Quantum dynamics as an analog of conditional probability

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Quantum theory can be regarded as a noncommutative generalization of classical probability. From this point of view, one expects quantum dynamics to be analogous to classical conditional probabilities. In this paper, a variant of the well-known isomorphism between completely positive maps and bipartite density operators is derived, which makes this connection much more explicit. This isomorphism is given an operational interpretation in terms of statistical correlations between ensemble preparation procedures and outcomes of measurements. Finally, the isomorphism is applied to elucidate the connection between no-cloning and no-broadcasting theorems and the monogamy of entanglement, and a simplified proof of the no-broadcasting theorem is obtained as a by-product.

I. INTRODUCTION

Quantum theory can be regarded as a noncommutative generalization of classical probability theory, in which density operators play the role of probability distributions and the Cartesian product of probability spaces becomes the tensor product of Hilbert spaces (or more generally of C^* algebras). This point of view has been highly influential in the developing field of quantum information theory [1], which studies the same questions that arise in classical information theory in the noncommutative context.

However, quantum theory, as it is usually formulated, is not directly analogous to abstract probability theory in the sense of Kolmogorov [2], but is much closer to the theory of stochastic processes [3]. In nonrelativistic quantum mechanics, a quantum state is conceived as the state of a number of subsystems at a particular time and states at different times are related by dynamics, generally represented as a completely positive (CP) map. In the relativistic case, there are many such descriptions corresponding to different inertial frames, related to each other via unitary transformations. Nevertheless, the states are always defined on spacelike hyperplanes, so the underlying causal structure is still present in all of these descriptions. This type of theory is closely analogous to a classical stochastic process, in which a state is a probability distribution over a set of random variables representing the properties of a system at a given time and the states at different times are related by dynamics, given by a stochastic transition matrix.

In contrast, abstract probability spaces make no assumptions about the causal structure of the events on which probabilities are defined. Two disjoint events might refer to properties of two different subsystems at a given time, or they might refer to properties of the same subsystem at two different times. In full generality, events need have no interpretation in terms of causal structure at all. It is interesting to ask whether quantum theory can be reformulated as an abstract noncommutative probability theory in this sense. A first step along this road is to ask whether correlations between different subsystems and correlations between the same system before and after the application of a CP map can be expressed using an identical formalism. In the analogous classical case, both can be handled by conditional probabilities, so we are really asking whether a good quantum analog of conditional probability exists.

In this paper, the question is answered in the affirmative by deriving a variant of the isomorphism between bipartite states and CP maps discovered by Jamiołkowski [4] and Choi [5], which makes the connection to conditional probability much more explicit. An operational interpretation of this isomorphism is given by showing that the same sets of correlations can be obtained in each of the two cases.

This result is interesting from the point of view of quantum information, since many relationships have already been discovered between the properties of bipartite quantum states and those of noisy quantum channels [6,7]—i.e., trace-preserving CP maps. Some of these can be extended using our approach. In particular, it is shown that the various types of no-cloning and no-broadcasting theorems [8–10] can be associated directly to statements about the monogamy of entanglement for tripartite states [11]—i.e., the fact that if two subsystems are in a pure entangled state, neither of them can be entangled with any other subsystems. As a by-product of this, a simplified proof of the no-broadcasting theorem is obtained.

A. Prior work

The central question addressed in this paper was originally raised by Ohya [12,13]. Griffiths [14] suggested that the Jamiołkowski isomorphism might be extended by allowing a CP map to act on a more bipartite state. The suggestion was not pursued in that work, but D'Ariano and Lo Presti later developed it in the context of quantum process tomography [15]. The specific isomorphism developed here was very much inspired by some observations made by Fuchs [16,17]. During the preparation of this manuscript, I became aware of work by Asorey *et al.* [18], where a similar isomorphism to the one developed here is considered. The main novelties of the present work are the operational interpretation of the isomorphism and the application to no-cloning

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and no-broadcasting theorems. Also, the case of density operators that are not of full rank is treated more carefully here. Finally, Cerf and Adami have developed a different notion of quantum conditional probability [19–21], based on the definition of the conditional von Neumann entropy.

B. Outline

The remainder of this paper is structured as follows. As preparation for the quantum isomorphism, the different causal structures that can give rise to the same classical joint probabilities are reviewed in Sec. II. The standard version of the Jamiołkowski isomorphism is reviewed in Sec. III, and Sec. IV combines the ideas of Secs. II and III to obtain a different variant of the isomorphism for the quantum case. Section V develops the application of the isomorphism to the connection between cloning and broadcasting and the monogamy of entanglement. Finally, further potential applications and open questions suggested by this work are discussed in Sec. VI.

II. CAUSAL RELATIONS AND CLASSICAL JOINT PROBABILITIES

Given two integer-valued random variables X and Y, with joint probability distribution $P(X=i, Y=j)$, the marginal probability distributions for X and Y are given by

$$\forall \ j \ P(X=j) = \sum_{k} P(X=j, Y=k), \tag{1}$$

$$\forall \ k \ P(Y=k) = \sum_{j} P(X=j, Y=k).$$

As is conventional in probability theory, the notation $P(X)$ is used as a stand-in for $P(X=j)$, when j is an arbitrary unspecified integer. Similar definitions apply for $P(Y)$ and $P(X,Y)$. When a random variable appears as a free index in an equation involving probabilities, then it is implicit that the equation holds for all possible values that the variable can take and Σ_X is an instruction to sum over the possible values of X. With these conventions, Eq. (1) may be simplified to

$$P(X) = \sum_{Y} P(X,Y), \quad P(Y) = \sum_{X} P(X,Y), \tag{2}$$

and the conditional probability of Y, given X, is defined as

$$P(Y|X) = \frac{P(X,Y)}{P(X)}, \tag{3}$$

for all values of X such that $P(X) \neq 0$, and is undefined whenever $P(X)=0$. Clearly,

$$P(X,Y) = P(Y|X)P(X) \tag{4}$$

whenever the right-hand side is defined and $P(X,Y)=0$ otherwise.

Note that so far no mention has been made of how the correlations between X and Y arise. X and Y might refer to the same physical quantity at two different times, Y differing from X due to the dynamics of the system, or they might refer to quantities associated with distinct physical systems at the same time. Indeed, they may have no interpretation in terms of physical quantities at all. In other words, classical probability theory does not depend in any way on the causal ordering of variables, and in particular it does not depend on how or even whether the random variables are embedded in spacetime. Of course, if X and Y are given physical meaning, then this is likely to severely constrain the possible assignments of $P(X,Y)$ we are likely to entertain, but this happens at the level of the *application* of probability theory to physics, and not within the abstract theory itself.

In contrast, the quantum formalism handles correlations in very different ways depending on how they arise. Joint states of two subsystems are handled by taking the tensor product of the underlying Hilbert spaces, whereas correlations between the same physical quantity at differing times are not. This is a weak point in the analogy between quantum theory and classical probability, since the former cannot be viewed as a completely abstract theory that is independent of how the observables we are interested in are embedded in spacetime. The isomorphism of Sec. IV is intended to remove this deficiency, but before moving on to the quantum case, it is helpful to understand the different ways in which joint probability distribution $P(X,Y)$ may be described in distinct causal scenarios, the simplest of which are depicted in Fig. 1.

A possible situation in which case (c) might arise is if X and Y represent the values of some physical quantity, associated with two distinct subsystems. Z may then represent the state of a source, which produces the two subsystems and sends them flying out in opposite directions. In this situation, the values of X and Y could potentially be observed at spacelike separation from one another. $P(X,Y)$ then represents the joint state of the two subsystems, and the marginals $P(X)$ and $P(Y)$ represent their reduced states. This is entirely analogous to the quantum description of the joint state of two subsystems by a density matrix $\rho_{AB} \in \mathfrak{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$ and the descriptions of the reduced states of the two subsystems by the reduced density matrices $\rho_A = \mathrm{Tr}_B(\rho_{AB})$, $\rho_B = \mathrm{Tr}_A(\rho_{AB})$, where $\mathcal{H}_A$ and $\mathcal{H}_B$ are the Hilbert spaces associated with two subsystems A and B, and $\mathfrak{L}(\mathcal{H})$ denotes the space of linear operators on a Hilbert space $\mathcal{H}$.

