Reliable entanglement transfer between pure quantum states

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The problem of the reliable transfer of entanglement from one pure bipartite quantum state to another using local operations is analyzed. It is shown that in the case of qubits the amount that can be transferred is restricted to the difference between the entanglement of the two states. In the presence of a catalytic state the range of the transferable amount broadens to a certain degree.

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I. INTRODUCTION

One of the most important recent achievements of quantum information theory is that of establishing the paradigm of bipartite entanglement as an asymptotically fungible resource [1]. In the asymptotic limit there are no theoretical restrictions on entanglement manipulation: bipartite entanglement can be redistributed reliably (i.e., without losses) as desired. It can be, for example, concentrated in a small number of states or diluted into a larger number of states.

In real situations, however, we always deal with a finite number of entangled states, and it is of practical importance to know what kinds of manipulations of entanglement of a finite number of states are permitted. The finite number scenario puts severe limitations on the efficiency of entanglement manipulations. Most of the protocols are accompanied by inevitable entanglement loss.

In this paper I address the question of whether any nontrivial reliable manipulations of entanglement of a finite number of states are possible. In particular, I analyze the simplest case where parties share two pure entangled states and want to transfer a fraction of entanglement from one of them to another.

Let us imagine the following situation. Alice and Bob, who live very far from each other, share a pure entangled state $|\psi\rangle_{AB}$, where A and B are Alice's and Bob's quantum particles, respectively. Alice and Bob have two friends, Alex and Barbara, who also share a pure entangled state $|\phi\rangle_{ab}$ between them. Alex lives in the same city with Alice, so any joint task carried out by Alice and Alex can be regarded as local. Similar rules apply to Bob and Barbara. The problem is formulated as follows: Is it possible to design a LOCC (local operations and classical communication) protocol, which will transfer an amount of entanglement ΔE from the "donor" state $|\psi\rangle_{AB}$ [thereby reducing its entanglement to $E(\psi)-\Delta E$] to the "acceptor" state $|\phi\rangle_{ab}$ [thereby increasing its entanglement to $E(\phi)+\Delta E$]? And if yes, what are the conditions for such a transformation? (Note that both $|\psi\rangle$ and $|\phi\rangle$ are required to remain pure.)

The entanglement transfer scenario described above is relevant for many tasks in quantum information. Recently, it has been shown that the successful implementation of some nonlocal operations, such as nonlocal POVM measurements [2], requires entangled states that possess a particular (nonmaximal) amount of entanglement; a "catalyst" state, needed to make some entanglement transformations possible, must be nonmaximally entangled [3]. In the two above examples, if we are given the state more entangled than required, then we obviously will have to reduce it. It is always possible to reduce the amount of entanglement by losing part of it. However, as entanglement is an expensive resource, we might prefer to transfer the redundant part to another system for future use. An additional example is the entangling capacity of nonlocal Hamiltonians and nonlocal unitaries. It was shown that the maximal rate of entanglement creation is achieved when a nonlocal Hamiltonian or a unitary acts on qubits that are partially entangled [4,5]. Thus, in order to maintain the maximal rate of entanglement production, one would like to be able to "transfer" the generated gain in entanglement ("surplus value") to a different system after each application in order to keep the target state in its optimal form.

Clearly, there is a situation when the entanglement transfer is possible. Indeed, Alice and Bob can locally swap the states of A,a and B,b respectively, thereby transforming the total initial state $|\psi\rangle_{AB}\otimes|\phi\rangle_{ab}$ into $|\phi\rangle_{AB}\otimes|\psi\rangle_{ab}$, and transferring the amount of entanglement $\Delta E=E(\psi)-E(\phi)$. This trivial protocol is not really helpful, though, because it restricts the state $|\phi\rangle_{ab}$ to that which we want to obtain in the first place. A nontrivial and interesting situation occurs when the desired state is not possessed initially either by Alice and Bob or by Alex and Barbara, and when ΔE is not determined by the initial states.

On the other hand, it is clear that there are situations when entanglement transfer is impossible. For example, let us assume that both donor and acceptor states possess the same amount of entanglement equal to 0.5 ebit. If our team was able to transfer all the amount of entanglement of the donor state to the acceptor state (thus doubling the entanglement of the latter), then it would essentially mean that they reliably implemented the entanglement concentration in the two-copy scenario. Imagine that there are $n\geqslant 1$ such pairs of states, and the above hypothetic protocol is implemented on each pair of states separately. $2n$ nonmaximally entangled states will be concentrated into n maximally entangled states. Such a procedure would not only achieve the result of the collective entanglement concentration method [1] by acting on the states individually, but would even outperform it as no losses, even sublinear, will take place. Although it might be possible to use such *reductio ad absurdum* arguments based on asymptotic scenarios to deduce that certain single-copy transformations are impossible, I will not base my argument on the asymptotic case at all. Instead, I will use only results and theorems for a single-copy, making the analysis logically self-sufficient. I believe that this approach will give interesting and fundamental insight into the nature of entanglement of the final number of states.

