Reexamination of dynamical stabilization of matter-wave solitons

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We consider dynamical stabilization of Bose-Einstein condensates by time-dependent modulation of the scattering length. The problem has been studied before by several methods: Gaussian variational approximation, the method of moments, the method of modulated Townes soliton, and the direct averaging of the Gross-Pitaevskii equation. We summarize these methods and find that the numerically obtained stabilized solution has a different configuration than that assumed by the theoretical methods (in particular a phase of the wave function is not quadratic with r). We show that there is presently no clear evidence for stabilization in a strict sense, because in the numerical experiments only metastable (slowly decaying) solutions have been obtained. In other words, neither numerical nor mathematical evidence for a new kind of soliton solutions has been revealed so far. The existence of the metastable solutions is nevertheless an interesting and complicated phenomenon on its own. We try some non-Gaussian variational trial functions to obtain better predictions for the critical nonlinearity g_{cr} for metastabilization but other dynamical properties of the solutions remain difficult to predict.

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I. INTRODUCTION

The nonlinear Schrödinger equation (NLSE) appears in many models of mathematical physics and has numerous applications. The one-dimensional (1D) NLSE is famous due to its integrability and soliton solutions. The two-dimensional and three-dimensional versions do not have such properties and are much less explored.

In the last decade the dynamics of Bose-Einstein condensates (BEC's) has attracted an enormous amount of interest which in turn is causing a renewed growth of interest in the NLSE, since it is known that the NLSE [often called the Gross-Pitaevskii (GP) equation in that context] describes the dynamics of BEC's at zero temperature very well [1]. While early analytical studies of BEC's were concentrated on (quasi-)1D systems, (quasi-)2D and 3D systems are more important for real experiments. In 2D and 3D systems an analytical treatment of the NLSE is very difficult and one has to use approximate methods. One of the very interesting and complicated phenomena being studied recently is stabilization of BEC's by the oscillating scattering length in two and three dimensions. In the NLSE with attractive nonlinearity, in 1D geometry, bright soliton solutions are stable without a trapping potential. In 2D free space, the kinetic energy can balance the interaction energy at a certain critical value of the nonlinearity, g_{cr}, but the resulting solution (Townes soliton) is unstable. That is, if the nonlinearity is either increased or decreased (and kept fixed afterwards), the solution either expands or collapses correspondingly. It was shown by several authors that stabilized solutions are possible with the oscillating scattering length. The oscillations of the scattering length lead to the creation of pulsating condensates—i.e., some kind of breather solution. One can draw an analogy with the Kapitza pendulum (a pendulum with a rapidly oscillating pivot), where unstable equilibria of unperturbed system are stabilized by means of fast modulation. This idea was already applied to the stabilization of beams in nonlinear media [2]. Among many other applications in related fields, the atom wire trap suggested in Ref. [3] should be mentioned. In Refs. [4,5] the novel application of this stabilization mechanism to BEC physics was presented which in turn encouraged several other works on that subject [6–10].

We consider here the problem of stabilization of BEC's in 2D free space by means of rapid oscillations of the scattering length in a greater detail (the third dimension is assumed to be excluded from the dynamics, say, due to a tight confinement). The system is described by the GP equation

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\nabla^2\psi + \frac{\omega_r^2(t)}{2}r^2\psi + g(t)|\psi|^2\psi, \qquad (1)$$

where $r^2=x^2+y^2$ and $g(t)=(8\pi m\omega_z/\hbar)^{1/2}Na(t)$ describes the strength of the two-body interaction. The interaction $g(t)$ is rapidly oscillating, $g(t)=g_0+g_1\sin(\Omega t)$, while the confinement trap described by $\omega_r(t)$ is slowly turned off. References [4–7] suggest it is possible to obtain a dynamically stabilized bright soliton in free space in such a way. Interactions between such objects were very recently studied in Ref. [9]. This is a very interesting phenomenon not only in the context of BEC's but also from a broader scope of nonlinear physics.

Such a kind of stabilization in 3D has also been reported [10]. The latter finding is, however, in some disagreement with other investigations on this topic (for example, Ref. [6]). In Ref. [11] it was shown that the scattering length modulation may indeed provide for the stabilization in 3D, but only in combination with a quasi-1D periodic potential. So 3D geometry might need additional careful examination. In the present paper we concentrate on the quasi-2D case only, where also not everything is clear yet. Unlike conventional 1D solitons, higher-dimensional solitonic objects may decay. Therefore, it is interesting to investigate if there indeed exists a novel genuine breather solution behind the phenomenon of stabilization. As we show in this paper, it turns out that the phenomenon does not fit into the simple models suggested earlier. For a theoretical description of the process, several methods were used by different groups of authors: a variational approximation based on the Gaussian *Ansatz* [4,6], direct averaging of the GP equation [6], a method based on the modulated Townes soliton [6], and the method of moments [8]. Surprisingly, we find all the methods are not very satisfactory even for qualitative predictions. In brief, the direct averaging of the GP equation (the method which is the most interesting for us) has the disadvantage of omitting terms which are of the same order as those responsible for creation of the effective potential. The other methods, although very different, all rely on the unwarranted assumption of parabolic dependence of the phase of the stabilized wave function on r: $\arg\psi=\alpha(t)+\beta(t)r^2$. We find that the behavior of the exact numerical wave function is, however, completely different (see Fig. 2). The above-mentioned parabolic approximation (PA) of the phase factor is very popular because it is appealingly simple and indeed often appears in solutions of the time-dependent GP equation [12]. Usually it comes from self-similar time evolution of the condensate density; for example, in 3D the following dynamics of the condensate density is possible: $\rho(x,y,z) =[\lambda_1(t)\lambda_2(t)\lambda_3(t)]^{-1}\rho(x/\lambda_1(t),y/\lambda_2(t),z/\lambda_3(t))$, where the coefficients λ_i are coupled by nonlinear differential equations. It is the important finding of the present paper that in our problem a stabilized wave function does not have such parabolic phase factor and does not fit into the self-similar patterns implied by the above-mentioned methods. This qualitative difference between the exact numerical solution and all theoretical models considered so far was not mentioned earlier. Besides, we noticed the presence of steady outgoing flux of atoms in numerically stabilized solutions. So even numerically there is no 2D soliton so far, but some slowly decaying object instead. Section II reviews the above-mentioned theoretical methods. In Sec. III we give some results obtained using the variational approximation with non-Gaussian trial functions, including a "super-Gaussian *Ansatz*." It is shown that a better accuracy can be obtained for predicting the critical nonlinearity g_{cr}, but we were not able to determine accurately such dynamical properties as the frequency of slow oscillations. Additionally, we checked the super-Gaussian *Ansatz* for another problem—determination of the critical number of attractive BEC's in a parabolic well—and found it to be much more accurate than the usual Gaussian *Ansatz*. This example also demonstrates that the stabilization mechanism is essentially more complicated than that assumed by the present (PA-based) methods, because predictions of the super-Gaussian *Ansatz* for the dynamical properties of the stabilized solution are much less accurate than in static problems.

In Sec. IV numerical results are presented and compared with predictions of the theoretical methods discussed in Secs. II and III. The configuration of stabilized solutions is discussed and the dynamics of some integral quantities of the solution is investigated. In Sec. V concluding remarks are given. We mention the relation between the BEC stabilization problem and stabilization of optical solitons in a layered medium with sign-alternating Kerr nonlinearity.