A possible situation in which case (a) or (b) might arise is if X and Y represent the values of the same physical quantity, associated with the same physical system at two different times $t_1 < t_2$. In case (a), X is the value of the quantity at t_1 and Y is its value at t_2. The transition from X to Y is the result of the dynamics of the system, which may include a stochastic component due to random external influences or a lack of knowledge about the precise details of a deterministic dynamics. A general dynamics is therefore described by a stochastic matrix $\Gamma_{Y|X}$, where $(\Gamma_{Y|X})_{ij}$ is the probability of a transition from the state $X=j$ at t_1 to the state $Y=i$ at t_2. The general picture we obtain from this is that the state $P(X)$ is prepared at time t_1; then, the dynamics $\Gamma_{Y|X}$ occurs, resulting in a final state $P(Y)$ at t_2. This is summarized by the dynamical rule

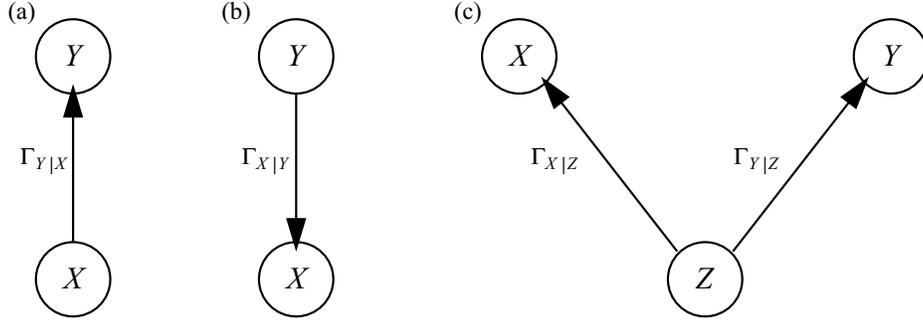$$P(Y=i) = \sum_{j} (\Gamma_{Y|X})_{ij} P(X=j). \tag{5}$$

FIG. 1. Distinct ways in which a general joint probability distribution $P(X,Y)$ may arise. (a) X is the cause of Y. The generation of Y must be in the temporal future of the generation of X. For example, Y may be the result of sending X through a noisy channel described by a stochastic matrix $\Gamma_{Y|X}$. (b) Y is the cause of X. The generation of X must be in the temporal future of the generation of Y. For example, X may be the result of sending Y through a noisy channel described by a stochastic matrix $\Gamma_{X|Y}$. (c) X and Y are the result of some common cause, described by a random variable Z. They may be observed at spacelike separation from one another, provided the points where this happens are both in the forward lightcone of the point where Z was generated.

In the quantum case, the analog of $\Gamma_{Y|X}$ is a trace-preserving completely positive (TPCP) map $\mathcal{E}_{B|A}:\mathfrak{L}(\mathcal{H}_A) \rightarrow \mathfrak{L}(\mathcal{H}_B)$, which can be used to describe the dynamics of a system that is interacting with its environment or when the dynamics is controlled by a random classical parameter (see [1] for further details). In this case, a density operator ρ_A is prepared at t_1 and then the system is subjected to a dynamical evolution according to the TPCP map $\mathcal{E}_{B|A}$ to obtain a density operator $\rho_B=\mathcal{E}_{B|A}(\rho_A)$ at t_2. Classically, there is no reason not to consider the two-time joint probability distribution $P(X,Y)$ that results from combining the preparation $P(X)$ with the dynamics $\Gamma_{Y|X}$. To do this, we need only define the conditional distributions $P(Y|X)$, since the joint probability is then given by Eq. (4). Comparing Eq. (5) with Eqs. (2) and (4), we see that setting

$$P(Y=i|X=j) = (\Gamma_{Y|X})_{ij}, \qquad (6)$$

for all i,j such that $P(X=j) \neq 0$, gives the desired result. Note that, for a fixed preparation $P(X)$, we may vary the dynamics arbitrarily for all values of X that have no support in $P(X)$, without affecting the conditional distribution $P(Y|X)$ or the joint probability $P(X,Y)$. Conversely, knowing $P(Y|X)$ or $P(X,Y)$ only specifies the dynamics on the support of $P(X)$.

Now, the set of joint probability distributions obtainable in cases (a) and (c) are precisely the same, so we can define an isomorphism between the pair of objects consisting of a preparation and a dynamics and the joint state of two subsystems:

$$(P(X),\Gamma^r_{Y|X}) \leftrightarrow P(X,Y). \qquad (7)$$

Here, $\Gamma^r_{Y|X}$ refers to the restriction of the dynamics $\Gamma_{Y|X}$ to the support of $P(X)$ and is in one-to-one correspondence with the conditional probability $P(Y|X)$. The left-hand side of Eq. (7) can be thought of as a description of a case (a) scenario and the right-hand side as a description of a case (c) scenario. This may seem like an unnecessarily complicated restatement of what is essentially the definition of conditional probability, but it is worth remarking upon because the new iso-

morphism of Sec. IV is the quantum analog of this. That is, we construct an isomorphism between the pair of objects consisting of a preparation and a dynamics and the joint state of two subsystems:

$$(\rho_A,\mathcal{E}^r_{B|A}) \leftrightarrow \tau_{AB}, \qquad (8)$$

where $\mathcal{E}^r_{B|A}$ denotes the restriction of a TPCP map $\mathcal{E}_{B|A}$ to the support of ρ_A. The object $\mathcal{E}^r_{B|A}$ is to be thought of as a quantum analog of conditional probability, playing the same role as $\Gamma^r_{Y|X}$ does in classical probability theory.

III. JAMIOŁKOWSKI ISOMORPHISM

In this section, the standard Jamiołkowski isomorphism is reviewed. This relates CP maps $\mathcal{E}_{B|A}$ to bipartite states τ_{AB}, without introducing the state ρ_A that appears in Eq. (8), and is later shown to be a special case of the more general isomorphism described in Sec. IV. Sections III A and III B give the mathematical statement of the isomorphism, and Sec. III C gives its operational interpretation. Comments about the isomorphism that will be important in what follows are made in Sec. III D. The discussion is intended to be self-contained, but the interested reader can find detailed overviews different aspects of the isomorphism in [7,22].

A. Operators and pure states

Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be Hilbert spaces of dimension d_A and d_B, respectively, and let $\{|j\rangle_A\}$ be an orthonormal basis for $\mathcal{H}_A$. An operator $R_{B|A}:\mathcal{H}_A \rightarrow \mathcal{H}_B$ is isomorphic to a (generally unnormalized) pure state $|\Psi_R\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$ given by

$$|\Psi_R\rangle_{AB} = \frac{1}{\sqrt{d_A}}\sum_{j=1}^{d_A} |j\rangle_A \otimes R_{B|A}|j\rangle_{A'} = I_A \otimes R_{B|A}|\Phi^+\rangle_{AA'},$$

$$(9)$$

where A' denotes an additional system with the same Hilbert space as A,[1] $|\Phi^+\rangle_{AA'} = \frac{1}{\sqrt{d_A}}\Sigma_{j=1}^{d_A}|j\rangle_A \otimes |j\rangle_{A'}$ is a maximally entangled state on $\mathcal{H}_A \otimes \mathcal{H}_{A'}$, and I_A is the identity operator on $\mathcal{H}_A$.[2]

To see that this is an isomorphism, note that the action of $R_{B|A}$ on a pure state $|\psi\rangle_A$ can be recovered from $|\Psi_R\rangle_{AB}$ via

$$R_{B|A}|\psi\rangle_A = d_A \langle\Phi^+|_{AA'}|\psi\rangle_A \otimes |\Psi_R\rangle_{A'B}. \quad (10)$$

B. Completely positive maps and mixed states

The isomorphism can be extended from operators to CP maps, $\mathcal{E}_{B|A}:\mathfrak{L}(\mathcal{H}_A) \to \mathfrak{L}(\mathcal{H}_B)$, where $\mathfrak{L}(\mathcal{H})$ is the space of linear operators on a Hilbert space $\mathcal{H}$. An arbitrary CP map can be characterized by a set of linear operators $R_{B|A}^{(\mu)}:\mathcal{H}_A \to \mathcal{H}_B$, known as Kraus operators. The action of $\mathcal{E}_{B|A}$ on density operators $\rho_A \in \mathfrak{L}(\mathcal{H}_A)$ is given by

$$\mathcal{E}_{B|A}(\rho_A) = \sum_\mu R_{B|A}^{(\mu)}\rho_A R_{B|A}^{(\mu)\dagger}, \quad (11)$$

where $\dagger$ denotes the conjugate transpose. Note that a CP map typically has more than one decomposition into Kraus operators.

If $\Sigma_\mu R_{B|A}^{(\mu)\dagger}R_{B|A}^{(\mu)} = I_A$, then the map is called a trace preserving completely positive (TPCP) map and it can be implemented with certainty by introducing an ancilla, performing a unitary transformation and then taking the partial trace over a subsystem (see [1] for details). On the other hand, if $\Sigma_\mu R_{B|A}^{(\mu)\dagger}R_{B|A}^{(\mu)} < I_A$, then the action of the CP map gives the (unnormalized) updated state after obtaining a particular outcome in a generalized measurement, and it cannot be implemented with certainty. In what follows, the main focus is on TPCP maps, but comments on the general case are made in Sec. III D.