The structure of the article is as follows. In Sec. II I will analyze the case of a disentangled acceptor state for the quantum system of any finite dimensionality. In Sec. III the case of an entangled acceptor for qubits is analyzed. In Sec. IV the possibility of catalytic transformation is taken into account. Section V demonstrates how all restrictions might be relaxed if a probabilistic transformation is allowed. Finally, Sec. VI shows that all restrictions are overcome in the asymptotic limit.

II. ENTANGLEMENT TRANSFER TO A DIRECT-PRODUCT STATE

Let us start with considering the case when $|\phi\rangle_{ab}$ is a direct product, i.e., $E(\phi_{ab})=0$. The results of this section can be applied to quantum systems of any finite dimensionality.

Proposition II.1. Given a single copy of a bipartite pure entangled state $|\psi\rangle$, it is impossible to transfer part of the amount of entanglement possessed by $|\psi\rangle$ to different quantum systems, which are initially disentangled, by means of LOCC without changing the Schmidt number of $|\psi\rangle$.

Proof. As a consequence of the *majorization condition* [6], the Schmidt number of a quantum state cannot be increased by LOCC. The hypothetical transformation under question leads to the inevitable increase of the Schmidt number, and therefore is forbidden. Indeed, before the transformation, the total Schmidt number is equal to the Schmidt number of $|\psi\rangle$. If the Schmidt number of $|\psi\rangle$ does not change, then after the transformation the total Schmidt number equals the Schmidt number of $|\psi\rangle$ times the Schmidt number of $|\phi\rangle$ (the state which the entanglement was transferred to). ■

Corollary. For two-qubit and two-qutrit states the entanglement can only be transferred in full because a two-qubit entangled state can only have the Schmidt number 2, while the next number below is 1 for product states. For qutrit states the maximal Schmidt number of 3 can also be reduced only to 1 (not to 2). The task can be trivially accomplished simply by two local SWAP operations.

The corollary and Proposition II.1 also apply to situations when we do allow some amount of entanglement to be lost during the transfer. The results of this section are consistent with the approach taken in the broadcasting of entanglement [7] (see a more detailed discussion in Sec. VII). Indeed, here it has been shown that the entanglement of a single pure state cannot be split into two less-entangled pure states. The only open possibility is that the states involved are mixed—exactly the case that was analyzed in Ref. [7].

Although the results of the next section are more general and include direct product acceptor states as a special case, Proposition II.1 stands as an important result on its own. The arguments used in the proof are simpler than those in Sec. III. Besides, in Sec. III we assume reliable protocols for qubits, while Proposition II.1 is valid for quantum systems of any dimensionality and for a more general scenario when we do allow entanglement losses.

III. ENTANGLEMENT TRANSFER TO AN ENTANGLED STATE

In this section I will analyze the general case of the entangled acceptor state $|\phi\rangle_{ab}$ for qubits. Let me write the donor state in its Schmidt decomposition as

$$|\psi_\beta\rangle_{AB} = \cos\beta|\mu\rangle_A|\nu\rangle_B + \sin\beta|\mu^\perp\rangle_A|\nu^\perp\rangle_B, \qquad (1)$$

where it is assumed that all phases are absorbed by local basis states $|\mu^\perp\rangle_A$ and $|\nu^\perp\rangle_B$. These phases, as well as actual local basis states, are not important as we will be interested below only in the values of the Schmidt coefficients. Similarly, I can write the acceptor state (of the qubits a and b) as

$$|\phi_\alpha\rangle_{ab} = \cos\alpha|\xi\rangle_a|\eta\rangle_b + \sin\alpha|\xi^\perp\rangle_a|\eta^\perp\rangle_b. \qquad (2)$$

We denote the amounts of entanglement possessed by $|\psi_\beta\rangle_{AB}$ and $|\phi_\beta\rangle_{ab}$ as $E(\psi_\beta)$ and $E(\phi_\alpha)$.

