Collective excitations of trapped Fermi or Bose gases

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A method is developed to calculate all excitations of trapped gases using hydrodynamics at zero temperature for any equation of state $\mu=\mu(n)$ and for any trapping potential. It is shown that a natural scalar product can be defined for the mode functions, by which the wave operator is Hermitian and the mode functions are orthogonal. It is also shown that the Kohn modes are exact for harmonic trapping in the hydrodynamic theory. Excitations for fermions are calculated in the Bardeen-Cooper-Schrieffer–Bose-Einstein condensation transition region using the equation of state of the mean-field Leggett model for isotropic harmonic trap potential.

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I. INTRODUCTION

Several experiments on trapped ultracold gases probed in the past decade the collective excitations of atomic gases. Earlier measurements on bosons [1,2] and more recent measurements on fermions [3,4] near Feshbach resonances can be explained rather satisfactorily using hydrodynamics at zero temperature. In his seminal paper [5] Stringari applied first hydrodynamics for trapped bosons undergoing Bose-Einstein condensation. His predictions were confirmed by experiments [1,2]. Later, using the same approach, he predicted [6] also the qualitative behavior of low-lying modes for fermions in the whole crossover region from a BCS-type superfluid Fermi gas to a molecular Bose-Einstein condensation (BEC) [7–9]. Now, several recent theoretical papers appeared in the literature [10–16] using hydrodynamic theory to better explain the measurements on the BCS-BEC transition. In general, no exact solution to the hydrodynamic equations are known, except when the equation of state has the polytropic form $\mu(n)\propto n^{\gamma}$ [11].

The hydrodynamic approach leads to a wave equation for the density oscillations. In principle, this wave equation can be solved for a single oscillating mode if the boundary conditions for the density oscillations are known. Bulgac *et al.* [13] has written the eigenvalue equation for a single mode in such a way that the two sides were Hermitian, but did not address the question of the function space to which all the excitations should belong. Here we use a different approach. For a general equation of state $\mu=\mu(n)$ it is usually very difficult to prescribe appropriate boundary conditions at the surface of the gas. There are a few examples where this problem is circumvented using some ansätze on the spatial forms of the excitations [12,15]. Here we shall introduce a natural scalar product, by which the wave operator itself is Hermitian and the mode functions are square integrable functions. The scalar product we shall use automatically ensures particle conservation. After calculating the matrix elements of the wave operator with the natural scalar product the eigenvalues give the squared frequencies of the excitations. We shall demonstrate the procedure for the mean-field model of the BCS-BEC transition for isotropic trap potentials. Finite-temperature effects [16,17] are briefly discussed in Sec. IV.

II. THEORY

In hydrodynamic theory for trapped gases at zero temperature, density oscillations are given by the continuity equation and the Euler equation

$$\frac{\partial n}{\partial t}+\nabla\cdot(n\mathbf{u})=0, \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}=-\frac{1}{mn}(\nabla P)-\frac{(\nabla V)}{m}, \tag{2}$$

where $\mathbf{u}$ is the velocity field, n is the density, t is the time, P is the pressure, m is the particle mass, and V is the external trapping potential.

Knowing the equation of state $\mu=\mu(n)$ at zero temperature in the corresponding homogeneous system the equilibrium density in the trapped case can be determined from the local chemical potential

$$\mu=\mu(\mathbf{r})\equiv\mu(n_0(\mathbf{r}))=\mu_0-V(\mathbf{r}), \tag{3}$$

where μ_0 is the overall constant chemical potential. For confining potentials the solution of this equation for positive $n_0(\mathbf{r})$ supplies an equilibrium density, which has a finite support with a well-defined boundary. Typically, n_0 decreases to zero by approaching the boundary. Using the thermodynamic identity

$$\frac{\partial\mu}{\partial n}=\frac{1}{n}\left(\frac{\partial P}{\partial n}\right), \tag{4}$$

valid also at $T=0$, the gradient of Eq. (3) gives

$$-A_0(\mathbf{r})\frac{\nabla n_0(\mathbf{r})}{n_0(\mathbf{r})}=\nabla V(\mathbf{r}),\quad A_0(\mathbf{r})\equiv\left(\frac{\partial P}{\partial n}\right)_{n=n_0(\mathbf{r})}. \tag{5}$$