The state isomorphic to $\mathcal{E}_{B|A}$ is generally mixed and is given by

$$\tau_{AB} = \sum_\mu |\Psi_{R^{(\mu)}}\rangle_{AB}\langle\Psi_{R^{(\mu)}}|_{AB} \quad (12)$$

$$= \mathcal{I}_A \otimes \mathcal{E}_{B|A'}(|\Phi^+\rangle_{AA'}\langle\Phi^+|_{AA'}), \quad (13)$$

where $\mathcal{I}_A$ is the identity CP map on $\mathfrak{L}(\mathcal{H}_A)$. Note that the state τ_{AB} depends only on $\mathcal{E}_{B|A}$ and not on a particular decomposition into Kraus operators. The form in Eq. (12) gives different pure-state decompositions of the same density operator as the Kraus decomposition is varied and all pure state decompositions of τ_{AB} can be obtained in this way.

The reverse direction of the isomorphism is similar to Eq. (10). The action of $\mathcal{E}_{B|A}$ on an arbitrary $\sigma_A \in \mathfrak{L}(\mathcal{H}_A)$ is given by

$$\mathcal{E}_{B|A}(\sigma_A) = d_A^2 \langle\Phi^+|_{AA'}\sigma_A \otimes \tau_{A'B}|\Phi^+\rangle_{AA'}. \quad (14)$$

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[1] Generally, $X', X'', \ldots$ are used as labels for ancillary systems with the same Hilbert space as the system labeled by X, and $\mathcal{H}_{X'}, \mathcal{H}_{X''}, \ldots$ are synonyms for $\mathcal{H}_X$.

[2] Generally, I_X denotes the identity operator on $\mathcal{H}_X$ for an arbitrary system label X.

C. Operational interpretation

So far, the isomorphism has been stated as a mathematical fact. For TPCP maps, it obtains operational meaning via *noisy gate teleportation*, the obvious extension of a protocol described in [23] for unitary gates.

Suppose Alice holds an unknown state σ_A and that Bob wishes to end up with the transformed state $\mathcal{E}_{B|A}(\sigma_A)$, where $\mathcal{E}_{B|A}$ is a TPCP map. They also share a copy of the isomorphic state $\tau_{A'B}$ and they wish to achieve the task via local operations and classical communication (LOCC). If the map $\mathcal{E}_{B|A}$ is just the identity $\mathcal{I}_A$, then the task can be achieved via the usual teleportation protocol, since $\tau_{A'B}$ is maximally entangled in this case. In general, the task can be achieved with probability of success at least $\frac{1}{d_A^2}$, since if Alice makes a measurement in a basis that includes the state $|\Phi^+\rangle_{AA'}$, then Bob will receive the correct transformed state whenever Alice gets this outcome, as can be seen from Eq. (14). For certain special maps, Bob can correct his state when Alice does not get the right outcome, as in the teleportation protocol, but this is not possible in general.

D. Remarks

The following facts about the Jamiołkowski isomorphism are important in what follows. First, note that the isomorphism is basis dependent, since the definition of the state $|\Phi^+\rangle$ makes use of a particular basis. The association between a bipartite state and a CP map is unique, up to this choice of basis.

Second, if $\mathcal{E}_{B|A}$ is a TPCP map, then the state τ_{AB} always has the maximally mixed state as the reduced density operator for system A—i.e., $\mathrm{Tr}_B(\tau_{AB}) = \frac{I_A}{d_A}$. This can be deduced from Eq. (13) and the fact that the state $|\Phi^+\rangle_{AA'}$ is maximally mixed on A. To obtain an arbitrary state via the isomorphism, one has to use the more general CP maps that cannot be implemented deterministically. For example, the pure product state $|00\rangle_{AB}$ corresponds to the projection operator $|0\rangle_B\langle 0|_A$ that results from obtaining the $|0\rangle$ outcome of a measurement in the computational basis and then relabeling system A to B. A major difference between the standard isomorphism and the new variant described in Sec. IV is that in the new version, all bipartite states are obtained with just TPCP maps.

IV. VARIANT OF THE JAMIOŁKOWSKI ISOMORPHISM

In this section, our isomorphism is described. It is constructed and shown to be an isomorphism in Sec. IV A. Section IV B gives the operational interpretation of the isomorphism. Finally, Secs. IV C and IV D describe some properties of the isomorphism that are exploited in the applications that follow.

A. Construction of the isomorphism

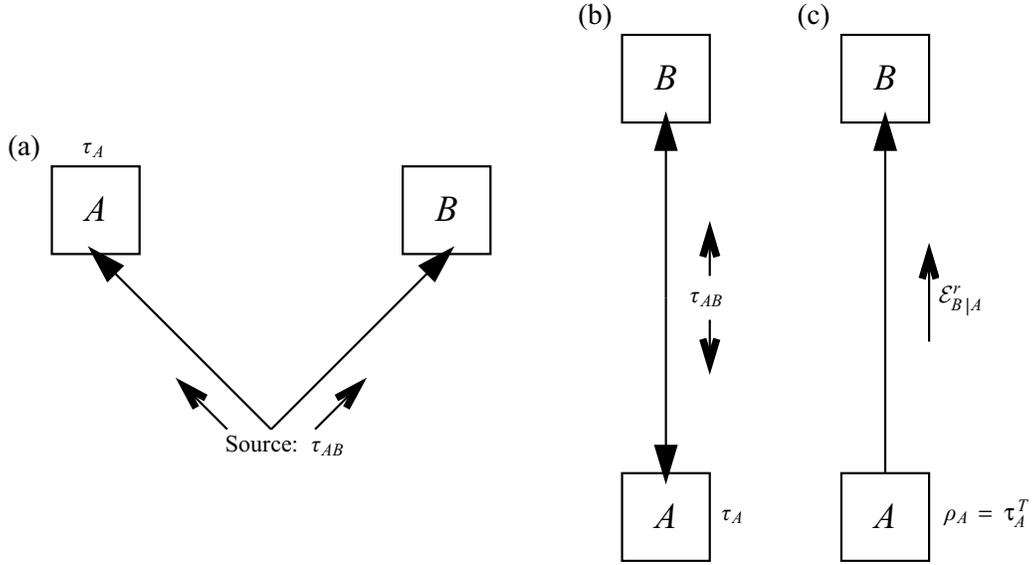Recall from Sec. II that the aim is to construct an isomorphism

FIG. 2. In these diagrams, time flows up the page. Starting from (a), the space and time axes are interchanged and the diagram is "stretched out" to arrive at (b). This does not describe a possible experiment, since we cannot send system A backwards in the time direction. In order to arrive at a feasible experiment, some arrows must be reversed, giving rise to (c). The transpose on τ_A is an artifact of this time reversal.

$$(\rho_A, \mathcal{E}^r_{B|A}) \leftrightarrow \tau_{AB}, \tag{15}$$

where $\mathcal{E}^r_{B|A}$ denotes the restriction of a TPCP map $\mathcal{E}_{B|A}$ to the support of ρ_A.

We begin by describing the forward direction of the isomorphism, $(\rho_A, \mathcal{E}^r_{B|A}) \rightarrow \tau_{AB}$. First, construct the state

$$|\Phi\rangle_{AA'} = \sqrt{d_A}(\rho_A^T)^{1/2} \otimes I_{A'}|\Phi^+\rangle_{AA'}, \tag{16}$$

where T denotes transpose in the basis used to define $|\Phi^+\rangle_{AA'}$. Note that this is a normalized state since $\langle\Phi|\Phi\rangle_{AA'} = \text{Tr}(\rho_A^T) = 1$. Also, $\text{Tr}_{A'}(|\Phi\rangle\langle\Phi|_{AA'}) = \rho_A^T$ and $\text{Tr}_A(|\Phi\rangle\langle\Phi|_{AA'})$ is identical to ρ_A, so that the action of $\mathcal{E}^r_{B|A'}$ on system A' is well defined for this state. Finally, define

$$\tau_{AB} = \mathcal{I}_A \otimes \mathcal{E}^r_{B|A'}(|\Phi\rangle\langle\Phi|_{AA'}), \tag{17}$$

which is a normalized state because $\mathcal{E}^r_{B|A'}$ is the restriction of a TPCP map.

For the reverse direction $\tau_{AB} \rightarrow (\rho_A, \mathcal{E}^r_{A \rightarrow B})$, begin by defining the state

$$\rho_A = \tau_A^T = \text{Tr}_B(\tau_{AB})^T. \tag{18}$$

To define the map $\mathcal{E}^r_{B|A}$, care must be taken if τ_A is not invertible. If this is the case, define τ_A^{-1} to be the inverse restricted to the support of τ_A. This means that its nonzero eigenvalues are the reciprocals of the nonzero eigenvalues of τ_A and they are associated with the same eigenvectors and that the zero eigenspaces of τ_A and τ_A^{-1} are the same. Now, define the state

$$\sigma_{AB} = \tau_A^{-1/2} \otimes I_B \tau_{AB} \tau_A^{-1/2} \otimes I_B. \tag{19}$$

This is a density operator with $\sigma_A = \text{Tr}_B(\sigma_{AB}) = \frac{1}{d_A^r}P_A$, where P_A is the projector onto the support of τ_A and d_A^r is the rank

of τ_A. The associated subspace $P_A\mathcal{H}_A$ is also a Hilbert space, for which P_A is the identity operator, so σ_A is maximally mixed on this subspace. Thus, σ_{AB} is uniquely associated with a TPCP map $\mathcal{E}^r_{B|A}:\mathfrak{L}(P_A\mathcal{H}_A) \rightarrow \mathfrak{L}(H_B)$ via the standard Jamiołkowski isomorphism.

