\cdots N)}). \end{aligned} \tag{13}$$

Note that Theorem 1 was used for the proof of the last inequality.

B. Applications

1. Two-party correlation

The two-party correlation measure is defined as

$$M_C(\rho^{(12)}) = \tfrac{1}{2}\mathrm{Tr}|\rho^{(12)} - \rho^{(1)}\rho^{(2)}|. \tag{14}$$

The physical meaning of this measure is the distance between the state $\rho^{(12)}$ and its reduced product state $\rho^{(1)}\rho^{(2)}$ [2].

Let us apply the two-party correlation measure to the following two typical states. The first state is the maximally classical correlated two-qubit state

$$\rho_c^{(12)} = \tfrac{1}{2}(|00\rangle_{(12)}\,{}_{(12)}\langle 00| + |11\rangle_{(12)}\,{}_{(12)}\langle 11|), \tag{15}$$

we find the two-party correlation according to our measure is given by $M_C(\rho_c^{(12)})=1/2$. The second state is the maximally entangled Bell state

$$|B\rangle_{(12)} = \frac{1}{\sqrt{2}}(|00\rangle_{(12)} + |11\rangle_{(12)}), \tag{16}$$

for which our two-party correlation measure gives $M_C(|B\rangle_{(12)}\,{}_{(12)}\langle B|)=3/4$. The unique difference between the Bell state (16) and the maximally classical correlated two-qubit state (15) is the very existence of quantum coherence in $|B\rangle_{(12)}$. Our result shows that quantum coherence will increase the two-party correlation [18].

2. Three-party correlation

When dealing with three-party systems, our correlation measure becomes

$$M_C(\rho^{(123)}) = \tfrac{1}{2}\mathrm{Tr}|C(\rho^{(123)})|. \tag{17}$$

We again study several types of typical three-qubit states. Analogously, the first state is the maximally classical correlated three-qubit state

$$\rho_c^{(123)} = \tfrac{1}{2}(|000\rangle_{(123)}\,{}_{(123)}\langle 000| + |111\rangle_{(123)}\,{}_{(123)}\langle 111|), \tag{18}$$

for which our three-party correlation measure vanishes, i.e., $M_C(\rho_c^{(123)})=0$.

The second state we consider is the GHZ state [19]

$$|G\rangle_{(123)} = \frac{1}{\sqrt{2}}(|000\rangle_{(123)} + |111\rangle_{(123)}), \tag{19}$$

for which our three-party correlation measure gives $M_C(|G\rangle_{(123)}\,{}_{(123)}\langle G|)=1/2$.

These two specific examples indicate that just like the case of two parties, quantum coherence generally increases the three-party correlation. More specifically, we note that $M_C(\rho_c^{(123)})=0$, i.e., there exists no genuine three-party correlation in this state. Yet, even according to our definition of separate states included in the condition (2), this state is not really a noncorrelated state. This example thus shows that even for a three-qubit state, the first of the five conditions becomes a sufficient but not necessary condition, i.e., if the correlation measure $M_C(\rho^{(123)})$ is zero, we do not know for sure if this state is noncorrelated or not. Contrary to this for a mixed state, however, we find that for a three-qubit pure state, the following theorem is valid.

Theorem 3. If $\rho^{(123)}$ is a three-qubit pure state, and the correlation measure $M_C(\rho^{(123)})=0$, then the state is noncorrelated, i.e., it can be written as a direct product of two density matrixes of mutually independent parts.

The proof of this theorem is attached as an appendix.

Remarkably, this theorem implies the other typical three-qubit pure state, the so-called W state [20] possesses genuine three-qubit correlation.

3. Multiparty correlations in the N-qubit GHZ state

Let us study the correlations in the N-qubit GHZ state, which is defined by

$$|\mathrm{GHZ}^{(N)}\rangle = \frac{1}{\sqrt{2}}(|0\cdots 0\rangle_{(1\cdots N)} + |1\cdots 1\rangle_{(1\cdots N)}). \tag{20}$$

An analogous classically correlated state based on what we considered earlier is

$$\rho_c^{(N)} = \tfrac{1}{2}(|0\cdots 0\rangle_{(1\cdots N)(1\cdots N)}\langle 0\cdots 0| + |1\cdots 1\rangle_{(1\cdots N)(1\cdots N)} \times\langle 1\cdots 1|). \tag{21}$$