Without loss of generality, let me assume that $0\leqslant\alpha\leqslant\pi/4$, $0\leqslant\beta\leqslant\pi/4$ and denote the (decreasingly ordered) Schmidt coefficients of the donor state and the acceptor state by $\{c_\beta^2,s_\beta^2\}$ and $\{c_\alpha^2,s_\alpha^2\}$, respectively [8]. The reduction of entanglement of $|\psi_\beta\rangle$ by ΔE corresponds to a reduction of β by $\Delta\beta$. Subsequently, the increase of entanglement of $|\phi_\alpha\rangle$ by the same amount ΔE corresponds to an increase of the angle α by $\Delta\alpha$. Note that in general $\Delta\alpha\neq\Delta\beta$. Here we use the entropy of entanglement as an entanglement measure, thus $\Delta\alpha$ and $\Delta\beta$ are related by the formula

$$H[c_\beta^2] + H[c_\alpha^2] = H[c_{\beta-\Delta\beta}^2] + H[c_{\alpha+\Delta\alpha}^2], \qquad (3)$$

where $H[x]=-x\log_2 x-(1-x)\log_2(1-x)$ is the (Shannon) entropy of the probability distribution $\{x,1-x\}$.

Thus, there are three free parameters in the problem. I will fix β and $\Delta\beta$ and investigate which values of α are possible. As we want to avoid reducing the entanglement of $|\phi_\alpha\rangle$, the relevant range of α is $0\leqslant\alpha\leqslant(\pi/4-\Delta\alpha)$. $\Delta\alpha$ enters the problem as an implicit function of α, which is determined by Eq. (3).

In order to analyze the possibility of such a transformation, we employ the majorization condition [6], which states that the transformation is possible iff the ordered Schmidt coefficients (of the combined four-qubit system) before the transformation $\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}$ and after the transformation $\{\lambda_1',\lambda_2',\lambda_3',\lambda_4'\}$ satisfy the following three inequalities:

$$\lambda_1' \geqslant \lambda_1, \qquad (4)$$

$$\lambda_1'+\lambda_2' \geqslant \lambda_1+\lambda_2, \qquad (5)$$

$$\lambda_1'+\lambda_2'+\lambda_3' \geqslant \lambda_1+\lambda_2+\lambda_3, \qquad (6)$$

where the last inequality can be rewritten as $\lambda_4'\leqslant\lambda_4$.

We can write the Schmidt coefficients in terms of the parameters of the problem *up to the ordering of the second and the third coefficients*, which depends on the particular values of α, β, $\Delta\alpha$, and $\Delta\beta$. They are

$$\{\lambda_i\} = \{(c_\alpha c_\beta)^2, (c_\alpha s_\beta)^2, (s_\alpha c_\beta)^2, (s_\alpha s_\beta)^2\}, \tag{7}$$

$$\{\lambda_i'\} = \{(c_{\alpha+\Delta\alpha} c_{\beta-\Delta\beta})^2, (c_{\alpha+\Delta\alpha} s_{\beta-\Delta\beta})^2, (s_{\alpha+\Delta\alpha} c_{\beta-\Delta\beta})^2, (s_{\alpha+\Delta\alpha} s_{\beta-\Delta\beta})^2\}. \tag{8}$$

It will be more convenient to use the following inequalities that are equivalent to Eqs. (4)–(6):

$$f_1 \equiv \sqrt{\lambda_1'} - \sqrt{\lambda_1} \geqslant 0, \tag{9}$$

$$f_2 \equiv \sqrt{\lambda_1' + \lambda_2'} - \sqrt{\lambda_1 + \lambda_2} \geqslant 0, \tag{10}$$

$$f_3 \equiv \sqrt{\lambda_4} - \sqrt{\lambda_4'} \geqslant 0, \tag{11}$$

which will present no problem since all trigonometric functions used in Eqs. (7) and (8) are positive in the relevant range of parameters.

In the following paragraphs it will be shown that for any value of $\alpha \in [0, \pi/4 - \Delta\alpha]$ at least one of the functions f_1, f_2, or f_3 is negative, which will be sufficient for us to conclude that the transformation is impossible. The only exception is the point $\alpha_* = \beta - \Delta\beta$, which is the simultaneous solution of $f_1 = 0$, $f_2 = 0$, and $f_3 = 0$. Only for $\alpha = \alpha_*$ is the transformation possible.