At zero temperature $A_0(\mathbf{r})=mc(\mathbf{r})^2$, where $c(\mathbf{r})$ is the local speed of sound. In mechanical equilibrium, the equilibrium pressure P_0 must satisfy

$$\nabla P_0(\mathbf{r})=-n_0(\mathbf{r})\,\nabla\,V(\mathbf{r}), \tag{6}$$

otherwise the right-hand side of Eq. (2) will not vanish for $\mathbf{u}=0$ [Eq. (6) is the local form of the Archimedes law for a general external potential]. Close to equilibrium, $\mathbf{u}$ and

$\delta n(\mathbf{r},t)=n(\mathbf{r},t)-n_0(\mathbf{r})$ are small, and P can be expanded to first order in δn as

$$P(\mathbf{r},t) = P_0(\mathbf{r}) + \left(\frac{\partial P}{\partial n}\right)_0 \delta n(\mathbf{r},t). \tag{7}$$

Linearizing the continuity equation (1) and the Euler equation (2) in δn and $\mathbf{u}$ using Eqs. (5) and (6), the linearized hydrodynamic equations can be written as

$$\frac{\partial \delta n}{\partial t} + \boldsymbol{\nabla}\cdot(n_0\mathbf{u}) = 0, \tag{8}$$

$$\frac{\partial \mathbf{u}}{\partial t} = -\boldsymbol{\nabla}\left[\frac{A_0}{n_0 m}\delta n\right]. \tag{9}$$

Let us introduce a new field by

$$\Psi(\mathbf{r},t) = \sqrt{\frac{A_0(\mathbf{r})}{n_0(\mathbf{r})}}\delta n(\mathbf{r},t), \tag{10}$$

where Ψ has the same support as n_0, A_0, and δn. From now on, we allow complex field Ψ (which is more convenient for problems, where angular momentum is conserved). Eliminating $\mathbf{u}$ from Eqs. (8) and (9) gives a wave equation $\partial_t^2\Psi + \hat{G}_\Psi\Psi = 0$, where

$$\hat{G}_\Psi = -\sqrt{\frac{A_0(\mathbf{r})}{n_0(\mathbf{r})}}\cdot\boldsymbol{\nabla}\cdot\frac{n_0(\mathbf{r})}{m}\cdot\boldsymbol{\nabla}\cdot\sqrt{\frac{A_0(\mathbf{r})}{n_0(\mathbf{r})}}. \tag{11}$$

The main advantage of the field Ψ is that its wave operator $\hat{G}_\Psi$ is manifestly Hermitian [19] with respect to the scalar product

$$\langle\Psi_1|\Psi_2\rangle = \int_{n_0(\mathbf{r})>0} d^3r\Psi_1^*(\mathbf{r})\Psi_2(\mathbf{r}). \tag{12}$$

The scalar product (12) is trivially the correct one for a homogeneous system with periodic boundary conditions. It was used for a weakly interacting trapped Bose gas [20], where $\mu(n)\propto n$. The same idea of finding a proper scalar product for eigenmodes of a trapped, noninteracting Bose or Fermi gas at finite temperature using the hydrodynamic approach was applied in Ref. [21]. A single eigenmode $\Psi_i(\mathbf{r},t)=\sin(\omega_i t + \phi_0)\Psi_i(\mathbf{r})$ fulfills

$$\omega_i^2\Psi_i(\mathbf{r}) = \hat{G}_\Psi\Psi_i(\mathbf{r}). \tag{13}$$

Solutions to Eq. (13) can be chosen to be orthonormal as

$$\delta_{ij} = \int_{n_0(\mathbf{r})>0} d^3r\Psi_i^*(\mathbf{r})\Psi_j(\mathbf{r}). \tag{14}$$

$\Psi_0(\mathbf{r})=\text{const}\times\sqrt{n_0(\mathbf{r})/A_0(\mathbf{r})}$ is always a formal solution of Eq. (13) with $\omega_0=0$. An important restriction for the density mode δn_i is particle conservation $\int d^3r\delta n_i(\mathbf{r},t)=0$. Equation (10) implies that $\delta n_i=\Psi_0\Psi_i$, thus the orthogonality relation (14) shows that all the modes with $i\neq 0$ are automatically particle conserving and the mode Ψ_0 should be canceled from the solutions.