Since all reduced density matrices of the above two states are exactly the same, the difference of their respective cumulants simply consists of the off-diagonal terms

$$C(|\mathrm{GHZ}^{(N)}\rangle) - C(\rho_c^{(N)}) = \tfrac{1}{2}(|0\cdots 0\rangle_{(1\cdots N)(1\cdots N)}\langle 1\cdots 1| + |1\cdots 1\rangle_{(1\cdots N)(1\cdots N)}\langle 0\cdots 0|). \tag{22}$$

Furthermore, the cumulant of the state $\rho_c^{(N)}$ is given by

$$C(\rho_c^{(N)}) = c_N\Big(\sum \rho_{\mathrm{even}} - \sum \rho_{\mathrm{odd}}\Big), \tag{23}$$

where

$$\rho_{\mathrm{even}} = \sum_{\{a_i\}\in\{0,1\}}^{\Sigma_i a_i\in \mathrm{even}} |a_1\cdots a_N\rangle_{(1\cdots N)(1\cdots N)}\langle a_1\cdots a_N|,$$

$$\rho_{\mathrm{odd}} = \sum_{\{a_i\}\in\{0,1\}}^{\Sigma_i a_i\in \mathrm{odd}} |a_1\cdots a_N\rangle_{(1\cdots N)(1\cdots N)}\langle a_1\cdots a_N|,$$

and the coefficient

$$c_N = \sum_{M=1}^{N}\sum_{i=0}^{M-1}\frac{(-1)^{M+i-1}(M-i)^N(M-1)!}{2^M i!(M-i)!} = (-1)\frac{\partial^{N-1}}{\partial\lambda^{N-1}}\left(\frac{1}{1+e^{\lambda}}\right)\Bigg|_{\lambda=0}. \tag{24}$$

Specifically, for any odd number N, $c_N=0$. Therefore for any odd number N, the N-party correlation measure gives

$$C_M(\rho_c^{(N)}) = 0, \tag{25}$$

$$C_M(|\mathrm{GHZ}^{(N)}\rangle) = \tfrac{1}{2}. \tag{26}$$

For an even number N, $c_N\neq 0$. For example, we find $c_2=1/4$, $c_4=-1/8$, $c_6=1/4$, $c_8=-17/16$, $c_{10}=31/4$, $c_{12}=-691/8$, $c_{14}=5461/4$, $c_{16}=-929\,569/32,\ldots$. The corresponding N-party correlation measure becomes

$$C_M(\rho_c^{(N)}) = 2^{N-1}|c_N|, \tag{27}$$

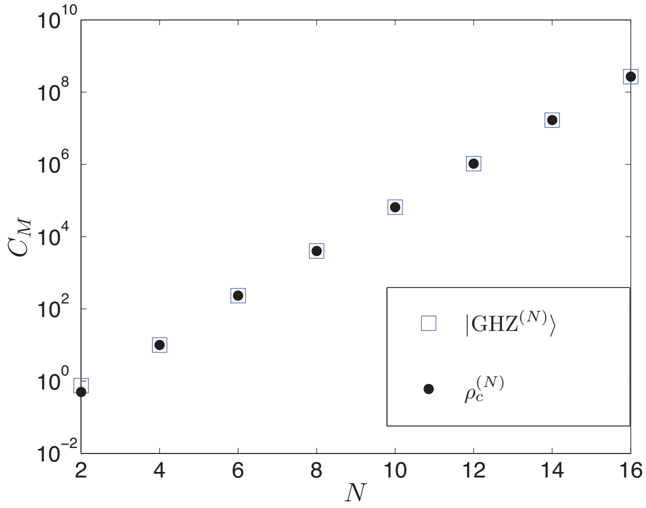

FIG. 1. (Color online) The N dependence of the total N-partition correlation for an N-qubit GHZ state and the maximal classically correlated N-qubit state.

$$C_M(|\mathrm{GHZ}^{(N)}\rangle) = 2^{N-1}|c_N| + \frac{\left|c_N + \frac{1}{2}\right| + \left|c_N - \frac{1}{2}\right| - 2|c_N|}{2}, \tag{28}$$

which for $|c_N| \geqslant 1/2$, gives $C_M(|\mathrm{GHZ}^{(N)}\rangle) = C_M(\rho_c^{(N)})$. In Fig. 1, we show the N dependence of the above correlation measures for the two states. Two interesting features are worthy of some attention. First, it is interesting to note that $C_M(|\mathrm{GHZ}^{(N)}\rangle) = C_M(\rho_c^{(N)})$ for $N \geqslant 8$ because $|c_N| \geqslant 1/2$ for $N \geqslant 8$. The physical meaning of this interesting equality is yet to be understood. Second, the exponentially increasing dependence of the total correlations on N in both cases reflects the exponentially increasing size of the Hilbert space.