Using Eqs. (7) and (8), we can write f_1 and f_3 unambiguously without any additional assumptions regarding the values of the parameters

$$f_1 = c_{\alpha+\Delta\alpha} c_{\beta-\Delta\beta} - c_\alpha c_\beta, \tag{12}$$

$$f_3 = s_\alpha s_\beta - s_{\alpha+\Delta\alpha} s_{\beta-\Delta\beta}. \tag{13}$$

There is an ambiguity regarding f_2 though. Depending on the ordering of the actual Schmidt coefficients, the three following regimes are obtained for f_2;

$$f_2 = \begin{cases} c_{\alpha+\Delta\alpha} - c_\alpha & \text{if } \alpha < \beta - \Delta\alpha - \Delta\beta, \\ c_{\beta-\Delta\beta} - c_\alpha & \text{if } \beta - \Delta\alpha - \Delta\beta \leqslant \alpha \leqslant \beta, \\ c_{\beta-\Delta\beta} - c_\beta & \text{if } \alpha > \beta. \end{cases} \tag{14}$$

In the first regime $\alpha < \beta - \Delta\alpha - \Delta\beta$, the function f_2 is obviously negative ($\Delta\alpha > 0$), while in the third regime $\alpha > \beta$, f_2 is constant and positive. (Different types of typical behavior of f_1, f_2, and f_3 are depicted in Fig. 1 for illustration.)

Let us take a closer look at the second regime $\beta - \Delta\alpha - \Delta\beta \leqslant \alpha \leqslant \beta$. First, solving $f_2 = c_{\beta-\Delta\beta} - c_\alpha = 0$ for α gives $\alpha_* = \beta - \Delta\beta$. Then, Eq. (3) immediately implies that $\Delta\alpha = \Delta\beta$, and therefore $f_1 = f_3 = 0$, i.e., α_* is indeed the point where all three functions simultaneously cross the α axis. It is straightforward to see that f_2 is negative when $\beta - \Delta\alpha - \Delta\beta \leqslant \alpha < \alpha_*$ and positive when $\alpha_* < \alpha \leqslant \beta$.

Unlike f_2, it is not so easy to show when $f_1 < 0$. The main reason is that f_2 is expressed in terms of three parameters at most. f_1, however, involves all four parameters. Although only three of them are free parameters, they are related by the implicit equation (3) and there is no simple analytic way to express one of them, say $\Delta\alpha$, in terms of the others. Thus, the negativity of f_1 cannot be demonstrated by the simple substitution of $\Delta\alpha$ into Eq. (12). Therefore, I will tackle the problem in a different way. I will show that the first derivative of f_1, with respect to α, is negative in the whole interval $0 \leqslant \alpha \leqslant (\pi/4 - \Delta\alpha)$, i.e., f_1 is strictly decreasing. This result will lead me to the following conclusions: (a) The fact that the first derivative of the continuous function f_1 does not change the sign is sufficient to conclude that no other roots of $f_1 = 0$, except α_*, exist in the interval, i.e., f_1 crosses the α axis only at α_* [9], (b) f_1 is positive below α_* and negative above α_*. Now, let us find out the sign of the first derivative of f_1,

$$\frac{df_1}{d\alpha} = s_\alpha c_\beta - s_{\alpha+\Delta\alpha} c_{\beta-\Delta\beta}\left(1 + \frac{d\Delta\alpha}{d\alpha}\right). \tag{15}$$

From Eq. (3) we obtain a relation for $d\Delta\alpha/d\alpha$. The differentiation of Eq. (3) in respect to α gives

$$s_{2\alpha} \ln \tan \alpha = s_{2(\alpha+\Delta\alpha)} \ln \tan(\alpha + \Delta\alpha)\left(1 + \frac{d\Delta\alpha}{d\alpha}\right). \tag{16}$$

Now let us substitute Eq. (16) into Eq. (15),

$$\frac{df_1}{d\alpha} = s_\alpha\left[c_\beta - c_{\beta-\Delta\beta} \frac{c_\alpha}{c_{\alpha+\Delta\alpha}} \frac{\ln \tan \alpha}{\ln \tan(\alpha + \Delta\alpha)}\right]. \tag{17}$$

The second term in square brackets is a factor of three products. This term is larger than the first term, c_β, because the first factor, $c_{\beta-\Delta\beta}$, is larger than c_β, while the other two factors are larger than 1. That implies that $df_1/d\alpha < 0$.

To summarize, we have shown that for $0 \leqslant \alpha < \alpha_*$ the function f_2 is negative, while for $\alpha_* < \alpha \leqslant (\pi/4 - \Delta\alpha)$ the function f_1 is negative, and α_* is the only point where all three inequalities of the majorization condition are satisfied. This value corresponds to the situation of state swapping described in Sec. I. Indeed, Eq. (3) implies that $\Delta\alpha = \Delta\beta$, i.e., $\Delta E = E(\psi_\beta) - E(\phi_\alpha)$.

Note that we did not analyze here the sign of f_3 analytically. The signs of f_1 and f_2 were sufficient to prove the main result. The typical behavior of f_3 can be seen, though, from the numerical simulations presented in Fig. 1, and will be discussed in more detail in the next section.