II. SEVERAL APPROXIMATE METHODS TO STUDY THE PROBLEM: PA-BASED METHODS (GAUSSIAN VARIATIONAL APPROXIMATION, THE MODULATED TOWNES SOLITON, THE METHOD OF MOMENTS) AND THE DIRECT AVERAGING OF THE GP EQUATION

A. PA-based methods

1. Gaussian variational approximation

The variational approach based on the Gaussian approximation (GA) is one of the most often used in studying dynamics of the GP equation. In actual calculations this approximation, however, often gives a large error as compared to exact numerical results [7,13]. For example, in Ref. [13] the Gaussian approximation in the dynamics of attractive BEC's was compared to the exact numerical solution of the GP equation. It was found that in estimating the critical number $\mathcal{N}_c$ of the condensate (the maximal number of condensed particles in a trap before collapse occurs) the Gaussian approximation gives a 17% error and similar values of discrepancy for other dynamical quantities (as a useful test, in the Appendix we provide the corresponding results obtained with a super-Gaussian variational *Ansatz*). However, it seems that in this example the GA enables one to reproduce important features of the system at least qualitatively. The GA was also used in many other treatments of the GP equation using a variational technique. In particular, it was applied to the problem of BEC stabilization by the oscillating scattering length. The Lagrangian density corresponding to the GP equation (1) is

$$L[\psi]=\frac{i}{2}\left(\frac{\partial\psi}{\partial t}\psi^*-\frac{\partial\psi^*}{\partial t}\psi\right)-\frac{1}{2}\left|\frac{\partial\psi}{\partial r}\right|^2-\frac{1}{2}g(t)|\psi|^4. \quad (2)$$

The normalization condition for the wave function is $2\pi\int_0^\infty|\psi|^2 r dr=1$.

In Ref. [4], a variational method with the following Gaussian *Ansatz* was used:

$$\psi(r,t)=\frac{1}{\sqrt{\pi}R(t)}\exp\left[-\frac{r^2}{2R^2(t)}+i\frac{\dot{R}(t)}{2R(t)}r^2\right], \quad (3)$$

where $R(t)$ is the variational parameter that characterizes the size of the condensate and the phase factor of the wave function describes the mass current [4,5,14].

After substitution of expression (3) into the Lagrangian density (2) one obtains the effective Lagrangian $L=2\pi\int_0^\infty rL[\psi]dr$ and the corresponding Euler-Lagrange equations of motion. One can obtain then the equation of motion for $R(t)$ as

$$\ddot{R}(t)=\frac{1}{R^3(t)}+\frac{g_0+g_1\sin\Omega t}{2\pi R^3(t)}. \quad (4)$$

So the gist of the model is to represent the 2D BEC as a classical nonlinear pendulum with modulated parameters. It is important that other one-parameter PA-based *Ansätze* also give the same nonlinear pendulum $[\ddot{R}=(a+b\sin\Omega t)/R^3$, where a,b depend on the parameters $g_1,g_0,\Omega]$, but with different functional dependence of a,b on the parameters.

The authors of Ref. [4] use then the Kapitza averaging method to study behavior of the system with the rapidly oscillating scattering length. They assume that the dynamics of R can be separated into a slow part R_0 and a small rapidly oscillating component ρ: $R=R_0(t)+\rho(\Omega t)$. From the equations of motion for R_0 and ρ one extracts the effective potential for the slow variable $U(R_0)\approx\frac{A_2}{R_0^2}+\frac{A_6}{R_0^6}$ and determines its minimum

$$R_{min}=\left(\frac{-3}{4\pi(g_0+2\pi)}\right)^{1/4}\left(\frac{g_1}{\Omega}\right)^{1/2}. \tag{5}$$

From the expression for the effective potential for R_0 they obtained the dependence of the monopole moment $\langle r\rangle$ and the breathing-mode frequency ω_{br} on the parameters g_1 and Ω. The frequency of small oscillations (breathing mode) around the minimum is given by [4]

$$\omega_{br}^2=\frac{8\Omega^2}{3g_1^2}(g_0+2\pi)^2. \tag{6}$$

Their numerical calculations were done for $g_0=-2\pi$. One can see that theoretical predictions (5) and (6) based on the Gaussian approximation can catch the $(g_1/\Omega)^{1/2}$ dependence of the monopole moment $\langle r\rangle$ and (Ω/g_1) dependence of the breathing-mode frequency ω_{br} but cannot determine the corresponding coefficients of proportionality, of which the one in Eq. (5) becomes infinity while the one in Eq. (6) becomes zero for $g_0=-2\pi$, the value actually used in the numerical calculations. On the other hand, from numerical calculations they were able to determine the coefficients as 1.06 and 0.32 correspondingly (see Fig. 2 of Ref. [4]). It was also determined in Ref. [4] that in order to stabilize the bright soliton, $|g_0|$ must exceed the critical value of collapse $|g_{cr}|$. Their numerical estimate for $|g_{cr}|$ is ≈ 5.8 while the theoretical estimate based on the Gaussian approximation is $2\pi\approx 6.28$. The 2π estimate in fact corresponds to fitting the so-called Townes soliton by a Gaussian trial function as will be discussed below.

Inspired by the idea of comparing a numerical solution with a simple model nonlinear pendulum, one may ask if it is possible to obtain a better accord with the numerical experiments using different *Ansätze*. We study this question in Sec. III, and it seems that only the stationary Townes soliton can be fit accurately, but not the stabilized breather solutions.

2. Modulated Townes soliton

A method based on modulated Townes soliton used in Refs. [6,8] should be mentioned. The Townes soliton is a stationary solution to the 2D NLS equation with constant nonlinearity g_{cr}. In our notations $|g_{cr}|\approx 1.862\pi\approx 5.85$. This solution is unstable: if $|g|$ is slightly increased or decreased, the solution will start to collapse or expand correspondingly. If the value of g is close to g_{cr}, one may search for a solution of the problem with fast oscillating g in the form of a modulated Townes soliton, as described in Refs. [6,8]. A solution is sought in the form of

$$\Psi(r,t)\approx[a(t)]^{-1}R_T[r/a(t)]e^{iS},$$

$$S=\sigma(t)+\frac{r^2\dot{a}}{4a},\quad \dot{\sigma}=a^{-2}, \tag{7}$$

where R_T represents amplitude of the Townes soliton. Then, starting from the approximation (7), one can derive the evolution equation for $a(t)$ and so determine the dynamics of the system. Note that the approach is also PA based. It is inevitable if we are to use one-parameter self-similar trial function in the form of $|\psi(r,t)|=Af(r/a,t)$.

3. Method of moments

Another PA-based method we would like to mention here is the method of moments [8]. One introduces integral quantities $I_1,I_2,I_3,\ldots$ as

$$I_1=\int_0^\infty|\psi|^2 d\mathbf{r},\quad I_2=\int_0^\infty r^2|\psi|^2 d\mathbf{r},$$

$$I_3=i\int_0^\infty\left(\psi\frac{\partial\psi^*}{\partial r}-\psi^*\frac{\partial\psi}{\partial r}\right)r d\mathbf{r},$$

$$I_4=\frac{1}{2}\int_0^\infty\left(|\nabla\psi|^2+\frac{n}{2}g(t)|\psi|^4\right)d\mathbf{r},$$

$$I_5=\frac{n}{4}\int_0^\infty|\psi|^4 d\mathbf{r}, \tag{8}$$

where $n=2,3$ is the dimension of the problem. In 2D, $d\mathbf{r}=2\pi r dr$, and in 3D, $d\mathbf{r}=4\pi r^2 dr$.