In the above construction, Eq. (19) can be viewed as a direct analog of the definition of conditional probability $P(Y|X) = P(X,Y)/P(X)$, since conjugation by $\tau_A^{-1/2}$ reduces to elementwise division in the case where τ_{AB} is diagonal in a basis $\{|\phi_j\rangle_A \otimes |\psi_k\rangle_B\}$, where the $|\phi_j\rangle_A$ form an orthonormal basis for $\mathcal{H}_A$ and the $|\psi_k\rangle$ form an orthonormal basis for $\mathcal{H}_B$. The introduction of the transpose in Eq. (18) is due to a time reversal implicit in the construction, which is illustrated by Fig. 2. Note that in the case where $\rho_A = \frac{I_A}{d_A}$, this construction reduces to the standard Jamiołkowski isomorphism.

Theorem IV.1. The construction described above is an isomorphism.

Proof. The above relations define an isomorphism if it can be shown that one obtains the same pair $(\rho_A, \mathcal{E}^r_{B|A})$ on applying the forward and reverse directions in sequence. To check this for the state ρ_A, let $R^{(\mu)}_{B|A'}$ be a set of Kraus operators for $\mathcal{E}^r_{B|A'}$. Then,

$$\tau_{AB} = \sum_{jk\mu} (\rho_A^T)^{1/2}|j\rangle\langle k|(\rho_A^T)^{1/2} \otimes R^{(\mu)}_{B|A'}|j\rangle\langle k|R^{(\mu)\dagger}_{B|A'}. \tag{20}$$

Taking the partial trace gives

$$\tau_A = \sum_{jk\mu} (\rho_A^T)^{1/2}|j\rangle\langle k|(\rho_A^T)^{1/2}\langle k|\sum_\mu R^{(\mu)\dagger}_{B|A'}R^{(\mu)}_{B|A'}|j\rangle. \tag{21}$$

Since $\mathcal{E}^r_{B|A}$ is trace preserving, $\sum_\mu R^{(\mu)\dagger}_{B|A'}R^{(\mu)}_{B|A'} = I_{A'}$, so

$$\tau_A = (\rho_A^T)^{1/2} \sum_j |j\rangle\langle j| (\rho_A^T)^{1/2} = (\rho_A^T)^{1/2}(\rho_A^T)^{1/2} = \rho_A^T, \quad (22)$$

which gives the correct state for the reverse direction.

To check the map $\mathcal{E}_{B|A}^r$, note that the state $|\Phi^+\rangle_{AA'}$ that appears in Eq. (16) can be replaced by the state $|\Phi^+\rangle_{AA'}^r = \sqrt{\frac{d_A}{d_A^r}} P_A \otimes I_{A'} |\Phi^+\rangle_{AA'}$, provided $\sqrt{d_A}$ is also replaced by $\sqrt{d_A^r}$ in Eq. (16). This is because $(\rho_A^T)^{1/2}$ only has support on the subspace that P_A projects onto, so the state $|\Phi\rangle_{AA'}$ obtained will be the same. The action of $\mathcal{E}_{B|A'}^r$ is well defined on $|\Phi^+\rangle_{AA'}^r$ and the two steps of the construction commute, so that the same state τ_{AB} is obtained by applying the CP map to $|\Phi^+\rangle_{AA'}^r$, followed by conjugation with $(\rho_A^T)^{1/2}$. The state $|\Phi^+\rangle_{AA'}^r$ is maximally entangled on the Hilbert space $P_A\mathcal{H}_A \otimes P_A\mathcal{H}_A$, and so the state $\sigma_{AB} = \mathcal{I}_A \otimes \mathcal{E}_{B|A'}^r(|\Phi^+\rangle^r\langle\Phi^+|_{AA'}^r)$ is obtained from applying the standard Jamiołkowski isomorphism to $\mathcal{E}_{B|A}^r$. On applying the reverse construction, the same state σ_{AB} is obtained in Eq. (19), and because states and maps are uniquely related by the standard isomorphism, the map $\mathcal{E}_{B|A}^r$ that we started with is recovered from this procedure. ∎

B. Operational interpretation

Unlike the standard Jamiołkowski isomorphism, our isomorphism does not have an immediate operational interpretation in terms of noisy gate teleportation. However, there is a sense in which τ_{AB} and the pair $(\rho_A, \mathcal{E}_{B|A}^r)$ are operationally indistinguishable. To understand this, we need to recall the role of positive-operator-valued measures (POVM's) in describing generalized quantum measurements [1] and explain their correspondence to ensemble preparations of density operators.

A POVM is a set of positive operators that sum to the identity. Here, POVM's are denoted by uppercase letters $M, N, \dots$. The operators within a POVM are denoted by the corresponding boldface letter—e.g., $M = \{\boldsymbol{M}^{(m)}\}$, where the superscript m is a positive integer used to distinguish the operators within POVM.

POVM's are normally used to compute the probabilities for the possible outcomes of generalized measurements. Let the possible outcomes be labeled by the same integers as the POVM elements, so that the generalized Born rule is

$$P(M = m) = \text{Tr}(\boldsymbol{M}^{(m)}\rho). \quad (23)$$

Note that the symbol M, which stands for a collection of operators, is also being used to denote the random variable generated by the measurement. It should be clear from the context which of the two meanings is intended.

It is convenient to extend the random variable notation used in Sec. II to POVM's, by leaving the index m implicit. With this, the POVM is written as $M = \{\boldsymbol{M}\}$, and Eq. (23) reduces to

$$P(M) = \text{Tr}(\boldsymbol{M}\rho). \quad (24)$$

Generally, a POVM only describes the outcome statistics of a measurement and does not specify how the state is to be updated on obtaining a particular outcome. The update rule that should be used depends on the details of the interaction between the system being measured and the measuring device. Among the possible update rules, a particularly natural choice is

$$\rho(M) = \frac{\sqrt{\boldsymbol{M}}\rho\sqrt{\boldsymbol{M}}}{P(M)}, \quad (25)$$

since this reduces to the Lüders–von Neumann projection postulate [24] in the case where each $\boldsymbol{M}$ is a projection operator. If a POVM M is measured on a state ρ, generating the probability distribution of Eq. (24), and the state is updated according to Eq. (25), then we refer to this an M measurement of ρ. The update rule for an M *measurement* is not important for the operational interpretation developed in this section, but it is used in Sec. IV C and in the applications of Sec. V.

Although POVM's are normally used to describe measurements, they can also be used to describe the different methods of preparing a density operator ρ. This is demonstrated by the following lemma.

Lemma IV.2. Let ρ be a density operator and $M = \{\boldsymbol{M}\}$ a POVM. Define $P(M) = \text{Tr}(\boldsymbol{M}\rho)$ and let $\rho(M) = \frac{\sqrt{\rho}\boldsymbol{M}\sqrt{\rho}}{p(M)}$ whenever $P(M) > 0$. Then, $\rho = \Sigma_M P(M)\rho(M)$ is an ensemble decomposition of ρ into a convex combination of density operators. Conversely, any ensemble decomposition of ρ is related to a POVM in this way.

Proof. It is clear from the definition of a POVM that $0 \leq p(M) \leq 1$ and $\Sigma_M p(M) = 1$. The operators $\rho(M)$ are positive, since they are of the form $A^\dagger A$ for $A = \sqrt{\boldsymbol{M}}\sqrt{\rho}$. They also have unit trace, so they are density operators. Furthermore, $\Sigma_M P(M)\rho(M) = \Sigma_M \sqrt{\rho}\boldsymbol{M}\sqrt{\rho} = \rho$, by virtue of the fact that POVM operators sum to the identity.

