Generalization of the Margolus-Levitin bound

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The Margolus-Levitin lower bound on the minimal time required for a state to be transformed into an orthogonal state is generalized. It is shown that for some initial states the new bound is stronger than the Margolus-Levitin one.

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A useful measure of the evolution speed of quantum systems is the minimal time t_1 required for a state to be transformed into an orthogonal state. There exist two basic estimates of t_1.

First, t_1 obeys

$$t_1 \geqslant \frac{\pi\hbar}{2\Delta E} \tag{1}$$

where ΔE is the energy dispersion of the initial state. Equation (1) follows easily from the inequality derived by Mandelstam and Tamm [1] and was studied by many authors [2–7].

The second estimate was derived a few years ago by Margolus and Levitin [8]. It is valid for Hamiltonians bounded from below and reads

$$t_1 \geqslant \frac{\pi\hbar}{2\langle E - E_0\rangle}; \tag{2}$$

here $\langle\ \rangle$ denotes the initial-state expectation value while E_0 is the ground-state energy.

Both Eqs. (1) and (2) can be derived using similar arguments [8,9]. By virtue of the spectral theorem one writes

$$\begin{aligned}\langle\Psi|e^{-itH/\hbar}|\Psi\rangle &= \int e^{-itE/\hbar} d\langle\Psi|P_E|\Psi\rangle \\ &= \int \cos\left(\frac{tE}{\hbar}\right) d\langle\Psi|P_E|\Psi\rangle \\ &\quad - i\int \sin\left(\frac{tE}{\hbar}\right) d\langle\Psi|P_E|\Psi\rangle\end{aligned} \tag{3}$$

where P_E is the spectral measure which enters the spectral decomposition of H, $H=\int E\, dP_E$. Therefore, denoting $\langle\Psi|A|\Psi\rangle \equiv \langle A\rangle$, one gets

$$\left\langle \cos\left(\frac{t_1 H}{\hbar}\right)\right\rangle = 0 = \left\langle \sin\left(\frac{t_1 H}{\hbar}\right)\right\rangle. \tag{4}$$

Now, consider an inequality of the form

$$f(x) \geqslant A \sin x + B \cos x \tag{5}$$

which is assumed to hold for all $x \geqslant 0$ [actually, in order to prove (1) one requires (5) to hold for all x]. Denoting by E_0 the lower energy bound one finds from Eq. (5)

$$\left\langle f\left(\frac{t(H-E_0)}{\hbar}\right)\right\rangle \geqslant A\left\langle \sin\left(\frac{t(H-E_0)}{\hbar}\right)\right\rangle + B\left\langle \cos\left(\frac{t(H-E_0)}{\hbar}\right)\right\rangle \tag{6}$$

provided $|\Psi\rangle$ belongs to the domain of $f(t(H-E_0)/\hbar)$. Indeed, Eq. (6) follows easily from the inequality (5) by noting that the expectation value of the non-negative function is non-negative:

$$\begin{aligned}&\left\langle f\left(\frac{t(H-E_0)}{\hbar}\right) - A\sin\left(\frac{t(H-E_0)}{\hbar}\right) - B\cos\left(\frac{t(H-E_0)}{\hbar}\right)\right\rangle \\ &= \int \left[f\left(\frac{t(E-E_0)}{\hbar}\right) - A\sin\left(\frac{t(E-E_0)}{\hbar}\right) \right. \\ &\quad \left. - B\cos\left(\frac{t(E-E_0)}{\hbar}\right)\right] d\langle\Psi|P_E|\Psi\rangle \geqslant 0.\end{aligned} \tag{7}$$

In particular, Eqs. (4) and (6) imply that

$$\left\langle f\left(\frac{t_1(H-E_0)}{\hbar}\right)\right\rangle \geqslant 0 \tag{8}$$

which imposes some restrictions on t_1.

In order to derive a new bound on t_1 we use the following inequality:

$$x^\alpha - \frac{\pi^\alpha}{2} + \frac{\pi^\alpha}{2}\cos x + \alpha\pi^{\alpha-1}\sin x \geqslant 0 \tag{9}$$

which holds for all $x \geqslant 0$ and $\alpha > 0$. Note that Eq. (9) provides a generalization of the inequality used in Ref. [8]; it reduces to the latter for $\alpha = 1$. By virtue of Eq. (8), Eq. (9) leads to the following bound on t_1:

$$t_1 \geqslant \frac{\pi\hbar}{2^{1/\alpha}\langle(E-E_0)^\alpha\rangle^{1/\alpha}}, \quad \alpha > 0 \tag{10}$$

provided $|\Psi\rangle$ belongs to the domain of $(H-E_0)^\alpha$. Equation (10) provides a generalization of the Margolus-Levitin bound which is attained for $\alpha = 1$.

The estimate (10) is for fixed $\alpha \neq 1$ neither weaker nor stronger than the Margolus-Levitin one. Indeed, although the convexity (concavity) of $x \to x^\alpha$ for $\alpha > 1$ ($\alpha < 1$) allows us to claim that $\langle E^\alpha\rangle^{1/\alpha} \geqslant \langle E\rangle$ ($\langle E^\alpha\rangle^{1/\alpha} \leqslant \langle E\rangle$); the additional factor $2^{1/\alpha}$ makes *a priori* estimate impossible. Obviously, one could take the supremum over all $\alpha > 0$ of the right-hand side of (10). However, this is only possible for $|\Psi\rangle$ belonging to the domains of all $(H-E_0)^\alpha, \alpha>0$.