For all t, we have $I_1=1$. For the remaining I_i one can write down the dynamical equations of motion as [8]

$$\dot{I}_2=I_3,\quad \dot{I}_3=I_4,\quad \dot{I}_4=g\frac{n-2}{n}I_5+\dot{g}I_5, \tag{9}$$

$$\dot{I}_5=\frac{n\pi 2^n}{8}\int_0^\infty\frac{\partial|\psi|^4}{\partial r}\frac{\partial \arg\psi}{\partial r}r^{n-1}dr. \tag{10}$$

The system of equations for the momenta is not closed because of I_5, and one should make some approximation in order to close it. In Ref. [8] it was assumed that

$$\arg\psi=\frac{I_3 r^2}{4I_2}; \tag{11}$$

i.e., the phase factor is proportional to r^2 (so that again it is a PA-based method) and the coefficient of proportionality is given by the ratio of I_3 and I_2. Then the system (9) possesses the dynamical invariants [8]

$$Q_1=2(I_4-gI_5)I_2-\frac{1}{4}I_3^2, \tag{12}$$

$$Q_2=2I_2^{n/2}I_5. \tag{13}$$

With the help of these invariants, the system becomes

$$\ddot{I}_2 - \frac{1}{2I_2}(\dot{I}_2)^2 = 2\left(\frac{Q_1}{I_2} + g\frac{Q_2}{I_2^{n/2}}\right). \qquad (14)$$

Introducing $X(t)=\sqrt{I_2(t)}$ one obtains [8]

$$\ddot{X} = \frac{Q_1}{X^3} + g(t)\frac{Q_2}{X^{n+1}}. \qquad (15)$$

The equation is analogous to that obtained by other PA-based methods. One can investigate the obtained equation (15) using various methods of nonlinear dynamics. The simplest Kapitza averaging method can be used again, but of course it is better to use rigorous averaging technique since modern averaging methods are available [15] which have been extensively used already in plasma physics, hydrodynamics, and classical mechanics [16]. The authors of Ref. [8] fulfilled a rigorous analysis of the model (15) using the results of Ref. [17]. It is important to have in mind that the relation between the exact dynamics of the full system and that of the model (15) of the method of moments remains unclear; therefore, one cannot determine sufficient conditions for stabilization, etc. In Ref. [8] it was noticed that the correspondence between numerical simulation of full 2D GP equation and dynamics of the model system (15) is not good. As is seen from Fig. 3 of Ref. [8], neither the frequency of slow oscillations nor the position of the minimum of the effective potential is predicted correctly. Nevertheless, we found that in numerically stabilized solutions the magnitudes of Q_1 and Q_2 are often well conserved; i.e., they oscillate about some mean value (see Sec. IV).

B. Direct averaging of the GP equation

Reference [6] also explores the Gaussian variational approximation. Beside that, a very promising method of directly averaging the GP equation was investigated. It is based on an analogous method used for the one-dimensional NLSE with periodically managed dispersion (in the context of optical solitons) [18]. In Ref. [6] the solution is sought as an expansion in powers of $1/\Omega$ (in our notation):

$$\psi(r,t) = A(r,T_k) + \Omega^{-1}u_1(A,\zeta) + \Omega^{-2}u_2(A,\zeta) + \cdots, \qquad (16)$$

with $\langle u_k\rangle=0$, where $\langle\cdots\rangle$ stands for the average over the period of the rapid modulation and $T_k\equiv\Omega^{-k}t$ are the slow temporal variables $(k=0,1,2,\ldots)$, while the fast time is $\zeta=\Omega t$. Then, for the first and second corrections the following formulas were obtained:

$$u_1 = -i[\mu_1 - \langle\mu_1\rangle]|A|^2A,$$

$$\mu_1 \equiv \int_0^{\zeta}[g(\tau) - \langle g_1\rangle]d\tau,$$

$$u_2 = [\mu_2 - \langle\mu_2\rangle][2i|A|^2A_t + iA^2A_t^* + \Delta(|A|^2A)] - |A|^4A\left(\frac{1}{2}[(\mu_1 - \langle\mu_1\rangle)^2 - 2M] + \langle g\rangle(\mu_2 - \langle\mu_2\rangle)\right),$$

$$\mu_2 = \int_0^{\zeta}(\mu_1 - \langle\mu_1\rangle)ds, \quad M = \frac{1}{2}(\langle\mu_1^2\rangle - \langle\mu_1\rangle^2). \qquad (17)$$

Using these results, the following equation was obtained for the slowly varying field $A(r,T_0)$, derived up to the order of Ω^{-2}:

$$-i\frac{\partial A}{\partial t} = \Delta A + |A|^2A + 2M\left(\frac{g_1}{\Omega}\right)^2[|A|^6A - 3|A|^4\Delta A + 2|A|^2\Delta(|A|^2A) + A^2\Delta(|A|^2A^*)]. \qquad (18)$$

The above equation was represented in the quasi-Hamiltonian form

$$\left[1 + 6M\left(\frac{g_1}{\Omega}^2|A|^4\right)\right]\frac{\partial A}{\partial t} = -\frac{\delta H_q}{\delta A^*},$$

$$H_q = \int dV\left[|\nabla A|^2 - 2M\left(\frac{g_1}{\Omega}\right)^2|A|^8 - \frac{1}{2}|A|^4 + 4M\frac{g_1}{\Omega}|\nabla(|A|^2A)|^2\right)^2. \qquad (19)$$

However, some contribution was missed while deriving Eq. (18). Let us take into account the third correction $u_3(A,\zeta)$:

$$\psi(r,t) = A(r,T_k) + \Omega^{-1}u_1(A,\zeta) + \Omega^{-2}u_2(A,\zeta) + \Omega^{-3}u_3(A,\zeta) + \cdots. \qquad (20)$$

Then, up to terms of order Ω^{-2} it changes nothing on the right-hand side (RHS) of Eq. (18) (spatial part), but it adds to the LHS of Eq. (18) an undetermined term $\Omega^{-2}\partial u_3/\partial\zeta$. This term has the same order Ω^{-2} as the terms from the second correction. So we do not get here a consistent equation for the slow field A because we do not have a closed set of equations for the second-order corrections (the third-order correction becomes a second-order correction after differentiating in time), and so the quasi-Hamiltonian (19) contains an undetermined error of the second order in Ω^{-1}. The influence of the contribution is not very clear but require additional investigation. Nevertheless, formally the omitted terms have the same order as those responsible for the creation of the effective potential. Having in mind how many difficulties arise in averaging of systems of *ordinary* differential equations [15], the rigorous direct averaging of the GP equation constitutes a very interesting and challenging open problem, since in principle it could reveal a true periodic solutions in such oscillating objects.