IV. ENTANGLEMENT TRANSFER WITH CATALYSIS

Some transformations that are impossible under LOCC become possible in the presence of a *catalytic* state [3]. In this section I address the question of whether catalysis can help in our case.

It was proved that catalysis can help only if the initial total state $|\psi_\beta\rangle_{AB} \otimes |\phi_\alpha\rangle_{ab}$ and the final total state $|\psi_{\beta-\Delta\beta}\rangle_{AB} \otimes |\phi_{\alpha+\Delta\alpha}\rangle_{ab}$ are *incomparable* [3]. For a 4×4-level system the necessary conditions for the possibility of catalytic transformation are

$$f_1 \geqslant 0, \quad f_2 < 0, \quad f_3 \geqslant 0. \tag{18}$$

From the results of the previous section it follows that the first two conditions are not satisfied if $\alpha > \alpha_*$.

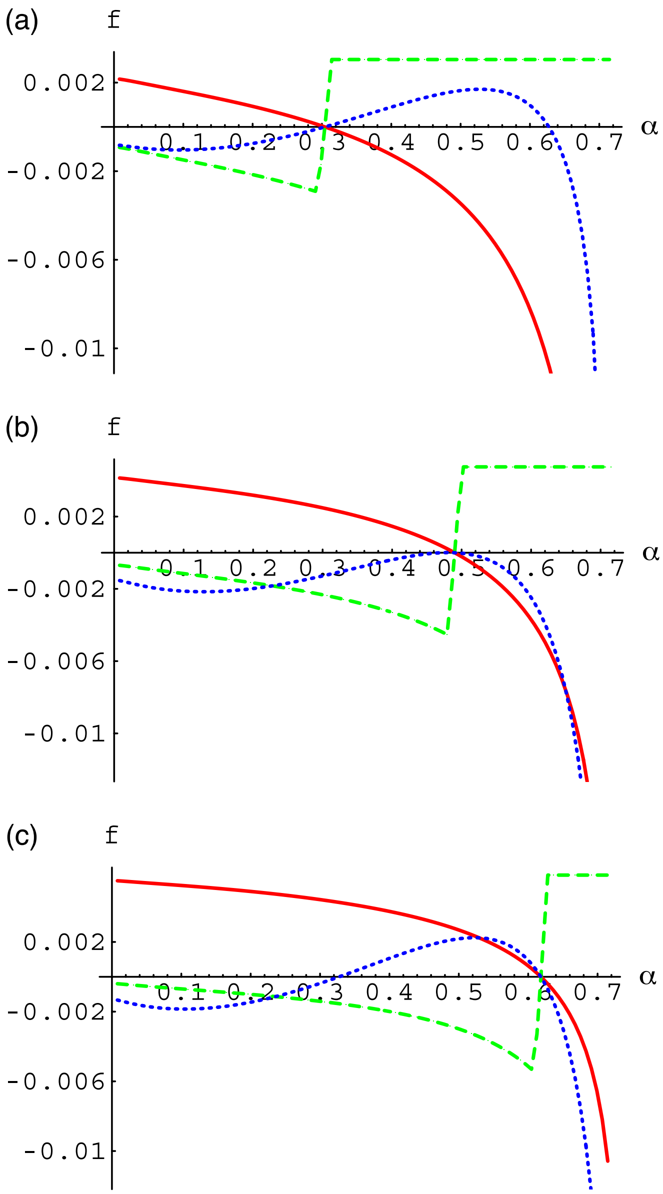

FIG. 1. (Color online) f_1 (solid line), f_2 (dashed line), and $f_3(\times 10)$ (dotted line) as functions of α for $\Delta\beta$=0.01: (a) $\beta=\pi/10$, (b) β=0.5, (c) $\beta=\pi/5$. The point α_* and the three different regimes of f_2 are clearly visible. Here and in all the following figures α, $\Delta\alpha$, β, and $\Delta\beta$ are measured in radians.