In order to show that, in some cases, the inequality (10) gives a much better bound for some $\alpha\neq 1$ one can use a simple example considered in Ref. [8]. Let us take the initial state of the form

$$|\Psi\rangle = \frac{a}{\sqrt{2}}(|0\rangle + |\varepsilon\rangle) + \frac{b}{\sqrt{2}}[|n\varepsilon\rangle + |(n+1)\varepsilon\rangle]; \qquad (11)$$

the normalization condition implies $|a|^2+|b|^2=1$.

One can easily check that

$$t_1 = \frac{\pi\hbar}{\varepsilon}. \qquad (12)$$

Computing the relevant expectation value one obtains

$$2^{1/\alpha}\langle(E-E_0)^\alpha\rangle^{1/\alpha} = \{1 + |b|^2[n^\alpha + (n+1)^\alpha - 1]\}^{1/\alpha}\varepsilon. \qquad (13)$$

Let us choose $b=\lambda/\sqrt{2}\sqrt[4]{n}$ with $\lambda\neq 0$ independent of n. Then, for $\alpha=\frac{1}{2}$, Eq. (10) gives in the limit of large $n(\sqrt{n}\gg 1)$

$$t_1 \geqslant \frac{\pi\hbar}{\varepsilon(1+|\lambda|^2)^2}. \qquad (14)$$

On the other hand, if $|\lambda|^2\sqrt{n}\gg 1$, Eq. (2) becomes

$$t_1 \geqslant \frac{\pi\hbar}{\varepsilon|\lambda|^2\sqrt{n}}. \qquad (15)$$

Also Eq. (1) gives in this limit the much weaker bound

$$t_1 \geqslant \frac{\sqrt{2}\pi\hbar}{\varepsilon|\lambda|\sqrt[4]{n^3}}. \qquad (16)$$

We see that for the above state our bound is $O(1)$ while (1) and (2) are $O(1/\sqrt[4]{n^3})$ and $O(1/\sqrt{n})$, respectively. Therefore, the new bound may be much better even for such very simple systems.

The above example may seem quite artificial. However, it is generic in the sense that it allows us to understand the status of bounds based on energy distribution moments. In fact, let us consider the following generalization of our example. We assume that the energy spectrum $(\mathcal{S})$ consists of a number of pairs of levels differing by the same energy amount ε: $\mathcal{S}(H)=\{E_0=0,\varepsilon,E_1,E_1+\varepsilon,E_2,E_2+\varepsilon,\ldots\}$. Consider the state for which both members of any "doublet" enter with the same amplitude, i.e.,

$$|\Psi\rangle = \sum_n \frac{a_n}{\sqrt{2}}(|E_n\rangle + |E_n+\varepsilon\rangle), \quad \sum_n |a_n|^2 = 1. \qquad (17)$$

Obviously, the orthogonalization time for this state is given by Eq. (12), irrespectively of the values of a_n and E_n, $n=0,1,2,\ldots$. On the other hand

$$2^{1/\alpha}\langle(E-E_0)^\alpha\rangle^{1/\alpha} = \left(\sum_n |a_n|^2[E_n^\alpha + (E_n+\varepsilon)^\alpha]\right)^{1/\alpha}. \qquad (18)$$

It is clearly seen from the above equation that our bound cannot be optimal except for a small number of states (see below). However, the advantage of it is that we have a free parameter α which can be manipulated to get the best possible estimate for a known spectrum. As we have shown explicitly above an appropriate choice of α can result in a much better bound than the Margolus-Levitin one.

The above reasoning also shows clearly that there exists no optimal bound based on energy distribution only. The relevant moments generically depend strongly on the values of E_k and a_k which, in turn, are completely irrelevant as far as the orthogonalization time is concerned. Therefore, it is desirable to have an *a priori* estimate that depends on a free parameter to be adjusted to "minimalize" the role of E_k and a_k.

Let us find the intelligent states saturating (10). To this end let us note that the left-hand side of Eq. (9) vanishes only for $x=0$ and π. Therefore, only two-level systems can saturate (10). One easily finds that they must be of the form

$$|\Psi\rangle = c_1|E_0\rangle + c_2|E_1\rangle, \quad |c_1| = |c_2| = \frac{1}{\sqrt{2}}. \qquad (19)$$

Finally, let us sketch how one can generalize our result to the mixed-state case. This can be done according to the lines of Ref. [10]. To this end, given two density matrices, one defines the fidelity

$$F(\rho,\rho') = (\mathrm{Tr}\sqrt{\sqrt{\rho}\rho'\sqrt{\rho}})^2. \qquad (20)$$

Given any Hamiltonian H and an initial state ρ,

$$\rho = \sum_n p_n|\phi_n\rangle\langle\phi_n|, \qquad (21)$$

we want to estimate the value of $F(\rho,\rho(t))$. To this end we consider some purification $|\chi\rangle$ of ρ,

$$|\chi\rangle = \sum_n \sqrt{p_n}|\phi_n\rangle|\xi_n\rangle. \qquad (22)$$

Assume that all states of an ancillary system evolve trivially in time. Then the total Hamiltonian governing the time evolution of $|\chi\rangle$ equals $H\otimes I$. Therefore, all energy distribution moments with respect to $|\chi\rangle$ coincide with those with respect to ρ. From Uhlmann's theorem [11] the following inequality holds:

$$F(\rho,\rho(t)) \geqslant |\langle\chi|\chi(t)\rangle|^2, \qquad (23)$$

which allows us to extend to the mixed-state case any bound based on energy distribution moments.

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