III. VARIATIONAL APPROXIMATION WITH NON-GAUSSIAN *ANSÄTZE*

Here we try to investigate the system more accurately using some non-Gaussian *Ansätze* and see if it is possible to get more accurate theoretical estimates. One may be interested in three dynamical quantities of the system: the value of critical nonlinearity g_{cr}, slow frequency of breathing oscillations of the stabilized soliton ω_{br}, and minimum of the

TABLE I. Variational predictions for the properties of stabilized solutions.

Ansatz	Amplitude part of the *Ansatz*	g_{cr}, analytical expression	g_{cr}, approximate value	κ_{br} linear prediction, $\omega_{br}=\kappa_{br}\Omega/g_1$
Gaussian	$A\exp\left(-\frac{r^2}{2R^2}\right)$	2π	6.283	$\sqrt{\frac{8}{3}}(g_0+2\pi)$
Super-Gaussian	$A\exp\left[-\frac{1}{2}\left(\frac{r}{R}\right)^\eta\right]$	$\pi 2^{\frac{1}{\ln 2}}\ln 2$	5.919	
Secanth	$A\,\mathrm{Sech}\left(\frac{r}{R}\right)$	$2\pi\ln 2\frac{2\ln 2+1}{4\ln 2-1}$	5.863	$\sqrt{\frac{8}{3}}\frac{\lvert 2\ln 2+1+g_0(4\ln 2-1/2\,\pi\ln 2)\rvert}{4\ln 2-1/2\,\pi\ln 2}$
Exponential	$A\left(1+\frac{r}{2R}\right)\exp\left(-\frac{r}{2R}\right)$	$\frac{144}{77}\pi$	5.875	

effective potential R_{min} about which the expectation value of the monopole moment $\langle r\rangle$ oscillates slowly.

Table I summarizes results of variational predictions for the critical nonlinearity g_{cr} and frequency of small breathing oscillations using several different *Ansätze*. Note that the phase dependence of a one-parameter trial function is not important for calculating g_{cr}. It is understood that if we choose a trial wave function with its amplitude in the form of $|\psi(r,t)|=Af[r/a(t)]$, then we need to use a phase factor with quadratic r dependence in order for the *Ansatz* to be self-consistent (i.e., the mass current generated by the changing parameter would be incorporated in the phase factor of an *Ansatz*). On the other hand, since the amplitude part of the trial function is just an approximation, one may try to use other forms of the phase factor with the same functional form of the amplitude.

When predicting the frequency of breathing oscillations from the corresponding effective potential, it is easy to obtain the result for small-amplitude linear breathing oscillations (given in Table I), but in actual stabilized solutions the amplitudes of the breathing oscillations are not so small.

It is possible to take into account the anharmonicity of breathing oscillations. As was mentioned earlier, all PA-based *Ansätze* produce the nonlinear pendulum $\ddot{R}+(a+b\sin\Omega t)/R^3$, with a corresponding effective potential having $R_{min}=\left(-\frac{3b^2}{2\Omega^2 a}\right)^{1/4}$, $\omega_{br}=\sqrt{\frac{8}{3}}\Omega|a/b|$, where ω_{br} is the frequency of the small-amplitude breathing oscillations (near the bottom of the effective potential). For larger breathing oscillations the (anharmonic) breathing frequency will be amplitude-dependent: $\omega_{br}^{anh}=2\pi\left\{\sqrt{-\frac{2}{h}}\left[\frac{x_3}{\sqrt{x_2-x_3}}\mathbf{K}(k)+\frac{x_2}{x_1}\sqrt{x_2-x_3}\mathbf{E}(k)\right]\right\}^{-1}$, with $k=\sqrt{\frac{x_2-x_1}{x_2-x_3}}$, where $x_1=R_1^2$, $x_2=R_2^2$ (R_1 and R_2 being the turning points), and x_3 is the third root of the equation $h=\frac{a}{2x}+\frac{b}{4\Omega^2 x^3}$, $\mathbf{K}$ and $\mathbf{E}$ are the complete elliptic integrals of the first and second kind. The magnitudes of x_1,x_2,x_3, and h can be determined from numerically obtained breathing oscillations (but the results depend on the choice of a particular *Ansatz*). Even this improvement is not helpful, simply because the parabolic approximation is not valid.

Finding g_{cr} only might be considered as an approximation to the stationary Townes soliton by a trial function so that the mass current term equals zero and that a phase factor may be skipped from the calculations. It is known that the Townes soliton $\psi_t=e^{it}R_T(r,t)$ at large r has asymptotic behavior for its amplitude in the form $R_T\sim e^{-r}/\sqrt{r}$. So that the Gaussian *Ansatz* is not very good for finding g_{cr} just because it is decaying too fast at large r. The super-Gaussian trial function provides a better approximation—namely, $g_{cr}=\pi 2^{1/\ln 2}\ln 2$ which corresponds to the super-Gaussian wave function with $\eta=\eta_T=2\ln 2<2$. Previously the super-Gaussian *Ansatz* was used to fit stationary solutions of some nonlinear problems including the NLS equation in the context of BEC's [19]. The superposition of two Gaussians in the form $A\exp\left(-\frac{r^2}{2R^2}\right)\cosh\left(\gamma\frac{r^2}{2R^2}\right)$ also enables one to obtain some improvement: $g_{cr}\approx 5.883$. The secanth *Ansatz*

$$\psi=\frac{A}{\cosh(r/R)}\exp[iS(\dot{R},R)r^2]$$

works better; with only one parameter it overcomes the above-mentioned two-parameter trial functions. A very good approximation is provided by the simplest *Ansatz* among all considered:

$$\psi=\frac{1}{3R\sqrt{\pi}}\left(1+\frac{r}{2R}\right)\exp\left\{-\frac{r}{2R}+iS(\dot{R},R)r^2\right\}. \quad (21)$$

It fits the Townes soliton adequately both at the origin and asymptotically at infinite r (a preexponential multiplier is not so important as the exponential factor). The preexponential factor is needed in order to fulfill the boundary condition in the origin $\lim_{r\to 0}\frac{1}{r}\psi_r<\infty$. Note that in the super-Gaussian *Ansatz* the former condition is not fulfilled; otherwise (if one included it in a similar way), the result would be better at the cost of more bulky calculations. The accuracy of the prediction implies that *Ansatz* (21) provides a very good approximation to the Townes soliton at fixed R and could approximately represent the modulated Townes soliton when R is time dependent and the phase factor with parabolic r dependence is used in accordance with the continuity condition.