To prove the converse, let $\rho = \Sigma_M P(M)\rho(M)$ be an ensemble decomposition of ρ. Then, define positive operators $\boldsymbol{M}^r = P(M)\rho^{-1/2}\rho(M)\rho^{-1/2}$, where $\rho^{-1/2}$ is the restricted inverse of $\rho^{1/2}$ as defined in Sec. IV A. These satisfy $\text{Tr}(\boldsymbol{M}^r\rho) = P(M)$ and $\frac{\sqrt{\rho}\boldsymbol{M}^r\sqrt{\rho}}{p(M)} = \rho(M)$. Clearly, $\Sigma_M \boldsymbol{M}^r = P_\rho$, where P_ρ is the projector onto the support of ρ. Let $P_\rho^\perp$ be the projector onto the orthogonal complement of this subspace and choose a set of positive operators $\boldsymbol{M}^s$ of the same cardinality as M, supported only on this orthogonal complement, which sum to $P_\rho^\perp$. Since $\boldsymbol{M}^s\rho^{1/2} = 0$, the operators $\boldsymbol{M} = \boldsymbol{M}^r + \boldsymbol{M}^s$ satisfy $\frac{\sqrt{\rho}\boldsymbol{M}\sqrt{\rho}}{p(M)} = \rho(M)$ and sum to the identity, as required. ∎

The above lemma shows that POVM's may be used to describe ensemble preparations of density operators as well as measurements. For a POVM M and a density operator ρ, an M *preparation* of ρ is defined to be the procedure of generating a classical random variable with distribution $P(M) = \text{Tr}(\boldsymbol{M}\rho)$ and then preparing the corresponding density matrix $\rho(M) = \frac{\sqrt{\rho}\boldsymbol{M}\sqrt{\rho}}{p(M)}$.

At this stage, the relation between the statistics obtainable from a bipartite state τ_{AB} and those obtainable from the isomorphic pair $(\rho_A, \mathcal{E}_{B|A}^r)$ can be stated and proved. It can be understood schematically from the idea that the isomorphism
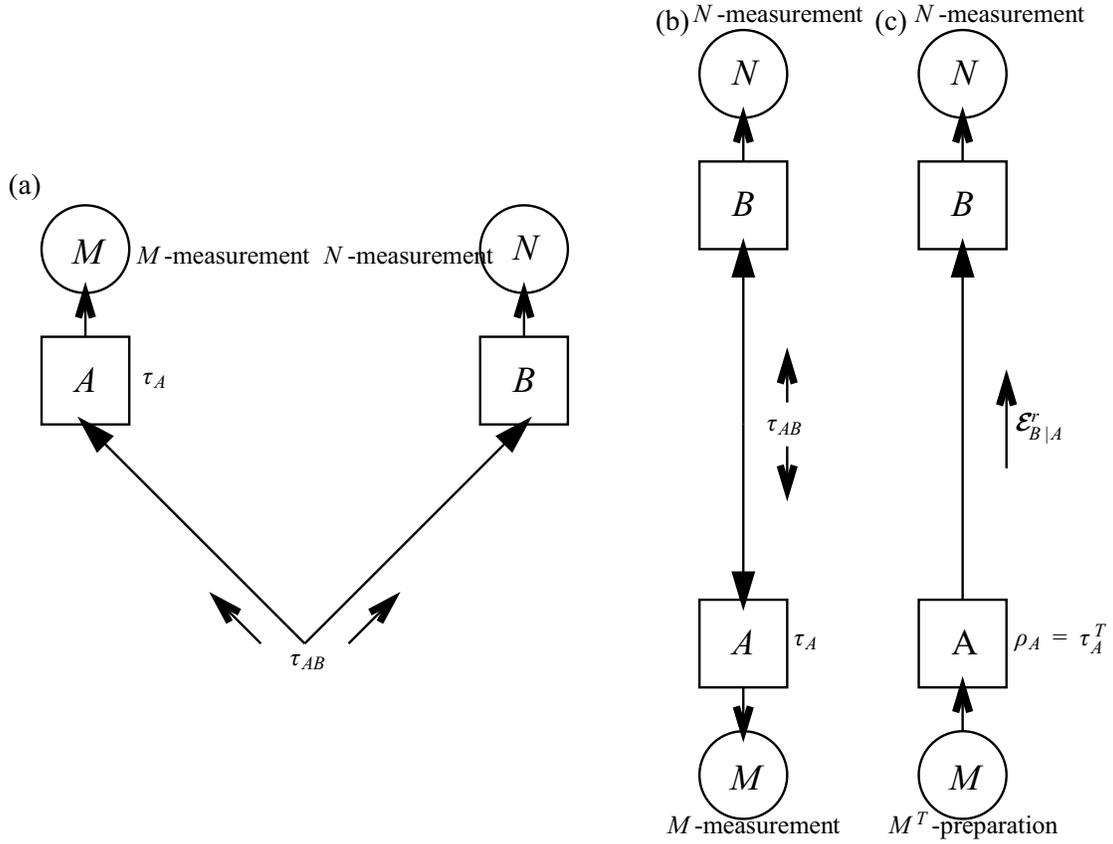
FIG. 3. The same experiments as Fig. 2, with the addition of measurements and preparations. (a) is obtained by simply adding M and N measurements to Fig. 2(a). In (b), the space and time axes have been swapped and the diagram has been "stretched out." As with Fig. 2(b) this does not represent a possible experiment. To obtain a feasible experiment, in addition to the transformations of Fig. 2, the M measurement must be transformed into a preparation, leading to (c). The transpose is an artifact of this time reversal.

represents an interchange of space and time axes, as shown in Fig. 3.

Theorem IV.3. Let τ_{AB} be a bipartite state that is related to a pair $(\rho_A, \mathcal{E}_{B|A})$ by the isomorphism of Sec. IV A. Let M and N be arbitrary POVM's, defined on $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, and let M^T be the POVM obtained by taking the transpose of all the operators in M with respect to the basis used to construct the isomorphism. Then, the joint probability distribution of M and N measurements on τ_{AB}, performed in parallel, is the same as the joint probability distribution of the sequence of operations consisting of an M^T preparation of ρ_A, followed by evolution according to $\mathcal{E}_{B|A}$, followed by an N measurement.

Proof. The proof is by direct computation. Let $P(M,N)$ $=\mathrm{Tr}(M_A \otimes N_B \tau_{AB})$ be the probability distribution of the two measurements performed in parallel on the state τ_{AB}, and let $Q(M,N)=\mathrm{Tr}_B[N_B \mathcal{E}^r_{B|A}(\rho_A^{1/2} M_A^T \rho_A^{1/2})]$ be the joint probability distribution of the M^T preparation, followed by evolution according to $\mathcal{E}^r_{B|A}$, followed by the N measurement. Let $\{|j\rangle\}$ be the basis of $\mathcal{H}_A$ in which the isomorphism is defined. Then,

$$P(M,N) = \mathrm{Tr}(M_A \otimes N_B \rho_{AB}) \qquad (26)$$

$$=\mathrm{Tr}(M_A \otimes N_B \tau_A^{1/2} \otimes I_B \sigma_{AB} \tau_A^{1/2} \otimes I_B), \quad (27)$$

where σ_{AB} is the state defined in Eq. (19). Since $\sigma_{AB}=\mathcal{I}_A$ $\otimes \mathcal{E}^r_{B|A'}(|\Phi^+\rangle^r \langle \Phi^+|^r_{AA'})$, this gives

$$P(M,N)$$
$$= \mathrm{Tr}\left(\tau_A^{1/2} M_A \tau_A^{1/2} \otimes N_B \frac{1}{d_A^r} \sum_{j,k=1}^{d_A^r} |j\rangle\langle k|_A \otimes \mathcal{E}^r_{B|A}(|j\rangle\langle k|_A) \right).$$
$$(28)$$

Let $\{R_{B|A}^{(\mu)}\}$ be a set of Kraus operators for $\mathcal{E}^r_{B|A}$. Substituting these and rearranging then gives

$$P(M,N) = \sum_{j,k=1}^{d_A^r} \langle k| \tau_A^{1/2} M_A \tau_A^{1/2} |j\rangle\langle k| R_{B|A}^{(\mu)\dagger} N_B R_{B|A}^{(\mu)} |j\rangle \qquad (29)$$

$$= \sum_{j=1}^{d_A^r} \langle j| (\tau_A^{1/2})^T M_A^T (\tau_A^{1/2})^T \sum_{k=1}^{d_A^r} |k\rangle\langle k| R_{B|A}^{(\mu)\dagger} N_B R_{B|A}^{(\mu)} |j\rangle.$$
$$(30)$$

Now, $\tau_A^T = \rho_A$ and $\sum_{k=1}^{d_A^r} |k\rangle\langle k| = P_A$, where P_A is the projector onto the support of ρ_A, so

$$P(M,N) = \sum_{j=1}^{d_A^r} \langle j | \rho_A^{1/2} M_A^T \rho_A^{1/2} P_A R_{B|A}^{(\mu)\dagger} N_B R_{B|A}^{(\mu)} | j \rangle. \quad (31)$$

However, $\rho_A^{1/2} P_A R_{B|A}^{(\mu)\dagger} = \rho_A^{1/2} R_{B|A}^{(\mu)\dagger}$, since $R_{B|A}^{(\mu)}$ is only defined on the support of ρ_A. Substituting this and rearranging gives

$$P(M,N) = \mathrm{Tr}_B \left(N_B R_{B|A}^{(\mu)} \sum_{j=1}^{d_A^r} | j \rangle \langle j |_A \rho_A^{1/2} M_A^T \rho_A^{1/2} R_{B|A}^{(\mu)\dagger} \right). \quad (32)$$

Now again $\sum_{j=1}^{d_A^r} | j \rangle \langle j |_A = P_A$ and $R_{B|A}^{(\mu)} P_A \rho_A^{1/2} = R_{B|A}^{(\mu)} \rho_A^{1/2}$, so

$$P(M,N) = \mathrm{Tr}_B [N_B \mathcal{E}_{B|A}^r (\rho_A^{1/2} M_A^T \rho_A^{1/2})] \quad (33)$$

$$= Q(M,N). \quad (34)$$

∎

C. Commutativity properties of the isomorphism

Two commutativity properties of the isomorphism are useful for the applications that follow. First, the isomorphism commutes with the partial trace for tripartite states. To describe this, it is useful to introduce the concept of a reduced map.

