Localized modes in arrays of boson-fermion mixtures

Yu. V. Bludov[1,*] and V. V. Konotop[1,2,†]

[1]*Centro de Física Teórica e Computacional, Universidade de Lisboa, Complexo Interdisciplinar, Avenida Professor Gama Pinto 2, Lisboa 1649-003, Portugal*

[2]*Departamento de Física, Universidade de Lisboa, Campo Grande, Ed. C8, Piso 6, Lisboa 1749-016, Portugal and Departamento de Matemáticas, ETS de Ingenieros Industriales, Universidad de Castilla–La Mancha, 13071 Ciudad Real, Spain*

(Received 12 June 2006; published 24 October 2006)

It is shown that the mean-field description of a boson-fermion mixture with a dominating fermionic component, loaded in a one-dimensional optical lattice, is reduced to the nonlinear Schrödinger equation with a periodic potential and periodic nonlinearity. In such a system there exist localized modes having peculiar properties. In particular, for some regions of parameters there exists a lower bound for a number of bosons necessary for creation of a mode, while for other domains small amplitude gap solitons are not available in the vicinity of either of the gap edges. We found that the lowest branch of the symmetric solution either does not exist or exists only for a restricted range of energies in a gap, unlike in pure bosonic condensates. The simplest bifurcations of the modes are shown and stability of the modes is verified numerically.

DOI: 10.1103/PhysRevA.74.043616 PACS number(s): 03.75.Kk, 05.40.−a, 03.75.Lm, 03.75.Ss

I. INTRODUCTION

Localized modes constitute an intrinsic feature of nonlinear systems with periodically varying parameters [1]. They also represent a signature of other fundamental physical phenomena like instabilities and phase transitions, in particular transition between superconducting and insulating states in condensates of atomic vapors. That is why during the last few years nonlinear modes in Bose-Einstein condensates (BECs) embedded in optical lattices attracted a great deal of attention (see e.g., Refs. [2,3] for review). Such modes were found to exist in single [4] and multicomponent [5] condensates, as well as in atomic-molecular condensates [6]. The properties of localized modes are governed by the lattice parameters and independently by the number of atoms which determines the position of the chemical potential in a gap of the lattice spectrum.

In the present paper we report the existence of localized modes in boson-fermion mixtures with a large number of spin-polarized, and thus noninteracting, fermions. The main features of these systems stem from the fact that the fermionic component is linear and at the same time modifies linear and nonlinear properties of the effective media for bosons. The role of fermions, when their number significantly exceeds the number of bosons, is of crucial importance for stability of mixtures [7,8], as well as for the possibility of existence of quasi-one-dimensional solitary waves [9] in spatially homogeneous traps and gap solitons in the presence of the optical lattice [10]. The gap solitons considered in Ref. [10] were however restricted to small amplitudes and to the case where the fermionic component does not result in the spatial variation of the nonlinearity. At the same time it was also shown that when the Fermi energy is of order of the amplitude of the lattice potential, it becomes strongly dependent on the spatial coordinate. If in such a situation the boson-fermion interaction is not negligible compared with boson-boson interaction, then in the mean-field approximation, the fermionic component significantly changes not only the linear potential but also the effective two-body interactions among bosons.

Departing from the known results and taking into account that the fermionic distribution itself is determined by the trap potential, one can predict that intrinsic localized modes in boson-fermion mixtures can exist, and that they can possess peculiar properties originated by fermions modifying the effective lattice potential and, what is most important, introducing a *nonlinear lattice* for bosons, i.e., periodic modulation of the nonlinearity governing boson-boson interactions. Description of such modes and of their properties is our main goal.

More specifically, in Sec. II we derive the nonlinear Schrödinger equation with periodic potential and periodically varying nonlinearity, as a model describing a quasi-one-dimensional boson-fermion mixture in an optical lattice. The diversity of localized modes and their properties are described in Sec. III. The outcomes of the work are summarized in the Conclusion.

II. EVOLUTION EQUATIONS

A. Mean-field approximation

We consider low-density bosons and spin-polarized fermions at large density embedded in an optical lattice. At zero temperature, the dynamics of fermions in the vicinity of the Fermi surface is described in the hydrodynamic approximation [11]. Boson-fermion interactions, which are relatively weak due to small density of bosons, can be accounted as corrections to the Gross-Pitaevskii (GP) equation for bosons. The respective mean-field equations were derived in Ref. [7]. In the present section we recover the main result of Ref. [7], by means of an alternative approach, based on the kinetic theory [11].