After obtaining estimates for g_{cr}, one can use the above-mentioned *Ansätze* in order to find an effective potential, its minimum, and the frequency of the breathing oscillations of the monopole moment about this minimum in the same way as was done for the Gaussian *Ansatz*. We checked the sech *Ansatz* and the super-Gaussian with quadratic phase dependence. In the super-Gaussian *Ansatz* the parameter η was fixed at the value of its "Townes-soliton-like" solution $\eta=\eta_T=2\ln 2$. In such a way the variational approximation with super-Gaussian *Ansatz* resembles the method of modulated Townes solitons. However, we find that such a trial function seriously underestimates the minimum of the effective potential (i.e., the mean value about which the monopole moment oscillates). Nevertheless, the result of the Gaussian *Ansatz* is even worse since for $g_0=2\pi$ it gives the diverging

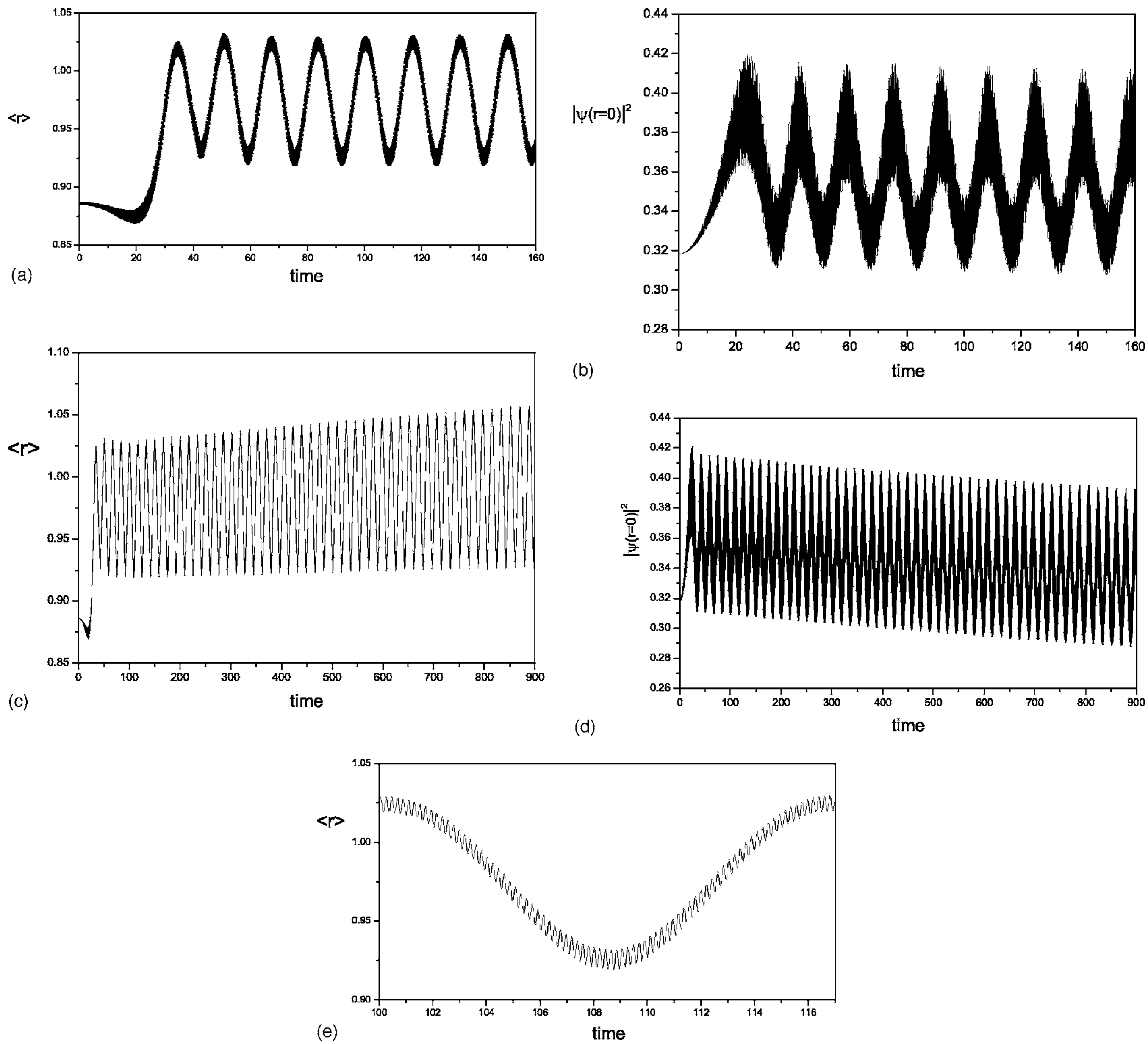

FIG. 1. (a) Oscillations of the monopole moment after turning off the trap. Parameters are $g_0=-2\pi$, $g_1=8\pi$, and $\Omega=30$. The trap was turned off completely at $T_{off}=30$. (b) Time evolution of the amplitude of the wave function at the origin. (c) Oscillations of the monopole moment on a longer time scale. (d) Time evolution of the amplitude of the wave function at the origin on longer times. (e) The oscillations of the monopole moment from the previous figure on a finer scale. Tiny high-frequency oscillations are seen.

expression for R_{min} and zero for the frequency of slow breathing oscillations ω_{br}, as mentioned in Sec. I and [4]. A natural idea for remedy is to use two-parameter trial functions to reproduce the nonparabolic phase factor dependence on r. In the super-Gaussian *Ansatz* it can be done by considering η as a dynamical (time-dependent) parameter. The problem is that it is difficult to obtain a self-consistent expression for the phase factor. We also try the super-Gaussian *Ansatz* with fixed η and with nonquadratic phase dependence (which is unfortunately not a self-consistent trial function) $\psi(r,t)=A\exp-(a+ib)r^{\eta_T}/2$, where A,a,b, and η are all functions of time, the parameter η is fixed at the value of its Townes-soliton-like solution $\eta=\eta_T=2\ln 2$. We find that such a modification drastically changes the dynamical parameters of the system. Still, the resulting model is the same classical nonlinear pendulum as in the Gaussian approximation, but with different parameters. The rigorous way to employ the two-parameter super-Gaussian *Ansatz* is to let η be a dynamical variable and construct a phase factor fulfilling continuity condition for the trial function. One could then obtain the two-dimensional effective potential within the same Kapitza approach.

As a useful test of applicability of the super-Gaussian *Ansatz*, we determine the critical number of attractive BEC's in the 3D parabolic trap studied in Ref. [13]. Their numerical result was $N_{cr}=1258.5$, while the Gaussian approximation yields $N_{cr}^{G}=1467.7$. We found the super-Gaussian prediction to be very accurate, $N_{cr}^{SG}=1236.1$.