Definition IV.4. For a linear map $\mathcal{E}_{BC|A} : \mathfrak{L}(\mathcal{H}_A) \to \mathfrak{L}(\mathcal{H}_B \otimes \mathcal{H}_C)$. The reduced map $\mathcal{E}_{B|A} : \mathfrak{L}(\mathcal{H}_A) \to \mathfrak{L}(\mathcal{H}_B)$ is given by composing the map with the partial trace—i.e., $\mathcal{E}_{B|A} = \mathrm{Tr}_C \circ \mathcal{E}_{BC|A}$.

Starting with a pair $(\rho_A, \mathcal{E}_{BC|A}^r)$, the isomorphism can be used to arrive at a tripartite state τ_{ABC}, and then the partial trace over C gives the bipartite reduced state τ_{AB}. This is the same bipartite state that one obtains by applying the isomorphism to the pair $(\rho_A, \mathcal{E}_{B|A}^r)$. This is summarized in the following diagram:

$$\begin{array}{ccc} \rho_{ABC} & = & (\rho_A, \mathcal{E}_{BC|A}^r) \\ \mathrm{Tr}_C \downarrow & & \downarrow \mathrm{Tr}_C \\ \rho_{AB} & = & (\rho_A, \mathcal{E}_{B|A}^r). \end{array} \quad (35)$$

The second commutativity property concerns M measurements. Starting with a pair $(\rho_A, \mathcal{E}_{B|A}^r)$, the isomorphism can be used to arrive at a bipartite state τ_{AB}, and then an M measurement can be applied to system A, giving a bipartite state $\sqrt{M}_A \otimes I_B \tau_{AB} \sqrt{M}_A \otimes I_B$, where the normalization factor has been omitted. This is the same bipartite state that one obtains by first performing an M^T measurement on ρ_A to obtain the pair $(\sqrt{M_A^T} \rho_A \sqrt{M_A^T}, \mathcal{E}_{B|A}^r)$ and then applying the isomorphism. This is summarized in the following diagram:

$$\begin{array}{ccc} \tau_{AB} & = & (\rho_A, \mathcal{E}_{B|A}^r) \\ M_A\text{-measurement} \downarrow & & \downarrow M_A^T\text{-measurement} \\ \sqrt{M}_A \otimes I_B \tau_{AB} \sqrt{M}_A \otimes I_B & = & (\sqrt{M_A^T} \rho_A \sqrt{M_A^T}, \mathcal{E}_{B|A}^r) \end{array} \quad (36)$$

These commutativity properties are straightforward to prove from the definition of the isomorphism, and so the proofs are omitted here.

D. Remarks

As with the standard isomorphism, our construction depends on the basis chosen for $|\Phi^+\rangle_{AA'}$. The forward direction takes a particularly simple form if this is chosen to be an eigenbasis of ρ_A, since this basis is then a Schmidt basis for $|\Phi\rangle_{AA'}$. Let $\rho_A = \sum_j \lambda_j |j\rangle\langle j|_A$ be an eigendecomposition of ρ_A. Then Eq. (16) can be written as

$$|\Phi\rangle_{AA'} = \sum_j \sqrt{\lambda_j} |j\rangle_A \otimes |j\rangle_{A'} \quad (37)$$

and Eq. (17) reduces to

$$\tau_{AB} = \sum_{jk} \sqrt{\lambda_j \lambda_k} |j\rangle\langle k|_A \otimes \mathcal{E}_{B|A'}^r (|j\rangle\langle k|_{A'}). \quad (38)$$

With this choice of basis, the cumbersome transpose can be eliminated, since $\rho_A^T = \rho_A$. Additionally, $M^T = M$ holds for any POVM M with operators M that are diagonal in this basis.

Note that if $\mathcal{E}_{B|A}$ is a unitary operation, then the state τ_{AB} is pure, regardless of the state ρ_A. If in addition ρ_A is of rank ≥ 2, then τ_{AB} has more than one Schmidt coefficient, so it is both pure and entangled.

V. APPLICATION: CLONING, BROADCASTING, AND THE MONOGAMY OF ENTANGLEMENT

The standard Jamiołkowski isomorphism is useful because it allows facts about CP maps to be recast as facts about bipartite states and vice versa. On the other hand, there are situations in which it is not necessary to know the action of a TPCP map on the whole Hilbert space, but only how it acts on a particular density matrix or, more generally, on an ensemble decomposition of a particular density matrix. In such cases, the present isomorphism is a more appropriate tool to use. A simple example of this is given by the no-cloning and no-broadcasting theorems [8–10]. In their original form, these theorems concern the possible action of a TPCP map on just a pair of noncommuting input states. The isomorphism allows these theorems to be recast as facts about the monogamy of entanglement in tripartite states. These terms are defined precisely in Sec. V A. Section V B gives a simple result for the no-universal broadcasting theorem, and Sec. V C describes the most general results, including a simplified proof of the no-broadcasting theorem that is derived as a by-product.

Before getting into the technical details, a word of warning about how to interpret the results described below. The goal is to relate properties of hypothetical TPCP maps of the form $\mathcal{E}_{BC|A} : \mathfrak{L}(\mathcal{H}_A) \to \mathfrak{L}(\mathcal{H}_B \otimes \mathcal{H}_C)$ to properties of hypothetical tripartite states τ_{ABC} via the isomorphism. The existence of such maps and states is known to be in contradiction with quantum mechanics via the no-cloning and no-broadcasting theorems and the monogamy of entanglement, respectively. Nevertheless, properties of τ_{ABC} are derived by assuming the existence of $\mathcal{E}_{BC|A}$ and the correctness of quantum mechanics as premises. This may seem meaningless, since any statement can be logically deduced from a contradiction, regardless of its truth or falsity. However, the commutativity of the isomorphism with the partial trace averts this conclusion. As

described below, the map $\mathcal{E}_{BC|A}$ is defined by placing constraints on the reduced maps $\mathcal{E}_{B|A}$ and $\mathcal{E}_{C|A}$. Quantum mechanics does in fact allow maps that satisfy these constraints when they are considered in isolation. The no-cloning and no-broadcasting theorems simply show that they cannot both be the reduced maps of some global map $\mathcal{E}_{BC|A}$. Similarly, the reduced states τ_{AB} and τ_{AC} of the hypothetical τ_{ABC} are in fact valid density operators; it is just that they cannot both be the reduced states of some valid global density operator τ_{ABC}.

The results below simply relate the properties of the valid reduced maps $\mathcal{E}_{B|A}$ and $\mathcal{E}_{C|A}$ to those of the valid reduced states τ_{AB} and τ_{AC} via the isomorphism. Thereby, no contradiction is involved, and a precise connection between the properties that make the reduced maps incompatible with a global map and those that make the reduced states incompatible with a global state is obtained.

A. Definitions

The definitions of broadcasting and cloning concern TPCP maps that have an output Hilbert space which is a tensor product of two copies of the input Hilbert space. With a view to applying the isomorphism, it is useful to continue distinguishing the three copies of the Hilbert space by assigning them different labels A, A', and A''. In this section, states denoted by the same Greek letter, differing only in their subsystem label, refer to the same state on different copies of the same Hilbert space; e.g., ρ_A, $\rho_{A'}$, and $\rho_{A''}$ all refer to the same state.

Definition V.1. A TPCP map $\mathcal{E}_{A'A''|A}: \mathfrak{L}(\mathcal{H}_A) \to \mathfrak{L}(\mathcal{H}_{A'} \otimes \mathcal{H}_{A''})$ is *broadcasting* for a state ρ_A if

$$\mathcal{E}_{A'A''|A}(\rho_A) = \sigma_{A'A''}, \qquad (39)$$

where

$$\mathrm{Tr}_{A''}(\sigma_{A'A''}) = \rho_{A'} \quad \text{and} \quad \mathrm{Tr}_{A'}(\sigma_{A'A''}) = \rho_{A''}.$$

Equivalently, the reduced maps of $\mathcal{E}_{A'A''|A}$ must satisfy

$$\mathcal{E}_{A'|A}(\rho_A) = \rho_{A'}, \quad \mathcal{E}_{A''|A}(\rho_A) = \rho_{A''}. \qquad (40)$$

Definition V.2. A TPCP map $\mathcal{E}_{A'A''|A}: \mathfrak{L}(\mathcal{H}_A) \to \mathfrak{L}(\mathcal{H}_{A'} \otimes \mathcal{H}_{A''})$ is *cloning* for a state ρ_A if

$$\mathcal{E}_{A'A''|A}(\rho_A) = \rho_{A'} \otimes \rho_{A''}. \qquad (41)$$

Cloning is a stronger requirement than broadcasting, since the output state is required to be a product. For example, if the input state is maximally mixed $\rho_A = \frac{1}{d_A} I_A$, then two possible output states for a broadcasting map are $|\Phi^+\rangle\langle\Phi^+|_{A'A''}$ and $\frac{1}{d_{A'}d_{A''}} I_{A'} \otimes I_{A''}$, but a cloning map can only output the latter. In the case of a pure state, cloning and broadcasting are equivalent, since the purity of the reduced output states on A' and A'' ensures that they must be a product. Only the broadcasting condition is needed below, but it is referred to as cloning when only pure states are being considered.