Taking a complete orthonormal basis, i.e., $\delta_{i,j}=\langle\varphi_i|\varphi_j\rangle$, the squared excitation frequencies ω^2 can be obtained from the eigenvalues of the matrix $G_{i,j}=\langle\varphi_i|\hat{G}_\Psi|\varphi_j\rangle$. The matrix elements require the knowledge or the numerical evaluation of spatial derivatives of the basis functions [19]. Usually this causes big numerical errors because the high-lying modes are rapidly oscillating functions. In practice, it is much better to apply the spatial derivatives to the (spatially varying) coefficients of the wave equation, which are usually not oscillating too much.

The wave operator (11) has the structure $\hat{G}_\Psi = -R\cdot\boldsymbol{\nabla}\cdot Q\cdot\boldsymbol{\nabla}\cdot R$. If there exists a similar system for which the boundary is the same and the wave operator also has the structure $\hat{G}^0=-R_0\cdot\boldsymbol{\nabla}Q_0\cdot\boldsymbol{\nabla}R_0$ but with known spectra and eigenfunctions

$$\hat{G}^0\varphi_i = \epsilon_i^{(0)}\varphi_i, \tag{15}$$

then one can eliminate the unwanted spatial derivatives of the basis functions in the matrix elements if the basis is given by φ_i $(i=0,1,\ldots)$. Let us introduce α and β by

$$\alpha=\alpha(\mathbf{r})=Q/Q_0, \quad \beta=\beta(\mathbf{r})=R/R_0, \tag{16}$$

then the matrix elements can be written as

$$G_{i,j} = \int d^3r\varphi_i^*(\mathbf{r})\varphi_j(\mathbf{r})G_{i,j}(\mathbf{r}), \tag{17}$$

where

$$G_{i,j}(\mathbf{r}) = \frac{\epsilon_i^{(0)}+\epsilon_j^{(0)}}{2}\alpha\beta^2 + R_0^2Q_0\alpha(\boldsymbol{\nabla}\cdot\beta)(\boldsymbol{\nabla}\cdot\beta) + \frac{1}{2}R_0^2\boldsymbol{\nabla}[Q_0\beta^2(\boldsymbol{\nabla}\alpha)]. \tag{18}$$

For an isotropic harmonic-trapping potential $V(\mathbf{r})=m\omega_0^2r^2/2$ and for any equation of state $\mu=\mu(n)$ a whole series of exact solutions of the wave equation can be given. If $\Psi(\mathbf{r})$ is chosen to be

$$\Psi(\mathbf{r}) = \text{const}\times\sqrt{\frac{n_0(\mathbf{r})}{A_0(\mathbf{r})}}r^lY_l^m(\vartheta,\phi), \quad l>0, \tag{19}$$

then this mode function fulfills the wave equation with eigenvalue $\omega^2=\omega_0^2l$. The three $l=1$ modes are the Kohn modes (see Refs. [13,16]) for isotropic trapping.

III. NUMERICAL RESULTS

As a specific, nontrivial model let us consider the mean-field model of Leggett [7] for the BCS-BEC transition. The Leggett model is fixed in homogeneous systems by the gap equation

$$\sum_{\mathbf{k}}\frac{1}{2}\left(\frac{1}{E_{\mathbf{k}}}-\frac{1}{\varepsilon_{\mathbf{k}}}\right) = -\frac{m}{4\pi\hbar^2a} \tag{20}$$