To this end we start with the Hamiltonian of the boson-fermion interactions $\hat{H}_{\text{int}}=g_{bf}\int d\mathbf{r}\hat{\Psi}^{\dagger}\hat{\Psi}\hat{\Phi}^{\dagger}\hat{\Phi}$, where $\hat{\Phi}$ and $\hat{\Psi}$

*Electronic address: bludov@cii.fc.ul.pt

†Electronic address: konotop@cii.fc.ul.pt

are the annihilation field operators for bosons and fermions, correspondingly, $g_{bf}=2\pi\hbar^2 a_{bf}/m$, $m=m_b m_f/(m_b+m_f)$, a_{bf} is a boson-fermion scattering length, and $m_{b,f}$ are atomic masses (subindexes b and f stand for bosons and fermions, respectively). Next we introduce the expectation values: the order parameter of bosons $\Psi=\langle\hat{\Psi}\rangle$ and the averaged density of fermions $\rho=\langle\hat{\Phi}^\dagger\hat{\Phi}\rangle$. Then after the standard approximation $\hat{\Psi}\approx\Psi$, the equation governing the dynamics of bosons acquires the form

$$i\hbar\frac{\partial\Psi}{\partial t}=-\frac{\hbar^2}{2m_b}\Delta\Psi+V_b(\mathbf{r})\Psi+g_{bb}|\Psi|^2\Psi+g_{bf}\rho\Psi, \quad (1)$$

where $g_{bb}=4\pi\hbar^2 a_{bb}/m_b$ and a_{bb} is the scattering length of boson-boson interactions. Hereafter $V_{b,f}(\mathbf{r})$ stand for the trap potentials of the two components.

It follows from the definition of $\hat{H}_{\text{int}}$ and from Eq. (1) that $\rho(\mathbf{r},t)$ and $|\Psi(\mathbf{r},t)|^2$ can be interpreted as external time-dependent potentials applied to bosons and to fermions, respectively. This in particular means that $\mathbf{F}=-\nabla|\Psi|^2$ is the average force which bosons exert on a fermion. Hence, the kinetic equation for the distribution function of the fermions $n(\mathbf{r},\mathbf{p},t)$ has the form ($\nabla_{\mathbf{p}}=\partial/\partial\mathbf{p}$)

$$\frac{\partial n}{\partial t}+\nabla n\nabla_{\mathbf{p}}\varepsilon-\nabla_{\mathbf{p}}n\nabla\varepsilon+\mathbf{F}\nabla_{\mathbf{p}}n=0. \quad (2)$$

Here $\varepsilon=\varepsilon(\mathbf{r},\mathbf{p},t)$ is the energy of a fermion and the link between the distribution function and average density of fermions is given by

$$\rho(\mathbf{r},t)=\int n(\mathbf{r},\mathbf{p},t)\frac{d\mathbf{p}}{(2\pi\hbar)^3}.$$

Following the standard procedure (see e.g., Ref. [11]), considering zero temperature, we represent the distribution function in a form of an unperturbed part $n_0(\varepsilon)$ dependent on the unperturbed Fermi distribution and considered as a function of the particle energy ε, and its excitation $\delta n(\mathbf{r},\mathbf{p},t)$: $n=n_0(\varepsilon)+\delta n(\mathbf{r},\mathbf{p},t)$. For the spin-polarized and thus noninteracting fermions in an external potential, we have $\varepsilon=\mathbf{p}^2/(2m)+V_f(\mathbf{r})$. Now we define the averaged momentum $\mathbf{P}(\mathbf{r},t)$ and the current $\mathbf{j}(\mathbf{r},t)$, respectively, by the formulas

$$\mathbf{P}(\mathbf{r},t)=\int\mathbf{p}n\frac{d\mathbf{p}}{(2\pi\hbar)^3}\quad\text{and}\quad\mathbf{j}(\mathbf{r},t)=\int\mathbf{v}\,\delta n\frac{d\mathbf{p}}{(2\pi\hbar)^3},$$

where $\mathbf{v}=\nabla_{\mathbf{p}}\varepsilon=\mathbf{p}/m_f$ is the velocity of a fermion. Taking into account that $\nabla\varepsilon=\nabla V_f$ and rewriting

$$\rho(\mathbf{r},t)=\rho_0(\mathbf{r})+\rho_1(\mathbf{r},t),\quad\rho_1(\mathbf{r},t)=\int\delta n\frac{d\mathbf{p}}{(2\pi\hbar)^3}, \quad (3)$$

we obtain in the leading order ($\alpha,\beta=x,y,z$)

$$\frac{\partial P_\alpha}{\partial t}+\rho_1\frac{\partial}{\partial x_\alpha}V_f(\mathbf{r})+\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\sum_\beta p_\alpha v_\beta\frac{\partial}{\partial x_\beta}\delta n+g_{bf}\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}n_0\frac{\partial}{\partial x_\alpha}|\Psi|^2=0. \quad (4)$$