Definition V.3. A *universal* broadcasting map is a TPCP map $\mathcal{E}_{A'A''|A}: \mathfrak{L}(\mathcal{H}_A) \to \mathfrak{L}(\mathcal{H}_{A'} \otimes \mathcal{H}_{A''})$ that is broadcasting for every possible input state. Equivalently, both the reduced maps $\mathcal{E}_{A'|A}$ and $\mathcal{E}_{A''|A}$ are the identity map.

The original no-cloning theorem [8,9] states that there is no TPCP map that is cloning for a pair of nonorthogonal and nonidentical pure states, and the original no-broadcasting theorem [10] states that there is no TPCP map that is broadcasting for a pair of noncommuting density operators. This obviously implies that universal broadcasting is impossible too, but it is worth considering as a special case because the connection between no universal cloning and the monogamy of entanglement is considerably simpler to prove than the general case.

Definition V.4. A TPCP map is broadcasting for an ensemble of states $\{(p_j, \rho_j)\}$ if it is broadcasting for every state ρ_j in the ensemble.

Strictly speaking, the weights p_j of the states in the ensemble are irrelevant to the definition, but introducing them is useful for deriving the connection to monogamy of entanglement. This is because the ensemble average state $\rho_A = \Sigma_j p_j \rho_j$ can be used along with the reduced maps $\mathcal{E}_{A'|A}$ and $\mathcal{E}_{A''|A}$ to construct bipartite states via the isomorphism.

Note that broadcasting and cloning are often defined in a superficially more general way than the definitions given here by allowing the input to include an arbitrary ancillary system in a standard state and the output to also include an ancillary system. However, the standard theorems about the representations of reduced dynamics by CP maps [1] ensure that the present definitions are equivalent.

The monogamy of entanglement refers to the fact that two bipartite states $\rho_{AA'}$ and $\rho_{AA''}$ cannot be arbitrarily entangled if they are the reduced states of a tripartite state $\rho_{AA'A''}$ [11]. Typically, there is a trade-off such that the greater the entanglement of $\rho_{AA'}$, according to some entanglement measure, the lower the entanglement of $\rho_{AA''}$ [25–29]. For present purposes, it is sufficient to note that if $\rho_{AA'}$ and $\rho_{AA''}$ are both pure, then they must both be product states in order to be compatible with a global state $\rho_{AA'A''}$.

B. Universal broadcasting

The standard Jamiołkowski isomorphism can be used to derive a connection between the no-universal broadcasting theorem and the monogamy of entanglement. Its proof is much simpler than the more general case described below, so it is included here for completeness.

Theorem V.5. Supposing the existence of a universal broadcasting map $\mathcal{E}_{A'A''|A}$ is equivalent to supposing the existence of a tripartite state $\tau_{AA'A''}$, where both the bipartite reduced states $\tau_{AA'}$ and $\tau_{AA''}$ are pure and maximally entangled.

Proof. By the assumption that $\mathcal{E}_{A'A''|A}$ is universal broadcasting, both the reduced maps $\mathcal{E}_{A'|A}$ and $\mathcal{E}_{A''|A}$ act as the identity on all input states. The state isomorphic to the identity by the standard isomorphism is the maximally entangled state $|\Phi^+\rangle$. Therefore, both $\tau_{AA'} = |\Phi^+\rangle\langle\Phi^+|_{AA'}$ and $\tau_{AA''} = |\Phi^+\rangle\langle\Phi^+|_{AA''}$.

Conversely, assume there is a tripartite state $\tau_{AA'A''}$, such that the reduced states $\tau_{AA'}$ and $\tau_{AA''}$ are pure and maximally entangled. By acting with independent local unitary transformations on the subsystems A' and A'', these states can be transformed to $|\Phi^+\rangle\langle\Phi^+|_{AA'}$ and $|\Phi^+\rangle\langle\Phi^+|_{AA''}$. The isomorphic

maps to these states are both the identity, so the map associated to the transformed tripartite state is universal broadcasting. ∎

Since this result uses the standard isomorphism, it should not be surprising that the converse has an interpretation in terms of teleportation. Indeed, if there existed a tripartite state $\tau_{AA'A''}$ with maximally entangled reduced states $\tau_{AA'}$ and $\tau_{AA''}$, then it would be possible to teleport any state to both A' and A'' simultaneously, which would provide a method of implementing a universal broadcasting map.

C. Ensemble broadcasting and cloning

In order to generalize this result to the ensemble broadcasting and cloning, some properties of the fixed point sets of TPCP maps are needed. Proofs of the quoted results can be found in [30,31]. These are used to provide a simple reduction of the no-broadcasting theorem to the no-cloning theorem. The mathematical structure uncovered in this proof is then used to derive the connection between ensemble broadcasting and the monogamy of entanglement. Finally, this is specialized to pure-state cloning, for which a stronger result is possible.

1. Fixed-point sets of TPCP maps

The set of density matrices invariant under any TPCP map that acts on $\mathfrak{L}(\mathcal{H})$ is a convex linear subspace of $\mathfrak{L}(\mathcal{H})$. There is a factorization of the Hilbert space $\mathcal{H}$ into a finite direct sum of tensor products

$$\mathcal{H} = \oplus_\alpha \mathcal{H}_{\alpha_1} \otimes \mathcal{H}_{\alpha_2}, \tag{42}$$

such that the invariant density operators are all those of the form

$$\sum_\alpha q_\alpha \mu_{\alpha_1} \otimes \nu_{\alpha_2}, \tag{43}$$

where $0 \leq q_\alpha \leq 1, \Sigma_\alpha q_\alpha = 1$. In this decomposition, the μ_{α_1}'s can be any density operators in $\mathfrak{L}(\mathcal{H}_{\alpha_1})$ and the ν_{α_2} are fixed density operators in $\mathfrak{L}(\mathcal{H}_{\alpha_2})$. For a pair of TPCP maps $\mathcal{E}$ and $\mathcal{F}$, the set of density operators invariant under both $\mathcal{E}$ and $\mathcal{F}$ is also of this form.

2. No-broadcasting theorem

Theorem V.6. A TPCP map $\mathcal{E}_{A'A''|A}$ that is broadcasting for a pair of states σ_1, σ_2, where $[\sigma_1, \sigma_2] \neq 0$, is cloning for a set of nonorthogonal and nonidentical pure states.

Proof. The set of density operators invariant under both the reduced maps $\mathcal{E}_{A'|A}$ and $\mathcal{E}_{A''|A}$ is of the form of Eq. (43). This set must include the density operators σ_1 and σ_2, by the assumption that $\mathcal{E}_{A'A''|A}$ is broadcasting, so they can be written as

$$\sigma_j = \sum_\alpha q_\alpha^{(j)} \mu_{\alpha_1}^{(j)} \otimes \nu_{\alpha_2}. \tag{44}$$

Since σ_1 and σ_2 do not commute, there must be at least one value β, such that for $\alpha = \beta$, the $\mathcal{H}_{\beta_1}$ factor in the decomposition of Eq. (42) is of dimension ≥ 2, $[\mu_{\beta_1}^{(1)}, \mu_{\beta_1}^{(2)}] \neq 0$ and

$q_{\beta_1}^{(1)}, q_{\beta_1}^{(2)} \neq 0$. Both maps $\mathcal{E}_{A'|A}$ and $\mathcal{E}_{A''|A}$ act as the identity on this factor, and hence any pure state on this factor is cloned by the map $\mathcal{E}_{A'A''|A}$. Since the factor is of dimension ≥ 2, there are nonorthogonal and nonidentical pure states within the factor. ∎

This result can be viewed as a simplified proof of the no-broadcasting theorem, since the no-cloning theorem itself is elementary to prove [8,9]. A similar strategy was used by Lindblad to prove a more general theorem [32], but the above is a more direct route to no-broadcasting.