and by the number equation

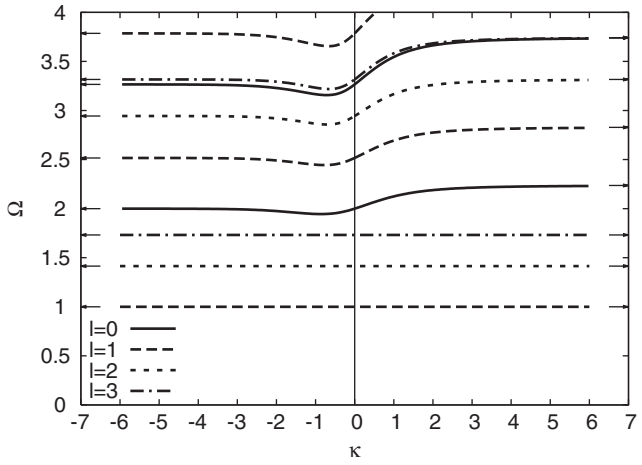

FIG. 1. Excitation frequencies $\Omega=\omega/\omega_0$ of the Leggett model as a function of $\kappa=d/(aN^{1/6})$ for isotropic harmonic trapping. Different curves belong to different l ($l=0$, full line; $l=1$, dashed line; $l=2$, dotted line; $l=3$, dashed-dotted line). Arrows on the left (right) denote the excitations for a noninteracting Fermi gas (for a weakly interacting Bose gas).

$$N=\sum_{\mathbf{k}}\left(1-\frac{\varepsilon_{\mathbf{k}}-\mu}{E_{\mathbf{k}}}\right), \tag{21}$$

where $E_{\mathbf{k}}=\sqrt{(\varepsilon_{\mathbf{k}}-\mu)^2+\Delta^2}$, $\varepsilon_{\mathbf{k}}=\hbar^2k^2/(2m)$, Δ is the pairing gap, and a is the s-wave scattering length. The equation of state $\mu=\mu(n)$ is implicitly given by the model. This model captures the essential features of the BCS-BEC transition. However, recent Monte Carlo data show [15] that there are corrections to the mean-field results, which should be taken into account for the equation of state, especially close to unitarity (i.e., around the $a=\infty$ point). Here we study the above model for simplicity. In the trapped case we use the $\mu(n)$ function taken from the model and solve Eq. (3) for the density profile keeping $N=\int d^3r\, n(\mathbf{r})$ to be fixed. The equilibrium pressure for any trap potential $V(\mathbf{r})$ can be calculated from the local form of the Archimedes law [Eq. (6)]. From the pressure and the density the calculation of $A_0(\mathbf{r})$ with the help of Eq. (5) is straightforward. The details of the full calculation for the Leggett model will be published elsewhere [22].

For isotropic harmonic trapping there is a dimensionless coupling parameter: $\kappa=d/(aN^{1/6})$, where $d=\sqrt{\hbar/m\omega_0}$ is the oscillator length. The spectra depends only on κ. In three cases the spectra is exactly known [6,10,11] because the equation of state has a polytropic form: $\mu\propto n^{\gamma}$. These particular values are $\kappa=-\infty$ (BCS limit), $\kappa=0$ (unitarity limit), and $\kappa=\infty$ (BEC limit). In these cases all the mode functions can be constructed exactly [10], even in the nonisotropic case. (The methods of Refs. [20,23] can be easily employed to the polytropic equation of state.) We used the $\kappa=-\infty$ mode functions [24] on the BCS side and the $\kappa=\infty$ mode functions [11] on the BEC side as basis functions. Our numerical results for different angular momentum l can be seen on Fig. 1. Arrows on both sides show the limiting well-known collective oscillation frequencies [6]. In Fig. 2 the behavior of the lowest $l=0$ quadrupole mode can be seen as a function of κ. This mode is the lowest κ-dependent mode in Fig. 1. The scaling ansatz approach [12] gives quite a good result for this particular mode. On the scale of Fig. 2 the two curves would be practically indistinguishable.

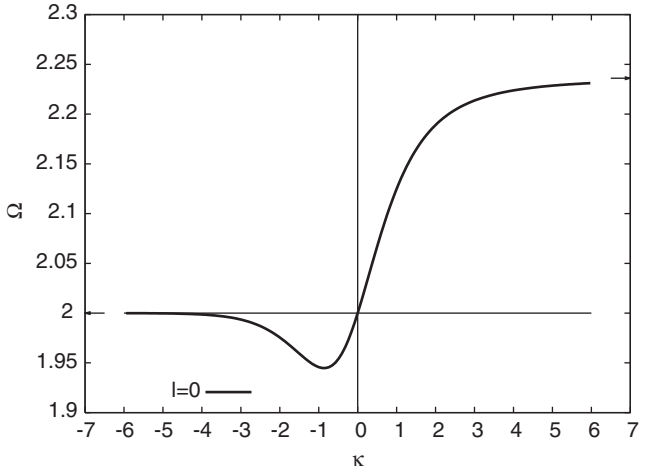

FIG. 2. The lowest $l=0$ excitation frequency $\Omega=\omega/\omega_0$ as a function of $\kappa=d/(aN^{1/6})$ on both sides of the BCS-BEC transition for isotropic harmonic trapping. The arrow on the left (right) denotes the excitation for a noninteracting Fermi gas (for a weakly interacting Bose gas).