In the right-hand side of this equation we have accounted only the leading terms. In the case at hand $\mathbf{j}=\mathbf{P}/m_f$ allows us, after differentiating with respect to time and retaining the terms of the leading order, to rewrite the continuity equation

$$\frac{\partial\rho}{\partial t}+\nabla\mathbf{j}=0$$

as follows:

$$\frac{\partial^2\rho_1}{\partial t^2}+\frac{1}{m_f}\sum_\alpha\frac{\partial}{\partial x_\alpha}\left(\rho_1\frac{\partial}{\partial x_\alpha}V_f(\mathbf{r})-\frac{1}{m_f}\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\sum_\beta p_\alpha p_\beta\frac{\partial}{\partial x_\beta}\delta n\right)+\frac{g_{bf}}{m_f}\nabla\rho_0\nabla|\Psi|^2=0.$$

Next we approach

$$\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\sum_\beta p_\alpha p_\beta\frac{\partial}{\partial x_\beta}\delta n\approx\frac{\delta_{\alpha\beta}}{3}\frac{\partial}{\partial x_\beta}p_F^2\rho_1,$$

where $\mathbf{p}_F$ is a momentum of the Fermi surface: $p_F^2=\hbar^2(6\pi^2\rho_0)^{2/3}$, and use the stationary Thomas-Fermi distribution [7,8]

$$\rho_0=\{2m_f[E_F-V_f(\mathbf{r})]\}^{3/2}/(6\pi^2\hbar^3). \quad (5)$$

Combining these approximations with (4) we obtain [7]

$$\frac{\partial^2\rho_1}{\partial t^2}=\nabla\left[\rho_0\nabla\left(\frac{(6\pi^2)^{2/3}\hbar^2}{3m_f^2\rho_0^{1/3}}\rho_1+\frac{g_{bf}}{m_f}|\Psi|^2\right)\right]. \quad (6)$$

Equations (1), (3), and (6) make up a system governing the mean-field dynamics of boson-fermion mixture (from the direct averaging of the equations for the field operators, this equation was earlier obtained in Ref. [7]).

In the present paper we are interested in specific solutions, representing localized excitations whose spatial dimension is of order of the lattice constant (contrary to gap solitons, considered in Ref. [10], which extend to many lattice periods). This is the case where the stationary approximation $\partial\rho_1/\partial t=0$ is valid. Then (6) immediately gives the relation

$$\rho_1=-\frac{3g_{bf}m_f}{(6\pi^2)^{2/3}\hbar^2}\rho_0^{1/3}(\mathbf{r})|\Psi|^2. \quad (7)$$

Substituting this formula in the expression for ρ and subsequently in (1), we arrive at [7]

$$i\hbar\frac{\partial\Psi}{\partial t}=-\frac{\hbar^2}{2m_b}\Delta\Psi+V(\mathbf{r})\Psi+g(\mathbf{r})|\Psi|^2\Psi, \quad (8)$$

where

$$V(\mathbf{r})=V_b(\mathbf{r})+g_{bf}\rho_0(\mathbf{r}) \quad (9)$$

is the effective lattice potential and

$$g(\mathbf{r}) = g_{bb} - \frac{3m_f g_{bf}^2 \rho_0^{1/3}}{(6\pi^2)^{2/3}\hbar^2} \tag{10}$$

is the effective nonlinearity.