4. Multiparty correlations in stabilizer states

The correlations (or entanglement) in stabilizer states have been discussed in Refs. [22–24]. We now apply our multiparty correlation measure (10) to characterize these correlations.

To compute our measure of Eq. (10) for a given state, we need first to obtain the corresponding reduced density matrices. This task is dramatically reduced if the given state is a stabilizer state. A stabilizer state is the unique simultaneous eigenstate of the maximal Abelian subgroup of the N-qubit Pauli group $\mathcal{G}_N$. The group $\mathcal{G}_N$ consists of all 4×4^N local operators of the form $M = \alpha_M M_1 \otimes \cdots \otimes M_N$, where $\alpha_M \in \{\pm 1, \pm i\}$ is an overall phase factor and M_i is either the 2×2 identity matrix I_i or one of the Pauli matrices X_i, Y_i, Z_i.

A stabilizer $\mathcal{S}$ in the Pauli group $\mathcal{G}_N$ is defined as an Abelian subgroup of $\mathcal{G}_N$ which does not contain $-I$. A stabilizer consists of 2^k Hermitian Pauli operators (i.e., they must have real overall phase factors ± 1), where k ($\leqslant N$) is the number of the generators of $\mathcal{S}$. As the operators in a stabilizer commute, they can be diagonalized simultaneously and, what is more, if $k = N$ then there exists a unique state $|\psi\rangle$ on N qubits such that $M|\psi\rangle = |\psi\rangle$ for every $M \in \mathcal{S}$. Such a state $|\psi\rangle$ is called a stabilizer state and the group $\mathcal{S} = \mathcal{S}(\psi)$ is called the stabilizer of $|\psi\rangle$. The expansion

$$|\psi\rangle\langle\psi| = \frac{1}{2^N} \sum_{M \in \mathcal{S}(\psi)} M, \tag{29}$$

which describes a stabilizer state as a sum of all elements in its stabilizer, can be readily verified.

The support supp(M) of an element $M = \alpha_M M_1 \otimes \cdots \otimes M_n \in \mathcal{S}(\psi)$ is the set of all $i \in \{1, \ldots, n\}$ such that M_i differs from the identity I_i. Let S be a subset of T_N. Tracing out all qubits of $|\psi\rangle$ outside S gives the reduced density matrix $\rho^{(S)}(\psi)$ associate with S, which is equal to

$$\rho^{(S)}(\psi) = \frac{1}{2^{|S|}} \sum_{M \in \mathcal{S}, \mathrm{supp}(M) \subseteq S} M. \tag{30}$$

This can easily be verified using the identity (29). This allows use to obtain the cumulant of the state by direct computation, with the result taking the following form:

$$C(|\psi\rangle\langle\psi|) = \frac{1}{2^N} \sum_{M \in \mathcal{S}} \beta_M M, \tag{31}$$

where β_M are real constants, and $\beta_M = 0$ when $M = \Pi_{i=1}^N I^i$. The eigenvalues of the cumulant $C(|\psi\rangle\langle\psi|)$ can be obtained as follows. First we take a generator S_G of the stabilizer group $\mathcal{S}$, which includes n independent elements in $\mathcal{S}$, denoted by G_i ($i = 1, 2, \ldots, n$). We take values of all the G_i as ± 1, which determines the values $v_{\{G_i\}}(M)$ of all the M. All the eigenvalues of the cumulant are

$$\frac{1}{2^N} \sum_{M \in \mathcal{S}} \beta_M v_{\{G_i\}}(M).$$

Therefore, our multiparty correlation measure for a stabilizer state becomes

$$M_C(|\psi\rangle\langle\psi|) = \frac{1}{2^{N+1}} \sum_{\{G_i\}} \left| \sum_{M \in \mathcal{S}} \beta_M v_{\{G_i\}}(M) \right|. \tag{32}$$

Without any complication, the above procedure we describe is easily extended for different partitions of the stabilizer states, or a reduced density matrix of the stabilizer states.