When $\alpha<\alpha_*$, however, the first two conditions in Eq. (18) are satisfied and the possibility of catalytic transformation depends on the sign of f_3. As we see from Fig. 1, f_3 can take positive values in some cases. The analytic analysis of the sign of f_3 would be more difficult than that of f_1 and f_2. I will combine numerical and analytical techniques instead. As we can see from Fig. 1, f_3 takes positive values at $\alpha<\alpha_*$ only if β is larger than a certain value. This critical β_c corresponds to the point where two roots of f_3=0 are degenerate [Fig. 1(b)]. We notice that for $\beta=\beta_c$, the derivative of f_3 in respect to α is zero at $\alpha=\alpha_*$. We will use this fact to deduce the value of β_c;

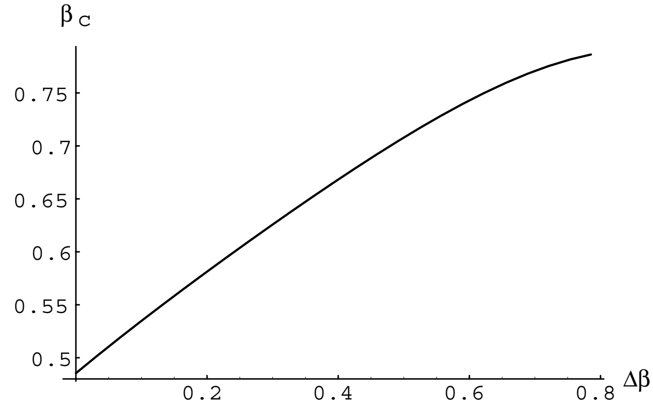

FIG. 2. β_c as functions of $\Delta\beta$.

$$\frac{df_3}{d\alpha}=c_\alpha s_\beta - c_{\alpha+\Delta\alpha}s_{\beta-\Delta\beta}\left(1+\frac{d\Delta\alpha}{d\alpha}\right). \tag{19}$$

Substituting Eq. (16) into Eq. (19), we obtain

$$\frac{df_3}{d\alpha}=c_\alpha\left(s_\beta-\frac{s_\alpha s_{\beta-\Delta\beta}}{s_{\alpha+\Delta\alpha}}\frac{\ln\tan\alpha}{\ln\tan(\alpha+\Delta\alpha)}\right). \tag{20}$$

Therefore, at $\alpha=\alpha_*$ and $\beta=\beta_c$ we get

$$\left(\frac{s_{\beta_c}}{s_{\beta_c-\Delta\beta}}\right)^2=\frac{\ln\tan(\beta_c-\Delta\beta)}{\ln\tan\beta_c}. \tag{21}$$

For a given $\Delta\beta$ we solve this equation numerically. For $\Delta\beta$=0.01 used previously in numerical examples (Fig. 1), β_c=0.49055. Figure 2 shows β_c as a function of $\Delta\beta$. β_c approaches 0.48557 as $\Delta\beta\to 0$.

Thus, only for $\beta>\beta_c$ the catalytic transformation is possible. The range of α, which allows this, is confined to the interval between two roots of f_3=0. The first (larger) root, as we have shown analytically, is α_*. The second (smaller) root can be obtained by solving simultaneous equations f_3=0 and Eq. (3) numerically. For example, in the case presented in Fig. 1(c), the range of allowed α is [0.3274, $\pi/5-0.01$]. Figure 3 presents both roots as a function of β for $\beta>\beta_c$ and four different values of $\Delta\beta$. We see that for a given $\Delta\beta$ the range of allowed α broadens towards larger β. The situation improves as $\Delta\beta$ decreases, and for very small $\Delta\beta$ the catalytic transformation becomes possible for essentially all values of α as β becomes close to $\pi/4$ (i.e., $|\psi_\beta\rangle$ is a nearly maximally entangled state).

V. PROBABILISTIC ENTANGLEMENT TRANSFER

So far we have seen that a reliable entanglement transfer is very restricted. To complete our analysis it is worth mentioning how the situation might be improved if we allow the transfer to be accomplished with some probability of success less than 1. How close to 1 can we get? To this end we use the extension of the majorization condition to the probabilistic transformations of a single copy [10,11], which in our case implies that the maximum probability of successful transformation is

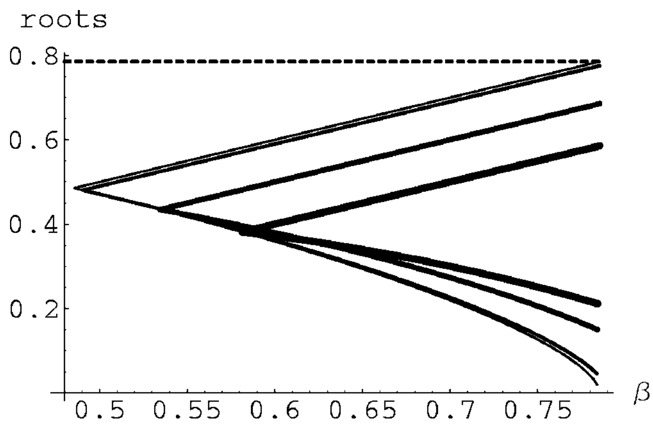

FIG. 3. Two roots of $f_3=0$ as functions of β for $\Delta\beta=0.2, 0.1, 0.01, 0.001$ (the thinner line corresponds to smaller $\Delta\beta$) and $\beta>\beta_c$. The upper line in each pair corresponds to α_*. The lower line corresponds to the second root. All α that lie between these two roots are allowed in the presence of the catalytic state. The dashed line corresponds to $\alpha=\pi/4$.