In order to give a quantitative measure about the quality of the latter approach we compare in Fig. 3 our excitation frequencies with those given by the scaling ansatz. In the isotropic harmonic-trapping-potential case ω_{sc}^2 is given [12,15] by $\omega_{sc}^2/\omega_0^2=9\langle n\partial\mu/\partial n\rangle/(2\langle V\rangle)-1$, where the average of a quantity like V is taken as $\langle V\rangle=\int d^3rV(\mathbf{r})n_0(\mathbf{r})$. From the figure it is clearly seen that the scaling ansatz is exact at $\kappa=0,\pm\infty$, but between these values it is not. However, the differences in the isotropic case are so small that they are much less than the experimental resolution. In Ref. [15] the scaling ansatz is further improved in several ways (see Fig. 2

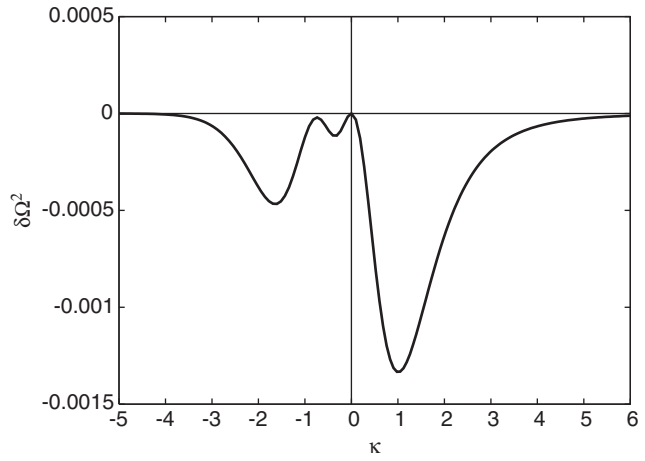

FIG. 3. Comparison of the exact frequencies and those given by the scaling ansatz as a function of $\kappa=d/(aN^{1/6})$ for the lowest $l=0$ monopole mode. $\delta\Omega^2$ is defined as $\delta\Omega^2=(\omega^2-\omega_{sc}^2)/\omega_0^2$ (see the text for ω_{sc}^2).

of Ref. [15]). We expect that a similar comparison with those modes gives much smaller values for the same differences.

We have preliminary data [22] for the excitation frequencies for the experimentally relevant axially symmetric harmonic trapping as well. Once again the scaling ansatz differs a little for the axial quadrupole mode for a general intermediate coupling κ. For such large anisotropies as in Refs. [3,4] however, the deviation $\delta\Omega^2$ for the radial mode is much bigger than in the isotropic case.

IV. DISCUSSION

Hydrodynamical approach is a long-wavelength approximation and gives good results for the elementary excitations at low frequency. Here we gave a straightforward method on how to solve the hydrodynamic equations at zero temperature in the trapped case if the equation of state is known. We introduced a natural scalar product for the transformed wave operator by which the operator is Hermitian. Collective excitations by our method can be found by the diagonalization of the matrix of the wave operator on some basis. We can predict the behavior of the excitations also for those modes for which no scaling ansatz is known. The method is not limited to a particular trap potential (isotropic or not), nor a given mode. There is no additional approximation; the method calculates the (numerically) exact modes given by the hydrodynamic theory at zero temperature. Extension of this work to finite temperature can be possible based on the results of Taylor and Griffin [16]. In that paper the coupling effects of the normal and superfluid part of the gas is discussed concentrating on nondissipative dynamics. We speculate that from the Lagrangian formalism of Ref. [16] one can also derive a multidimensional scalar product as here. Viscous effects, which can explain satisfactorily the experimentally observed dampings, are difficult to incorporate. The first step in that direction is the approach of Bruun and Smith [17] based on the Boltzmann equation, or that of Nikuni and Griffin [18].

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