B. One-dimensional limit

Consider now a one-dimensional (1D) lattice along the x axis and tight confinement in the transverse direction:

$$V_{b,f} = m_{b,f}\omega_{b,f}^2 r_\perp^2/2 + U_{b,f}(\kappa x).$$

Here $\omega_{b,f}$ are the linear oscillator frequencies in the transverse direction, $\mathbf{r}_\perp=(y,z)$, $\kappa=\pi/d$, d is the lattice constant, and $U_{b,f}(\kappa x)$ are π-periodic functions, $U_{b,f}(\kappa x)=U_{b,f}(\kappa x+\pi)$. Pauli's exclusion principle leads to significant transverse extension of the distribution $\rho_0(\mathbf{r})$. However, as it is shown in Ref. [9], strong confinement of the bosonic component and weakness of the two-body interactions allow transition to the simplified 1D evolution equation. Thus we require $a \ll d$, where $a=\sqrt{\hbar/m_b\omega_b}$ is the transverse linear oscillator length of bosons. The self-consistent way of the derivation of the 1D reduction is based on the multiple-scale expansion, which in the leading order yields

$$\Psi(\mathbf{r},t) = \zeta(\mathbf{r}_\perp)\Psi(x,t)\exp(-i\omega_b t),$$

where $\zeta=a^{-1}\pi^{-1/2}\exp[-r_\perp^2/(2a^2)]$ describes the linear transverse distribution and $\Psi(x,t)$ is a smooth envelope. Since the respective procedure was described in numerous works (see, e.g., Ref. [9] for fermion-boson mixtures) we skip the details. The result is a nonlinear Schrödinger (NLS) equation with a periodic potential and periodic nonlinearity, which in dimensionless variables $X=\kappa x$, $T=\kappa^2 a^2 t/2$, $\psi(X,T)=(\kappa N_b)^{-1/2}\Psi(x,t)$, and $\varrho=\rho_0/\kappa^3$, reads

$$i\frac{\partial\psi}{\partial T} = -\frac{\partial^2\psi}{\partial X^2} + \mathcal{U}(X)\psi + \mathcal{G}(X)|\psi|^2\psi. \tag{11}$$

The periodic coefficients are obtained from (9) and (10). They are explicitly determined by the stationary distribution of fermions $\varrho(X)$,

$$\mathcal{U}(X) = \mathcal{U}_0(X) + \mathcal{U}_1\varrho(X), \tag{12}$$

$$\mathcal{G}(X) = \mathcal{G}_0 - \mathcal{G}_1\varrho^{1/3}(X), \tag{13}$$

$\mathcal{U}_0(X)=U_b(X)/E_R$, $E_R=\hbar^2\kappa^2/2m_b$ is the boson recoil energy, $\mathcal{U}_1=4\pi\kappa a_{bf}m_b/m$, $\mathcal{G}_0=4a_{bb}N_b/\kappa a^2$, and $\mathcal{G}_1=2(6/\pi)^{1/3}\times(a_{bf}/a)^2(m_f m_b/m^2)N_b$. Both functions $\mathcal{U}(X)$ and $\mathcal{G}(X)$ are π-periodic: $\mathcal{U}(X)=\mathcal{U}(X+\pi)$ and $\mathcal{G}(X)=\mathcal{G}(X+\pi)$. To make all values to be of the unity order, above we introduced N_b, which determines the order of magnitude of the total number of bosons.

III. LOCALIZED MODES

A. Modulational instability and bifurcation of localized modes from the continuum spectrum

We start the analysis of (11) by the study of the stability of the linear Bloch states with respect to smooth modulations of their amplitude. This phenomenon is referred to as modulational instability (or alternatively as dynamical instability) and provides the information about small amplitude gap solitons. We designate by $\mathcal{E}_\alpha^{(\sigma)}$ and by $\varphi_\alpha^{(\sigma)}(X)$ the energy and the Bloch states of lower ("σ" stands for "−") and upper ("σ" stands for "+") edges of the α's band ($\alpha=1,2,\ldots$) of the spectrum of the operator $\mathcal{L}\equiv-\partial^2/\partial X^2+\mathcal{U}(X)$: $\mathcal{L}\varphi_\alpha^{(\sigma)}=\mathcal{E}_\alpha^{(\sigma)}\varphi_\alpha^{(\sigma)}$. Then the interval $(\mathcal{E}_\alpha^{(+)},\mathcal{E}_{\alpha+1}^{(-)})$refers to the α's gap and $\alpha=0$ corresponds to the semi-infinite gap. Next, we compute the inverse effective mass $(M_\alpha^{(\sigma)})^{-1}=(1/2)d^2\mathcal{E}_\alpha^{(\sigma)}/dk^2$ and the nonlinearity coefficient $\chi_\alpha^{(\sigma)}=\int_0^\pi\mathcal{G}(X)[\varphi_\alpha^{(\sigma)}(X)]^4dX$. Finally, we formulate the condition of the modulational instability, which is also the condition necessary for existence of small amplitude gap solitons, as

$$M_\alpha^{(\sigma)}\chi_\alpha^{(\sigma)} < 0 \tag{14}$$

(see e.g., Refs. [2,9] for more details).