$$p_{max}=\min\left\{\frac{1-\lambda_1}{1-\lambda_1'},\frac{1-\lambda_1-\lambda_2}{1-\lambda_1'-\lambda_2'},\frac{\lambda_4}{\lambda_4'}\right\}. \tag{22}$$

As an example, let us consider the combination of parameters described in Fig. 1(a). It can be easily checked that $p_{max}=\frac{1-\lambda_1}{1-\lambda_1'}$ for $\alpha>\alpha_*$, while $p_{max}=\frac{1-\lambda_1-\lambda_2}{1-\lambda_1'-\lambda_2'}$ for $\alpha<\alpha_*$. The p_{max} vs α dependence is depicted in Fig. 4. The probability of failure that must be accepted in order to allow the entanglement transfer is finite and increases with the distance between α and α_*.

VI. ASYMPTOTIC CASE

Not surprisingly, all limitations described in the previous sections disappear in the asymptotic limit. We assume that Alice and Bob possess $n\to\infty$ copies of $|\psi_\beta\rangle$ and Alex and Barbara possess n copies of $|\phi_\alpha\rangle$.

Clearly, Alice and Bob are able to obtain n copies of $|\psi_{\beta-\Delta\beta}\rangle$ without any help from Alex and Barbara simply by using the asymptotic entanglement concentration (distillation) method [1]: first they concentrate $|\psi_\beta\rangle^{\otimes n}$ into $nH(c_\beta^2)$ singlets and then distill them into $|\psi_{\beta-\Delta\beta}\rangle^{\otimes nH(c_\beta^2)/H(c_{\beta-\Delta\beta}^2)}$, thus apart from required n copies they obtain $n[H(c_\beta^2)/H(c_{\beta-\Delta\beta}^2)-1]$ additional copies of $|\psi_{\beta-\Delta\beta}\rangle$.

Alex and Barbara are able now to absorb the entanglement of these additional copies into their states. First, they distill (or concentrate, depending on the value of α) $|\phi_\alpha\rangle^{\otimes n}$ into $|\phi_{\beta-\Delta\beta}\rangle^{\otimes nH(c_\alpha^2)/H(c_{\beta-\Delta\beta}^2)}$. Now, acting collectively on these copies together with $n[H(c_\beta^2)/H(c_{\beta-\Delta\beta}^2)-1]$ copies of $|\psi_{\beta-\Delta\beta}\rangle$, they concentrate them into n copies of $|\phi_{\alpha+\Delta\alpha}\rangle$, where $\Delta\alpha$ satisfies Eq. (3). Thus, we obtain $(|\psi_{\beta-\Delta\beta}\rangle_{AB}\otimes|\phi_{\alpha+\Delta\alpha}\rangle_{ab})^{\otimes n}$ as desired.

This procedure is a clear demonstration that there are no restrictions on the redistribution of entanglement in the asymptotic limit.

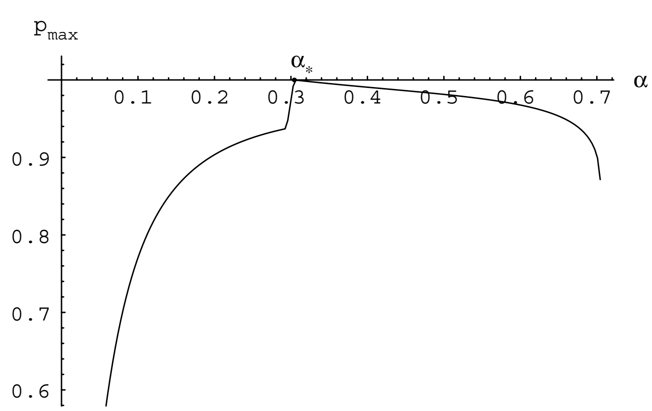

FIG. 4. p_{max} vs α dependence for $\Delta\beta=0.01$ and $\beta=\pi/10$. p_{max} reaches 1 at $\alpha=\alpha_*$.

VII. CONCLUSION

In this paper I have analyzed the question of reliable entanglement transfer between two bipartite pure states, which I call the donor and the acceptor states. The case of a disentangled acceptor state was considered for systems of any dimensionality. It was shown that no partial entanglement can be transferred if the Schmidt number of the donor state does not change.