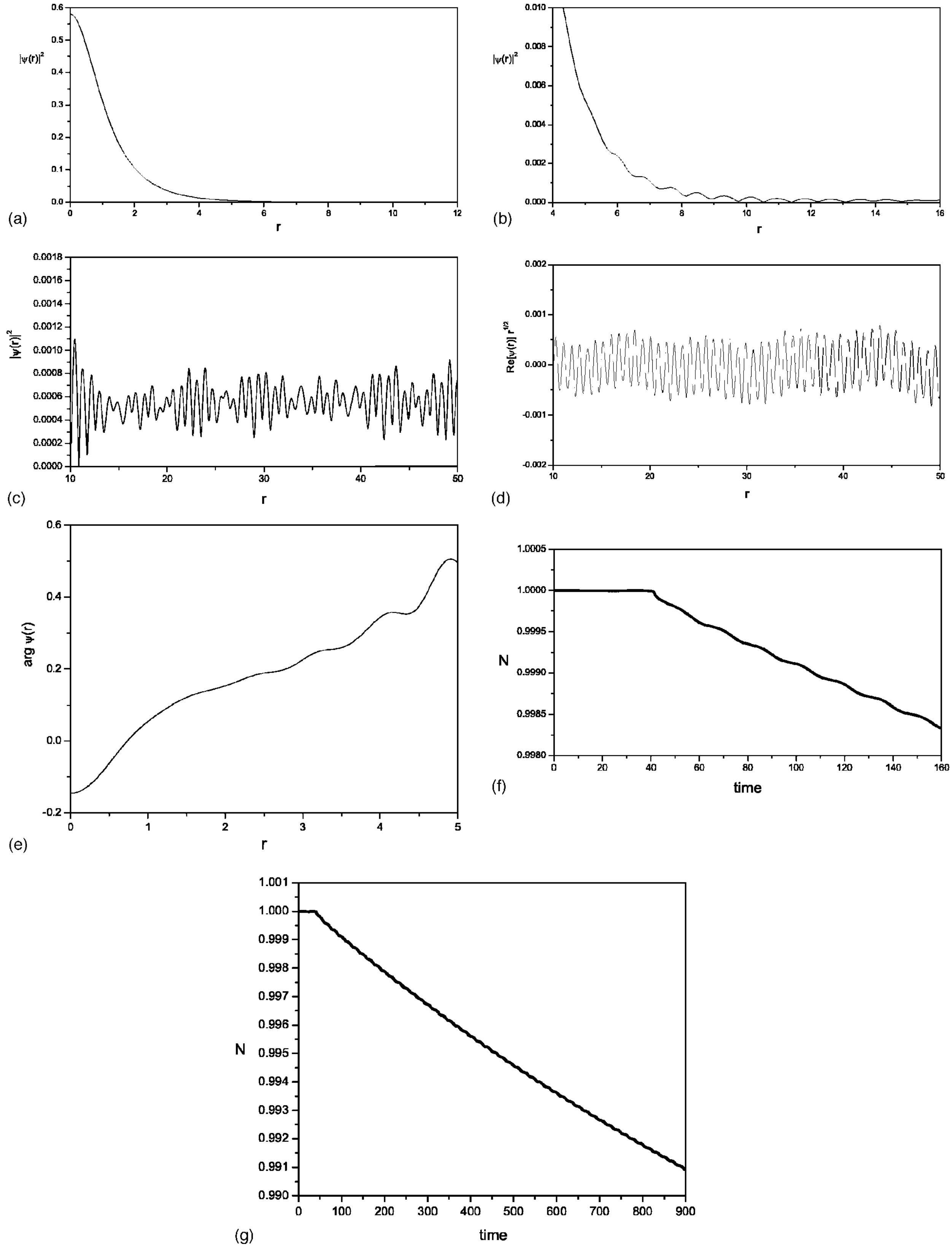

FIG. 2. Configuration of the quasistabilized wave function. Parameters are the same as in previous figures. (a) A snapshot of an amplitude profile. (b) Tiny oscillations in the tail of the quasistabilized solution. (c) Amplitude of the wave function far from the origin (the tail plus outgoing cylindrical wave). (d) Real part of the wave function far from the origin multiplied by $\sqrt{r}$. (e) Snapshot of the phase factor of the quasistabilized solution. It can be seen that it is parabolic only at very small r. The curve has an inflection point at $r \leq 1$. (f),(g) The slowly decaying norm of the solution. Although the trap was turned off at $t = T_{off} = 30$, the norm remains almost constant until the flux of atoms leaking from the core soliton reach the edge of the mesh and begin to disappear. After that it decreases slowly.

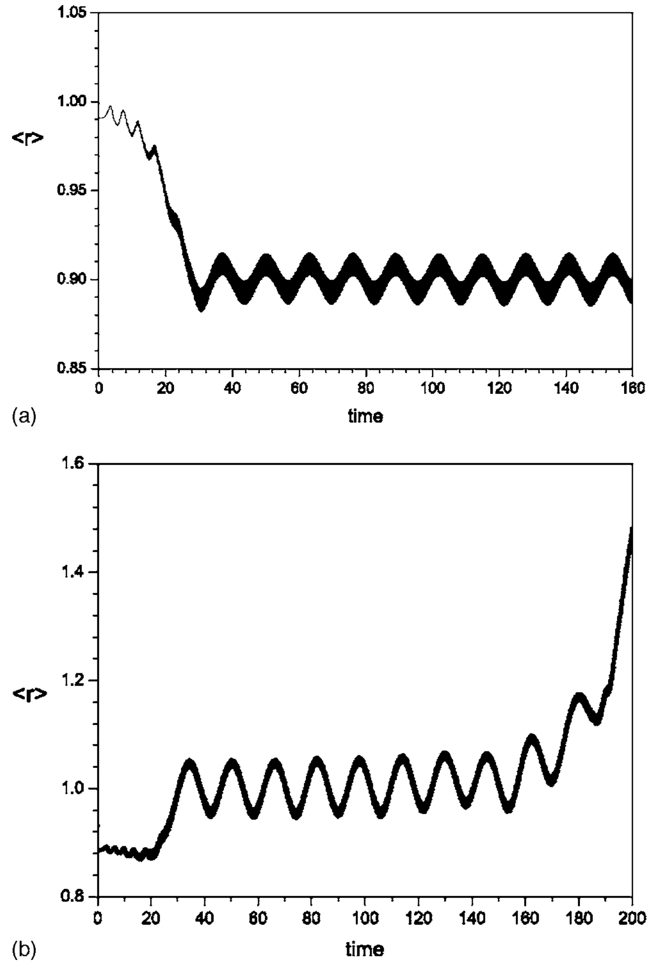

FIG. 3. Oscillations of the monopole moment. (a) $g_0=-6.5$, $\Omega=35$, and $g_1=10\pi$. Initial frequency of the parabolic trap is $\omega(0)=0.8$. (b) $g_0=-6.5$, $\Omega=30$, and $g_1=14.5$. Initial frequency of the parabolic trap is $\omega(0)=1$. The quasistabilized solution is destroyed after several oscillations.