3. Ensemble broadcasting

Theorem V.7. Suppose there existed a TPCP map $\mathcal{E}_{A'A''|A}$ that is broadcasting for a two-element ensemble of states $\{(p, \sigma_1), ((1-p), \sigma_2)\}$, such that $[\sigma_1, \sigma_2] \neq 0$. Let $\rho_A = p\sigma_1 + (1-p)\sigma_2$. The tripartite state $\tau_{AA'A''}$, isomorphic to $(\rho_A, \mathcal{E}_{A'A''|A}^r)$, would have to be such that it can be transformed with nonzero probability of success into a state that has pure, entangled reduced states on both AA' and AA'' by local operations.

Proof. For the states σ_1, σ_2, use the decomposition given in Eq. (44) and define β as before. Now consider the ensemble average density operator $\rho_A = p\sigma_1 + (1-p)\sigma_2$. This also has a decomposition of the form of Eq. (43):

$$\rho_A = \sum_\alpha q_\alpha \mu_{\alpha_1} \otimes \nu_{\alpha_2}, \tag{45}$$

where $q_\alpha = \mathrm{Tr}[pq_\alpha^{(1)}\mu_{\alpha_1}^{(1)} + (1-p)\mu_{\alpha_1}^{(2)}]$, and if $q_\alpha \neq 0$, then $\mu_{\alpha_1} = [pq_\alpha^{(1)}\mu_{\alpha_1}^{(1)} + (1-p)\mu_{\alpha_1}^{(2)}]/q_\alpha$. Now, μ_{β_1} must be of rank ≥ 2 because $[\mu_{\beta_1}^{(1)}, \mu_{\beta_1}^{(2)}] \neq 0$.

Let P_α be the projection operator onto $\mathcal{H}_{\alpha_1} \otimes \mathcal{H}_{\alpha_2}$. The set of P_α for all values of α is a POVM (in fact it is a projector-valued measure) that commutes with ρ_A.

Now consider the pair $(\rho_A, \mathcal{E}_{A'|A}^r)$ and construct the isomorphic state $\tau_{AA'}$. To do this, a basis must be chosen to define the state $|\Phi^+\rangle$ used to construct the isomorphism. Choose an eigenbasis of ρ_A to make use of the facts noted in Sec. IV D.

Recall that the state obtained from performing a P_α-measurement system A when the state is $\tau_{AA'}$ can be determined by applying the isomorphism to the pair $(P_\alpha \rho_A P_\alpha, \mathcal{E}_{A'|A}^r)$, where the T is omitted because $P_\alpha^T = P_\alpha$ and the square root is omitted because P_α is idempotent. Suppose the outcome β is obtained, which happens with nonzero probability of success. Then the updated state after the measurement is

$$P_\beta \rho_A P_\beta = \mu_{\beta_1} \otimes \nu_{\beta_2}. \tag{46}$$

Since the map $\mathcal{E}_{A'|A}$ acts as the identity on the factor $\mathcal{H}_{\beta_1}$, the isomorphic state on AA' is of the form $|\psi\rangle\langle\psi| \otimes \eta$, where $|\psi\rangle$ is a pure state on $\mathcal{H}_{\beta_1} \otimes \mathcal{H}_{\beta_1}$ and η is a state on $\mathcal{H}_{\beta_2} \otimes \mathcal{H}_{\beta_2}$. One copy of $\mathcal{H}_{\beta_1}$ and $\mathcal{H}_{\beta_2}$ belongs to system A and the other belongs to system A'. The state $|\psi\rangle$ is entangled, since ρ_{β_1} is of rank ≥ 2 and the rank of ρ_{β_1} is the number of Schmidt coefficients of $|\psi\rangle$.

Thus, starting from the state $\tau_{AA'}$, a pure entangled state can be obtained with nonzero probability of success by performing a P_α measurement on system A and discarding the two copies of the subsystem $\mathcal{H}_{\beta_2}$ if the β outcome is obtained. The same argument applies to $\tau_{AA''}$, the state isomorphic to $(\rho_A, \mathcal{E}^r_{A''|A})$, so we have the desired result. ∎

4. Ensemble cloning

For pure state ensemble cloning, a stronger result is possible, which removes the need to perform a measurement on system A.

Theorem V.8. Suppose there existed a cloning map $\mathcal{E}_{A'A''|A}$ for an ensemble of ≥ 2 pairwise nonorthogonal and nonidentical, pure states $\{(p_j, |\psi_j\rangle)\}$, $0 < p_j < 1$, $\Sigma_j p_j = 1$. Let $\rho_A = \Sigma_j p_j |\psi_j\rangle\langle\psi_j|$. The tripartite state $\rho_{AA'A''}$ isomorphic to $(\rho_A, \mathcal{E}^r_{A'A''|A})$ would have to be such that both reduced states $\rho_{AA'}$, $\rho_{AA'}$ are pure and entangled.

Proof. Each state $|\psi_j\rangle\langle\psi_j|$ is in the fixed-point set of both the reduced maps $\mathcal{E}_{A'|A}$ and $\mathcal{E}_{A''|A}$, and the common fixed-point set is of the form of Eq. (43). Therefore, each state $|\psi_j\rangle$ must be of the form $|\psi_j\rangle = |\phi_j\rangle_{\alpha_1} \otimes |\theta_j\rangle_{\alpha_2}$, where $|\phi_j\rangle_{\alpha_1}$ is a state on a factor $\mathcal{H}_{\alpha_1}$ and $|\theta_j\rangle_{\alpha_2}$ is a fixed state on a factor $\mathcal{H}_{\alpha_2}$. In fact, the factor α must be the same for all the states $|\psi_j\rangle$, since otherwise they would be orthogonal. That means that $|\theta_j\rangle_{\alpha_2}$ must be the same state $|\theta\rangle_{\alpha_2}$ for all j and that the $|\phi_j\rangle_{\alpha_1}$'s are nonorthogonal and nonidentical. The state ρ_A can then be written as $\rho_A = \Sigma_j \mu_{\alpha_1} \otimes |\theta\rangle_{\alpha_2}\langle\theta|_{\alpha_2}$, where $\mu_{\alpha_1} = \Sigma_j p_j |\phi_j\rangle\langle\phi_j|_{\alpha_1}$. Note that μ_{α_1} is of rank ≥ 2 and the map $\mathcal{E}_{A'|A}$ acts as the identity on $\mathcal{H}_{\alpha_1}$ and on the state $|\theta\rangle_{\alpha_2}$.

The isomorphic state $\rho_{AA'}$ is therefore of the form

$$|\xi\rangle\langle\xi| \otimes |\theta\rangle\langle\theta|_{\alpha_2} \otimes |\theta\rangle\langle\theta|_{\alpha_2} \qquad (47)$$

where $|\xi\rangle$ is an entangled pure state on $\mathcal{H}_{\alpha_1} \otimes \mathcal{H}_{\alpha_1}$. One copy of $\mathcal{H}_{\alpha_1}$ belongs to the subsystem A and the other to A', so the state is both pure and entangled. The same argument applies for the other reduced state $\rho_{AA''}$. ∎

VI. DISCUSSION

In this paper, an alternative variant of the Jamiołkowski isomorphism was derived and used to demonstrate the connection between the no-cloning and no-broadcasting theorems and the monogamy of entanglement. It is likely that our isomorphism can be applied in a variety of other parts of quantum information theory, whenever the action of a TPCP map on a particular ensemble of states is of interest, rather than its action on the entire Hilbert space. For example, this occurs in prepare-and-measure quantum key distribution schemes [33,34].

A possible future project would be to derive bounds on the maximum obtainable fidelity in approximate ensemble broadcasting from the known inequalities for the monogamy of entanglement [25–29]. It seems plausible that the closer the bipartite reduced states can be made to the ones obtained from the isomorphism, the better the fidelity of the broadcast copies would be. Fewer results are known about approximate broadcasting for mixed states [35–39] than for approximate cloning of pure states, so this could be a fruitful route to pursue. The main difficulty is that the entanglement measures used in monogamy inequalities are typically not related to fidelity in a straightforward way.

From a more foundational point of view, we have shown that the map $\mathcal{E}^r_{B|A}$ mimics the behavior of classical conditional probability very closely. The alternative definition of quantum conditional probability proposed by Cerf and Adami [19–21] shares a different set of properties with its classical counterpart, particularly the role of conditional probability in the definition of conditional entropy. One might ask whether there exists a unified notion of quantum conditional probability that shares all these properties or whether certain properties of conditional probability are mutually exclusive when raised to the quantum domain. The answer to this question could be of practical use, since there are several classical probabilistic structures that are usually defined in terms of conditional probabilities, such as Markov chains and Bayesian networks [40]. These might have more than one quantum generalization if the quantum analog of conditional probability is not unique.

More speculatively, the analogy to conditional probability offers some hope that a formalism for an abstract quantum probability without any background causal structures might be obtainable, perhaps within the framework recently proposed by Hardy [41]. One might hope that such a theory would give insights into how to apply quantum theory to cases in which the background causal structure is unknown *a priori*, as in quantum gravity.

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