In the case of qubit states it has been shown that the amount of entanglement allowed to be transferred reliably to an entangled acceptor state is very restricted. Without the presence of a catalytic state the transfer is possible only when the entanglement of the acceptor state $E(\phi)$ is smaller than the entanglement of the donor state $E(\psi)$. The amount that can be transferred is just the difference between the two, $\Delta E=E(\psi)-E(\phi)$. The task is accomplished by swapping the states, which can be done locally. In all other cases, the transfer is impossible.

In the presence of a catalytic state the above restrictions are relaxed to a certain degree. Transfer might be possible subject to the following conditions. The first condition is $E(\phi)\leqslant E(\psi)-\Delta E$. Note that this essentially implies that no entanglement can be transferred to an acceptor state that is more entangled than the donor state. The second condition is, for a given ΔE the entanglement of the donor state has to be larger than a certain threshold $E(\psi_{\beta_c})=f(\Delta E)$. The third condition is, $E(\phi)$ has to fall into a certain range, which broadens as $E(\psi)$ increases. As $E(\psi)$ tends to maximum and ΔE is small, the catalytic transformation becomes possible for all values of $E(\phi)$. Thus, using catalysis it is always possible to "chop" a small piece of entanglement from a maximally entangled donor state and transfer it to an acceptor state (providing the acceptor state has "room" for this amount of entanglement, of course). Alternatively, the entanglement of a maximally entangled donor state can be transferred in full.

The above restrictions were derived under the requirement of a reliable transfer. The possibility of a probabilistic transfer also has been discussed, and it was shown that the probability of a successful transfer cannot be made arbitrarily close to 1. Reliability can also be sacrificed by allowing part of the transferred entanglement to be lost. A preliminary analysis shows that such losses are not negligible.

Entanglement transfer, addressed in this paper, should be compared with the broadcasting of entanglement [7] and entanglement splitting [12]. To say that entanglement has been (partially) broadcasted is to say that two less-entangled states have been obtained from one more-entangled state by local operations. To say that entanglement has been split is to say that the entanglement of a pure state has been split into two branches, i.e., the second party had "shared" her entanglement with a third party, so they are both now entangled with the first party. There are two main differences between my approach and those of Refs. [7,12]. First, I require that the resulting states remain pure. In Ref. [7] this requirement was relaxed, and the separability criterion for mixed states was used to analyze the entanglement of the resulting states. In fact, the results of Sec. II of this paper imply that entanglement cannot be broadcasted to pure states. In Ref. [12] the requirement of purity obviously cannot be applied to a single branch. Second, entanglement is broadcasted to initially disentangled particles (nonentangled acceptor state), whereas entanglement transfer, analyzed here, takes place when the acceptor particles are entangled. In this sense, entanglement transfer is more general.

I believe that the results of this paper have shed more light on the nature of entanglement of a finite number of pure states. It will be interesting to generalize this argument to a quantum system of higher dimensionality. There is also a potential unexplored relation between entanglement transfer and the broadcasting of entanglement and entanglement splitting.

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[1] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, Phys. Rev. A **53**, 2046 (1996).
[2] B. Reznik, Int. J. Quantum Inf. **4**, 99 (2006).
[3] D. Jonathan and M. B. Plenio, Phys. Rev. Lett. **83**, 3566 (1999).
[4] W. Dür, G. Vidal, J. I. Cirac, N. Linden, and S. Popescu, Phys. Rev. Lett. **87**, 137901 (2001).
[5] M. S. Leifer, L. Henderson, and N. Linden, Phys. Rev. A **67**, 012306 (2003).
[6] M. A. Nielsen, Phys. Rev. Lett. **83**, 436 (1999).
[7] V. Buzek, V. Vedral, M. B. Plenio, P. L. Knight, and M. Hillery, Phys. Rev. A **55**, 3327 (1997); S. Bandyopadhyay and G. Kar, *ibid.* **60**, 3296 (1999); S. Adhikari, B. S. Choudhury, and I. Chakrabarty, J. Phys. A **39**, 8439 (2006).
[8] Here and in the following I will use the notation $c_x \equiv \cos x$, $s_x \equiv \sin x$.
[9] Although this statement is not rigorously proven mathematically, its validity seems obvious.
[10] H.-K. Lo and S. Popescu, Phys. Rev. A **63**, 022301 (2001).
[11] G. Vidal, Phys. Rev. Lett. **83**, 1046 (1999).
[12] D. Bruß, Phys. Rev. A **60**, 4344 (1999).