IV. NUMERICAL RESULTS

Numerical calculations reveal the fact that stabilized solutions do not have parabolic phase factors in contradiction to all the methods considered in Sec. II (except the method of direct averaging). The calculations were done using explicit finite-difference schemes. We use explicit finite differences of second and fourth order for spatial derivatives and the fourth-order Runge-Kutta method for time propagation. We use meshes varying from 2000 to 10 000 points, time steps $\Delta t=0.0001-0.0004$, and spatial steps $\Delta r=0.02-0.04$. In addition, we found that it is very important to use an absorbing (imaginary) potential at the edge of the mesh, in accordance with the conclusions of Ref. [8]. Without such an adsorbing potential, a wave reflected from the edge sometimes destroys the otherwise stable solution. Following [4], initially we start with a Gaussian wave packet in a parabolic trap. Then the trap was slowly turned off while the oscillating nonlinearity was slowly turned on in a way similar to Ref. [4]. In Fig. 1 one can see the creation of a stabilized soliton. In Figs. 1(b) and 1(d) oscillations of the amplitude of the wave function at the origin are shown. It decays very slowly. In fact, this is in accordance with the calculations of Ref. [4]: after a careful examination of the corresponding figures in that paper one notices the same behavior. The monopole moment grows very slowly [Figs. 1(a) and 1(c)]. We checked that in the case when the trap is not turned off completely, the norm is conserved during the same long time with a high accuracy (of order 10^{-8}), so decay is certainly not due to numerical errors. In Fig. 2 the configuration of the quasistabilized wave function is shown. One can see the smooth core pulse profile, tiny oscillations in the tail, and an outgoing cylindrical wave leaking from the core pulse. In Fig. 2(e) the behavior of the phase factor is shown. It is seen to differ from parabolic with r considerably. Figures 2(f) and 2(g) show the slow decay of the norm of the solution due to the flux of atoms from the core to infinity. We made a series of numerical experiments with different parameters. We found that the behavior of the matter-wave pulse is often unpredictable. When the Gaussian approximation predicts stabilization, in the corresponding numerical solution it does not necessarily occur. Nor can the method of moments give reliable predictions for the stabilization. We checked the latter method carefully. As was mentioned already in Secs. I and II, the method relies on the crucial approximation of Eq. (11). It is due to this approximation that one obtains the existence of the dynamical invariants Q_1 and Q_2 [see Eq. (13)]. As a result, the dynamics is determined by Eq. (15). Returning back to Fig. 2, we see a snapshot of the phase factor, arg ψ, of a stabilized solution. It clearly demonstrates that none of the PA-based methods reproduce the dynamics of the system adequately. Only at small r is the parabolic law fulfilled, while the deviation from this quadratic dependence is very strong even at $r\leq 1$, where the amplitude of the solution is not small at all (and is sufficient to drastically influence the dynamics of the system). Snapshots at other moments produce similar results: the phase of the solution is changing with time but remains very far from being parabolic in r. It is easy to check that the dynamical properties of the system within a variational approximation are very sensitive to the r dependence in the phase factor of a trial function. To check the dynamics further, we calculated time evolution of the "invariants" Q_1 and Q_2 in the stabilized solution. They are constants in the model but not in the exact numerical solution. We found that in the numerically quasistabilized solution these magnitudes oscillate around some mean value (see Fig. 3). Actually, it was already found in Ref. [8] that the method of moments does not work for Gaussian initial data; still, it is interesting to trace the dynamics of the relevant magnitudes. The time evolution of Q_1 and Q_2 and other magnitudes related to the method of moments are shown in Figs. 4 and 5. It is seen that the magnitudes of Q_1 and Q_2 related to a stabilized soliton undergo slow oscillations.

When calculating the values of Q_1 and Q_2 and other properties of the quasistabilized solution it is necessary to stop the integration at some reasonable value of $r=r_{max}$ [we take $r_{max}=20$ where the amplitude of the wave function becomes very small (of order 10^{-4} in our case]. In that way we separate the properties of the quasistabilized soliton from that of the tail which, although it has very small amplitude, can

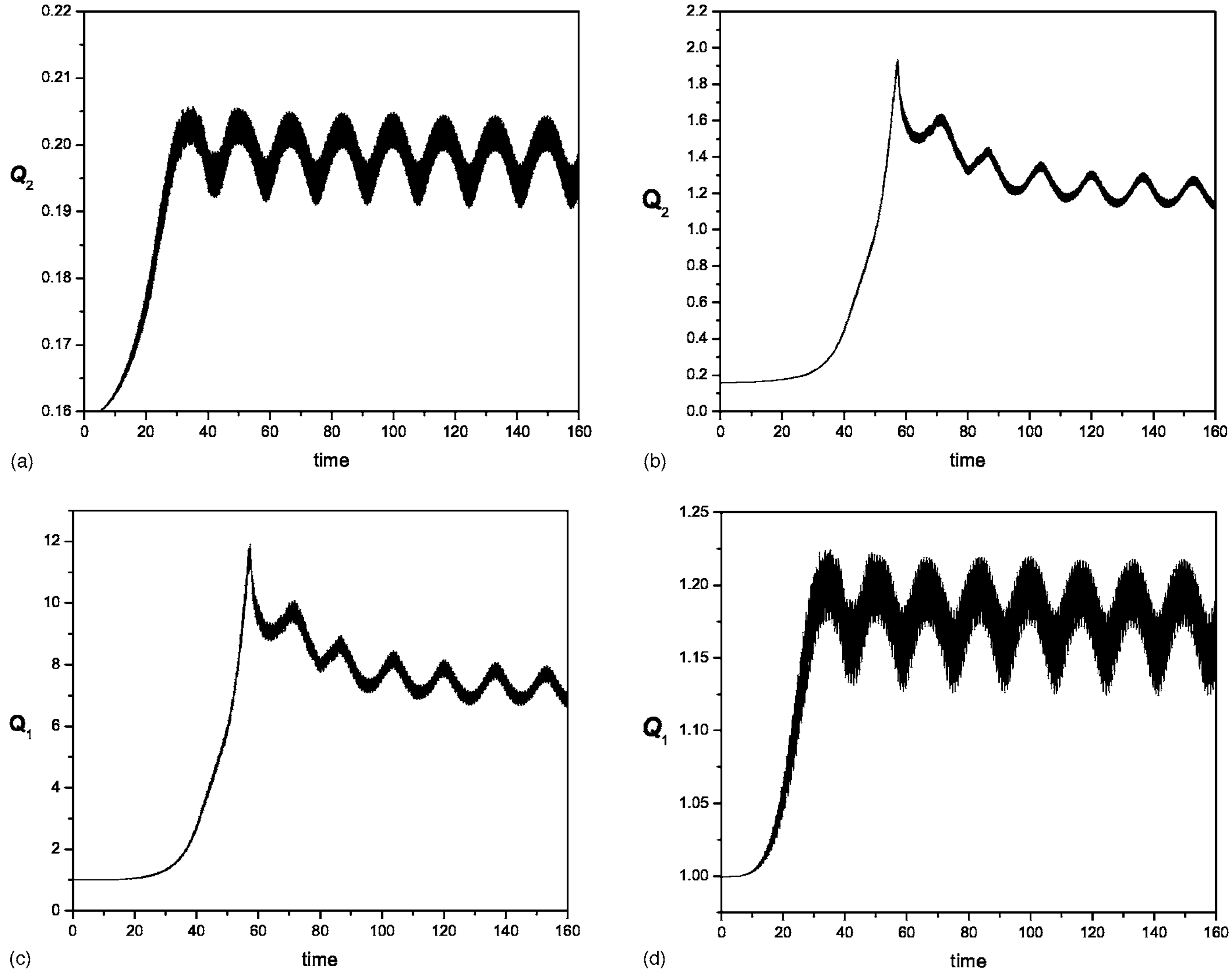

FIG. 4. Time evolution of the integral quantities Q_1, Q_2. (a) Oscillations of "shortened" Q_2 (designated as q_2). We integrate expressions entering Eq. (13) from $r=0$ to $r\approx 8$ so that it characterizes the core part of the solution (quasistabilized soliton) without the oscillating tail. (b) Time evolution of full Q_2. The expressions (13) were integrated from $r=0$ to $r\approx 120$ so that it includes a large contribution from the oscillating tail. (c) Time evolution of "shortened" Q_1 (designated as q_1). We integrate expressions entering Eq. (13) from $r=0$ to $r\approx 8$ so that it characterizes the core part of the solution. The dynamics of the core soliton for quite a long time is almost independent of the behavior of the tail which after reaching the edge of the grid begins to disappear. (d) Time evolution of the full Q_1 (including large contribution from the oscillating tail which depends on location of the absorbing potential and the mesh size).

carry large moments I_2 and I_3 and would give a large contribution to Q_1 and Q_2 [so that in the corresponding figures we presented these quantities for the core soliton and the whole solution (including tail) separately].