Let us demonstrate the above procedure with the three-qubit stabilizer state ($N = 3$), or the equivalent three-qubit GHZ state as considered earlier in Eq. (19). When viewed as a stabilizer state, its generators can be taken as follows:

$$G_1 = X_1 Z_2, \quad G_2 = Z_1 X_2 Z_3, \quad G_3 = Z_2 X_3.$$

The elements in the stabilizer group $\mathcal{S}$ can be written as

$$M_{a_1 a_2 a_3} = G_1^{a_1} G_2^{a_2} G_3^{a_3} = X_1^{a_1} Z_1^{a_2} Z_2^{a_1} X_2^{a_2} Z_2^{a_3} Z_3^{a_2} X_3^{a_3},$$

where $a_1, a_2, a_3 \in \{0, 1\}$. So the three-party density matrix is

$$\rho^{(123)} = \frac{1}{2^3} \sum_{a_1, a_2, a_3 = 0}^{1} G_1^{a_1} G_2^{a_2} G_3^{a_3}.$$

The two-party density matrices are

$$\rho^{(12)} = \frac{1}{2^2}\sum_{a_2=a_3=0} G_1^{a_1}G_2^{a_2}G_3^{a_3} = \frac{1}{2^2}\sum_{a_1} G_1^{a_1},$$

$$\rho^{(23)} = \frac{1}{2^2}\sum_{a_1=a_2=0} G_1^{a_1}G_2^{a_2}G_3^{a_3} = \frac{1}{2^2}\sum_{a_3} G_3^{a_3},$$

$$\rho^{(13)} = \frac{1}{2^2}\sum_{a_2=a_1+a_3=0} G_1^{a_1}G_2^{a_2}G_3^{a_3} = \frac{1}{2^2}\sum_{a_1+a_3=0} G_1^{a_1}G_3^{a_3},$$

and the one-party density matrices are

$$\rho^{(1)} = \frac{I_1}{2}, \quad \rho^{(2)} = \frac{I_2}{2}, \quad \rho^{(3)} = \frac{I_3}{2}.$$

The cumulant thus becomes

$$C(\rho^{(123)}) = \frac{1}{2^3}(G_1G_2G_3 + G_1G_2 + G_2G_3 + G_2),$$

and our three-party correlation measure is given by

$$C(\rho^{(123)}) = \frac{1}{2^4}\sum_{G_1,G_2,G_3=\pm 1} |G_1G_2G_3 + G_1G_2 + G_2G_3 + G_2| = \frac{1}{2},$$

the same result as obtained earlier in the line after Eq. (19) based on a direct calculation.

IV. DISCUSSION AND CONCLUSION

We proposed a multiparty correlation measure as the trace norm of the cumulant of the multiparty density matrix, as defined in Eq. (10). A natural question arises: Is this a unique multiparty correlation measure based on the cumulant? For example, by replacing the trace norm of Eq. (10) by a squared norm, can we obtain an alternative measure

$$M_C'(\rho^{(12\cdots N)}) = \mathrm{Tr}|C(\rho^{(12\cdots N)})|^2, \tag{33}$$

which then can be directly written as a sum of a complete set of squared correlation functions? We find that $M_C'(\rho^{(12\cdots N)})$ satisfies the conditions (1), (2), and (3) of a multiparty correlation measure. In particular, it is invariant under local unitary transformations, thus it is appropriate to call $M_C'(\rho^{(12\cdots N)})$ a local unitary invariant multiparty correlation function (LUI-MCF). We do not know whether the LUI-MCF satisfies the condition (5), i.e., whether M_C' is nonincreasing under general local operations. Unfortunately, We find it does not satisfy the additional condition (4) proposed by us, thus the LUI-MCF is not a legitimate correlation measure.

As mentioned in the introduction, a total correlation was previously defined by Vedral to measure the total correlation in a multiparty quantum state [6]. Our work above also suggests an alternative total correlation measure as the distance between the quantum state and its reduced completely noncorrelated state, i.e.,

$$M_{TC}(\rho^{(12\cdots N)}) = D\left(\rho^{(12\cdots N)}, \prod_{i=1}^{N}\rho^{(i)}\right). \tag{34}$$

It is easy to check that it satisfies the conditions (1), (2′), and (3)–(5), but does not satisfy the condition (2), which implies it is indeed a legitimate total correlation measure.

We expect our correlation measure will find applications not only in quantum-information science but also in many-body physics. This expectation is based on the observation that the usual correlation functions cannot characterize general correlations in a multiparty quantum state. On the one hand, our correlation measure recovers to the usual correlation function when there is only a single nonzero correlation function. On the other hand, our correlation measure satisfies the basic general requirements for a legitimate correlation measure, which implies they will faithfully characterize the multiparty correlation in a quantum state. In this paper, we define a multiparty correlation measure capable of capturing genuine multiparty correlation of a multiparty quantum state.