The main feature of the problem at hand, making it very different from the standard and well studied case of the nonlinear Schrödinger equation with a periodic potential, is that the obtained criterion (14) strongly depends on the fermionic distribution responsible for the spatial dependence of $\mathcal{G}(X)$ [see Eqs. (12) and (13)] not only through the effective mass $M_\alpha^{(\sigma)}$, but also through the effective nonlinearity $\chi_\alpha^{(\sigma)}$. In its turn, the spatial distribution $\mathcal{G}(X)$ strongly depends on the transverse frequency ω_f (particular numerical examples can be found in Ref. [10]). On the other hand, presence of the linear potential makes the problem very different from the situation studied recently in Refs. [12,13] where the spatially periodic nonlinearity was considered in absence of the linear potential. Namely, in a linearly homogeneous, $\mathcal{U}(X)\equiv 0$, nonlinear lattice one has $\varphi_\alpha^{(\sigma)}(X)\equiv 1$ and thus $\chi_\alpha^{(\sigma)}=\mathcal{G}_{av}$, where $\mathcal{G}_{av}$ is the spatial average of $\mathcal{G}(X)$. Thus $\chi_\alpha^{(\sigma)}=0$ when $\mathcal{G}_{av}=0$ and no small amplitude solitons can exist near gap edges [12]. This leads to existence of a lower bound for the number of particles necessary for excitation of a localized mode. In the presence of a linear lattice, $\mathcal{U}(X)\neq 0$, generally speaking $\chi_\alpha^{(\sigma)}\neq\mathcal{G}_{av}$ and even signs of $\chi_\alpha^{(\sigma)}$ and $\mathcal{G}_{av}$ may be different. Then, if (14) is satisfied, the lower boundary of the number of atoms is removed even for $M_\alpha^{(\sigma)}\mathcal{G}_{av}>0$.

In Fig. 1(b) we show the nonlinearity coefficients corresponding to the semi-infinite, $\chi_1^{(-)}$, and first lowest, $\chi_1^{(+)}$ and $\chi_2^{(-)}$, gaps for the ^{87}Rb-^{40}K mixture, in the region where they change the sign. It follows from the figure that small amplitude gap solitons can be created neither in the semi-infinite gap for $\kappa a_{bb}>\gamma_1^{(-)}$, nor in the vicinity of the upper edge of the first band for $\kappa a_{bb}<\gamma_1^{(+)}$, nor in the vicinity of the lowest edge of the second band at $\kappa a_{bb}>\gamma_2^{(-)}$. An unusual situation occurs for $\gamma_2^{(-)}<\kappa a_{bb}<\gamma_1^{(+)}$: *gap solitons do not exist in the vicinity of either of the edges of the first gap.*

B. Localized modes in the semi-infinite gap

We look for a stationary solution of Eq. (11) in the form $\psi=e^{-i\mathcal{E}T}\phi(X)$, where $\phi(X)$ is real and $\phi\to 0$ as $X\to\infty$. Starting with the semi-infinite gap, i.e., with $\mathcal{E}<\mathcal{E}_1^{(-)}$, we observe that if $\mathcal{G}_m=\min_X\mathcal{G}(X)<0$, then in the limit $\mathcal{E}\to-\infty$, there

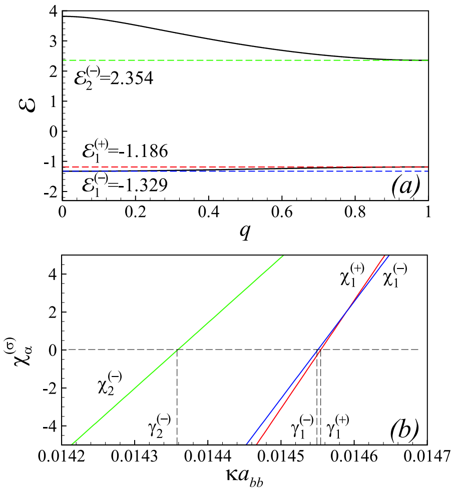

FIG. 1. (Color online) (a) The two lowest bands of the effective band structure for bosons and (b) dependence $\chi_\alpha^{(\sigma)}$ on κa_{bb} for a mixture of $N_f=10^4$ of K^{40} atoms per one cell and $N_b=5000$ of ^{87}Rb atoms, at $\kappa a_{bf}\approx 0.0314$. The lattices are $\mathcal{U}_b=-10\cos(2X)$ and $\mathcal{U}_f=-21.75\cos(2X)$ (it is taken into account that $\mathcal{U}_f/\mathcal{U}_b\approx m_b/m_f=2.175$ [14]). $\gamma_1^{(-)}\approx 1.4550\times 10^{-2}$, $\gamma_1^{(+)}\approx 1.4555\times 10^{-2}$, and $\gamma_2^{(-)}\approx 1.4359\times 10^{-2}$ indicate values of κa_{bb} where the nonlinearity coefficients become zero.

exists a soliton, which is strongly localized about $\mathcal{G}_m$. More specifically this happens when that soliton width, which is estimated as $1/\sqrt{\mathcal{E}}$, is much less than the lattice period, i.e., $\mathcal{E}\gg 1$. Now the periodic potential can be viewed as a perturbation and the respective solution can be approximated by

$$\phi_S(X)\approx\frac{\sqrt{2|\mathcal{E}-\mathcal{U}_m|/|\mathcal{G}_m|}}{\cosh(X\sqrt{|\mathcal{E}-\mathcal{U}_m|})},\qquad(15)$$

where $\mathcal{U}_m=\min_X\mathcal{U}(X)$.

This conclusion is confirmed by the numerical study reported in Fig. 2. When $\chi_1^{(-)}<0$, the normalized number of bosons, defined as $N=\int\phi^2(x)dx$, is a decreasing function of $\mathcal{E}$ (the line $\chi_1^{(-)}=-5.37$ in the upper panels). It approaches zero as the energy detuning decreases according to the law $N\propto\sqrt{\mathcal{E}_1^{(-)}-\mathcal{E}}$. This is the standard situation [2]. If, however $\chi_1^{(-)}>0$, N achieves its minimum $N_m\approx 1.108N_b$ in the vicinity of the gap edge and starts to grow with the decrease of the energy detuning (the line $\chi_1^{(-)}=0.41$ in the upper panels). Thus there exists a *nonzero minimal number* of bosons necessary for creation of a localized mode (in a linearly homogeneous NLS equation with a periodic nonlinearity this effect was observed in Ref. [12]), which corroborates with the above conclusion about the impossibility of existence of small-amplitude gap solitons at $M_1^{(-)}\chi_1^{(-)}>0$. This phenomenon manifests itself in shapes of the localized modes: cf. the profiles in panels *A* and *B*, corresponding to the same energy. The solutions in panels *C* and *D* display the approach to the limit $\phi_S(X)$ obtained above. From the upper panels of Fig. 2 one can see that for equal energies the number of the bosons in localized modes at $\chi_1^{(-)}>0$ exceeds one at $\chi_1^{(-)}<0$.

By direct numerical solution of Eq. (11) we have verified that the localized modes displayed in the panels *A*, *B*, *C*, and *D* of Fig. 2 are stable. More specifically we perturbed the mode profiles by the factor $1+0.1\cos(21X)$ and computed the dynamics until $T=500$, observing the oscillations of the soliton amplitudes of order of 10% of their averaged values.

C. Gap solitons in the first gap

Considering the localized modes of the first gap (see Fig. 3), the first intriguing property which one can observe at $\kappa a_{bb}>\gamma_1^{(+)}$, i.e., for the parameters giving $\chi_1^{(+)}>0$ and $\chi_2^{(-)}>0$, is a zig-zag type dependence of the boson number on the energy [Fig. 3(a)]. The dependence of this type can be viewed as a set of successive bifurcations of the branches in the points $\mathcal{E}_*^{(j)}$ $(j=1,2,\ldots)$, the first two indicated in Fig. 3(a). Moreover there exists a critical energy, coinciding with the first bifurcation point $\mathcal{E}_*^{(1)}$, such that no lowest-branch gap solitons exist at $\mathcal{E}_*^{(1)}<\mathcal{E}<\mathcal{E}_2^{(-)}$ (according to our numerical