From a theoretical viewpoint, one open problem is how to extract the quantum part of the correlation from our correlation measure. This will then provide a measure of multiparty entanglement. In addition, as an open question for further investigation, it will be interesting to find out how our correlation measure is related to quantum entanglement measures [23–25] of interest in quantum-information science. In many-body physics, we are especially interested in finding out what is really responsible for the quantum phase transition: Is it the quantum correlation, classical correlation, or the total correlation [26–29]?

In summary, we have proposed a multiparty correlation measure based on the cumulant of multiparty density matrix, which is capable of characterizing genuine multiparty correlation. We proved that our correlation measure is a legitimate multiparty correlation because it satisfies the five basic requirements for a multiparty correlation measure. The fourth requirement is suggested by us based on physical and operational considerations of multiparty correlation. As an application, we find an efficient algorithm to compute the multiparty correlations for all stabilizer states.

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APPENDIX: PROOF OF THEOREM 3

Under local unitary transformations, any three-qubit state can be most economically expressed as equivalent to [21]

$$|\psi\rangle = a_0 e^{i\phi}|000\rangle + b_1|100\rangle + b_2|010\rangle + b_3|001\rangle + a_1|111\rangle,$$

where the parameters satisfy

$$a_0, a_1, b_1, b_2, b_3, \phi \in R,$$

$$a_0^2 + a_1^2 + b_1^2 + b_2^2 + b_3^2 = 1.$$

Let us denote the cumulant $C(\rho^{(123)})$ as $\mathcal{C}$, i.e.,

$$\mathcal{C} = \rho^{(123)} - \rho^{(1)}\rho^{(23)} - \rho^{(2)}\rho^{(13)} - \rho^{(3)}\rho^{(12)} + 2\rho^{(1)}\rho^{(2)}\rho^{(3)}.$$

We first consider the element

$$\langle 100|\mathcal{C}|010\rangle = b_1b_2(1-2a_0^2)(a_1^2+b_3^2)=0.$$

According to the symmetry of the state, we obtain

$$b_1b_2(1-2a_0^2)(a_1^2+b_3^2)=0,$$

$$b_2b_3(1-2a_0^2)(a_1^2+b_1^2)=0,$$

$$b_1b_3(1-2a_0^2)(a_1^2+b_2^2)=0.$$

If

$$a_1^2+b_3^2=0,$$

or

$$a_2^2+b_3^2=0,$$

or

$$a_1^2+b_2^2=0,$$

then it is easy to check that the state is a product state. Therefore we only check two cases

$$1-2a_0^2=0$$

or

$$b_1b_2=0,$$

$$b_2b_3=0,$$

$$b_1b_3=0.$$

We need to further compute the following matrix element:

$$\langle 111|\mathcal{C}|111\rangle = a_1^2a_0^2(1-2a_1^2)+2b_1^2b_2^2b_3^2+2a_1^2(b_1^2b_2^2+b_2^2b_3^2 \\ +b_1^2b_3^2)=0.$$

If

$$a_0^2=\tfrac{1}{2}$$

then

$$a_1^2a_0^2(1-2a_1^2)\geqslant 0=0,$$

which gives

$$a_1^2=\tfrac{1}{2}$$

or

$$a_1=0.$$

When $a_1^2=1/2, a_1^2=1/2$, we find

$$b_1=b_2=b_3=0,$$

which is the GHZ state, whose cummulant takes the maximum value. For $a_1=0$, the state is a product state. If

$$a_0^2\neq\tfrac{1}{2},$$

then

$$a_1^2a_0^2(1-2a_1^2)=0,$$

which leads to

$$a_1a_0=0,$$

or

$$a_1^2=\tfrac{1}{2}.$$

For the former case the state is a product state. Thus we only need to prove the theorem for the specific state of the latter case

$$|\psi\rangle = a_0e^{i\phi}|000\rangle + b_1|100\rangle + \frac{1}{\sqrt{2}}|111\rangle.$$

We only need to check the element

$$\langle 000|\mathcal{C}|111\rangle = \frac{1}{\sqrt{2}}a_0e^{i\phi} - a_0e^{i\phi}b_1b_1\frac{1}{\sqrt{2}}=0,$$

or

$$a_0(1-b_1^2)=0,$$

which gives $a_0=0$, i.e., the state is also a product state.

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