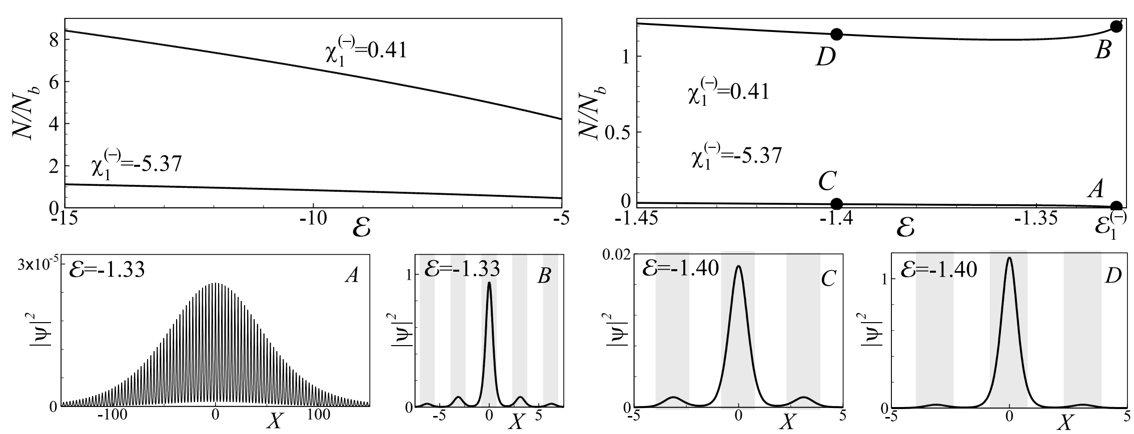

FIG. 2. The number of bosons, N/N_b, of the localized modes vs energy $\mathcal{E}$ in the semi-infinite gap $-\infty<\mathcal{E}<\mathcal{E}_1^{(-)}$ (upper panels, $\mathcal{E}_1^{(-)}$ coincides with the right boundary of the panel) for the ^{87}Rb-^{40}K mixture with the same parameters as in Fig. 1, except $\kappa a_{bb}=1.4445\times 10^{-2}$ (the curves $\chi_1^{(-)}\approx -5.37$) and $\kappa a_{bb}=1.4558\times 10^{-2}$ (the curves $\chi_1^{(-)}\approx 0.41$). The lower panels show solitonic shapes corresponding to the points *A*, *B*, *C*, and *D* [gray and white colors correspond to half-periods with $\mathcal{U}(X)<0$ and $\mathcal{U}(X)>0$]. Only the lowest branch of the solutions is shown.

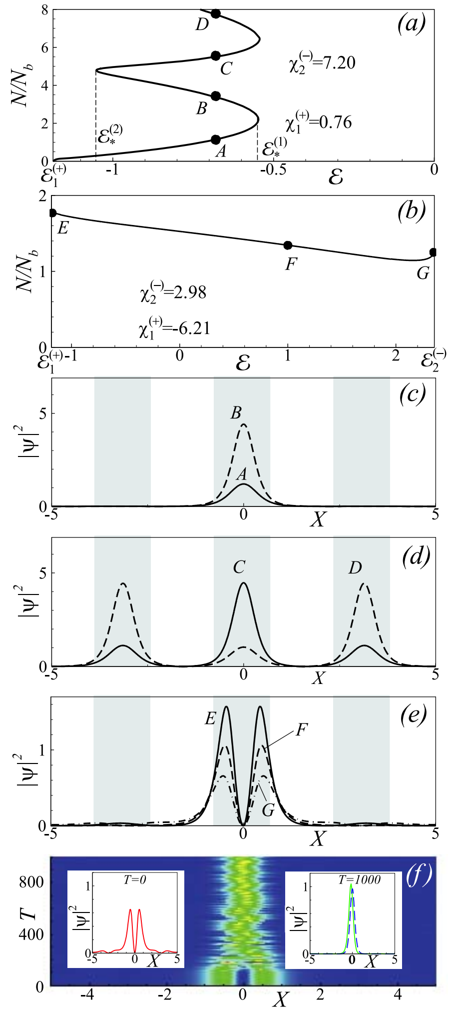

FIG. 3. (Color online) The number of bosons N/N_b vs energy $\mathcal{E}$ in the first gap $\mathcal{E}_1^{(+)}<\mathcal{E}<\mathcal{E}_2^{(-)}$ [panels (a) and (b)] for the case of the ^{87}Rb-^{40}K mixture with the same parameters as in Figs. 1 and 2, except $\kappa a_{bb}=1.4568\times 10^{-2}$ [panel (a)] and $\kappa a_{bb}=1.4445\times 10^{-2}$ [panel (b)]. (Only the lowest branch is shown.) The panels (c), (d), and (e) show explicit shapes of the modes corresponding to different points of the solution branches, while panel (f) shows the dynamics of mode G [in the insets the initial, $T=0$, and final, $T=1000$, shapes are shown by solid lines; the numerically calculated shape of the mode with $\mathcal{E}=-7.66$ is depicted on the right-hand inset in panel (f) by the dashed curve].

results a similar statement applies also for upper branches of the solutions). Motion along the curve $N(\mathcal{E})$ is accompanied by the redistribution of atoms among the potential minima as it is illustrated in Figs. 3(c) and 3(d). The unlimited growth of N can be understood as follows. When the energy detuning toward the gap grows, the amplitude of the mode grows and the width decreases. Such a mode is localized in the vicinity $\mathcal{G}_m<0$. Increase of the number of particles leads also to occupation of the regions where not only $\mathcal{U}(X)>0$, but also $\mathcal{G}(X)>0$, i.e., where the nonlinearity enhances the repelling of the atoms, thus allowing storage of a larger number of atoms, compared with nonlinearly homogeneous models. The modes A and B in Fig. 3(a) were found to be dynamically stable. As in the previous case we integrated Eq. (11) with the initial condition corresponding to the modes A and B perturbed by the factor $1+0.1\cos(21X)$. We observed the oscillations of the mode amplitudes of order of $\pm 5\%$ until $T=500$. At the same time the numerical simulations evidenced that modes C and D are not stable. Their dynamics is accompanied by the traveling of the atoms among minima of the lattice.

Another situation, which does not exist in nonlinearly homogeneous structures, corresponds to the interval $\gamma_1^{(+)}<\kappa a_{bb}<\gamma_2^{(-)}$ where $\chi_1^{(+)}<0$ and $\chi_2^{(-)}>0$ [see Fig. 3(b)]. Here we concentrate on antisymmetric excitations. Similarly to the case of the semi-infinite gap, there exists *a minimal number* of bosons, which is necessary for creation of a localized mode: the lowest branch depicted in Fig. 3(b) does not reach zero.

By direct simulation of Eq. (11) we found that solutions corresponding to the part of the branch, where $dN/d\mathcal{E}<0$ [e.g., modes E and F shown in Fig. 3(e)], are dynamically stable, while solutions corresponding to $dN/d\mathcal{E}>0$ [mode G in Fig. 3(e)] are unstable (similar to the conventional Vakhitov-Kolokolov criterion [15]). In Fig. 3(f) we show the dynamics of an unstable solution [it corresponds to mode G in Fig. 3(e)] which displays transformation of an asymmetric mode into a pulsating symmetric distribution at time $T\approx 150$. By comparing the initial, $T=0$, and final, $T=1000$, distributions, which are shown in the insets in Fig. 3(f), one can suggest that the antisymmetric mode was transformed into a symmetric state which is localized about the local minimum of the potential $\mathcal{U}(X)$ and nonlinearity $\mathcal{G}(X)$. In order to check this conjecture we estimated the energy of the emerging mode (it is $\mathcal{E}=-7.66$) and compared its shape with the shape of the stationary mode obtained for the same energy. The result depicted by the dashed line in the right inset of Fig. 3(f), illustrating close similarity between the slightly oscillating dynamical solution and the stationary localized mode, strongly supports the above supposition.

IV. CONCLUSION

We have shown that bosonic component of quasi-one-dimensional boson-fermion mixtures, loaded in relatively deep optical lattices, can be described by the NLS equation with a periodic potential and a periodic nonlinearity. The fermionic component modifies the effective lattice for bosons and originates modulation of the interaction among bosons. In such a system there exist localized mode solutions, with properties very different from those of the known models with homogeneous nonlinearity and/or potential. In particular, we established regions of parameters where existence of the localized modes requires a minimal number of bosons and is not limited by some upper bound (these phenomena resemble properties of the quintic NLS equation [16]). The respective dependence of the number of the mode particles on the energy, unlike in any other known NLS models with periodic coefficients, has a zig-zag behavior, originated by the set of bifurcations. We have also found situations where no small amplitude gap solitons can exist near either of the gap edges, thus making both Bloch states bordering the gap to be modulationally stable, and where not for all energies in a gap solitary wave solutions are available. Most of the symmetric modes were found to be dynamically stable.

ACKNOWLEDGMENTS

One of the authors (Y.V.B.) was supported by the FCT Grant No. SFRH/PD/20292/2004. One of the authors (V.V.K.) acknowledges support of the Secretaria de Stado de Universidades e Investigación (Spain) under Grant No. SAB2005-0195. The work was supported by the FCT and European program FEDER under Grant No. POCI/FIS/56237/2004.

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