Similar features can be seen in Fig. 6 where calculations with $g_0=-7.0$ are presented. Several snapshots of the phase factor at different moments are presented in order to demonstrate that the nonquadratic behavior of the phase factor is typical. Time evolution on a very long time is traced. We find that sometimes magnitudes of Q_1 and Q_2 of stabilized solutions are almost conserved (undergoing small oscillations about their mean value) despite the strongly nonquadratic behavior of the phase factor. It suggests that the method of moments developed in [8] might provide a useful perspective for studying the problem and it would be fruitful to extend it, taking into account the nonparabolicity of the phase factor.

V. CONCLUDING REMARKS

Although there are many publications dedicated to the stabilization of a trapless BEC by the rapidly oscillating scattering length, it seems that the strong nonparabolic behavior of the phase of the stabilized wave function has not been brought to attention yet. However, it should be noted that the role of deviation of the phase profile of NLSE solutions from the parabolic shape was addressed previously in the contexts of solitons in optical fibers in Refs. [20,21]. In particular, in Ref. [20] the *Anzatz* $u=A\,\mathrm{sech}(\tau/W)\exp[i\phi+ib\tanh^2(\tau/W)]$ was used which models the phase saturation. Despite the fact that several independent methods were used previously, we have seen that three of the four theoretical methods used rely on the unwarranted parabolic approximation, while the fourth method (direct averaging of the GP equation) is, strictly

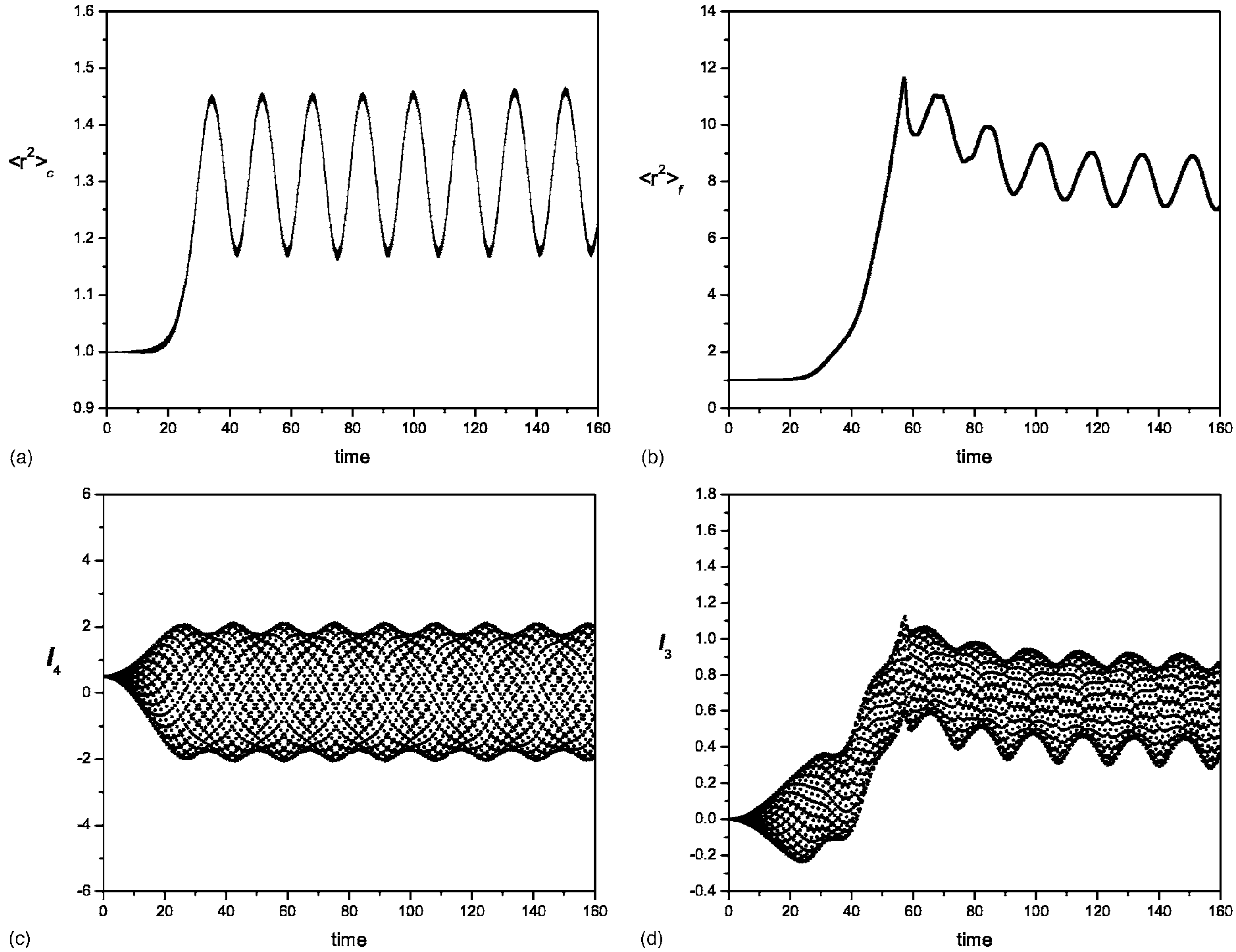

FIG. 5. Time evolution of the moments (I_2,I_3,I_4). (a) Oscillations of the second moment $\langle r^2\rangle$ of the core soliton (designated as i_2). The boundary of the core of the quasistabilized soliton was taken to be $r\approx 8$. (b) Time evolution of the second moment $I_2=\langle r^2\rangle$ of the whole solution including tail (this magnitude depends on mesh size, here $r_{max}\approx$ 120) (c) Time evolution of I_4. (d) Time evolution of I_3.

speaking, incorrect, despite its inspiring motivation (in the sense that the omitted terms have the same order as those responsible for the creation of the effective potential). Besides, we find that there is no evidence presently for stabilization in a strict sense. It seems that the numerical examples presented so far deal with quasistable solutions which slowly decay due to the leaking of atoms from the core pulse as an outgoing cylindrical wave. It means that even from a numerical point of view there are no evidence for true 2D solitons (breathers) yet.

It should be mentioned also that the phenomenon of BEC stabilization has its counterpart in nonlinear optics. As was studied in Ref. [2], in the periodically alternating Kerr media stabilization of beams is possible. Mathematically, one deals with a similar NLSE. Instead of the time dependence of the scattering length of BEC's one has a dependence of the media nonlinearity coefficient on the coordinate z along which a beam propagates:

$$iu_z + \frac{1}{2}\nabla_{tr}^2 u + \gamma(z)|u|^2 u = 0, \tag{22}$$

where the diffraction operator ∇_{tr}^2 acts on the transverse coordinates x and y. The nonlinearity coefficient $\gamma(z)$ jumps between constant values $\gamma_\pm$ of opposite signs inside the layers of widths $L_\pm$. The analysis of this problem was done using a variational approximation based on a natural sech *Ansatz* $U=A(z)\exp[ib(z)r^2+i\phi(z)]\mathrm{sech}[r/w(z)]$. However, the behavior of the phase factor was not checked *a posteriori*, therefore it would be useful to investigate the problem of (2+1)-dimensional solitons in a layered medium with sign-alternating Kerr nonlinearity in greater detail. The interplay between the phenomenon of stabilization in Kerr media and BEC's was addressed also in Ref. [22] in the context of the stabilization of (3+1)-dimensional optical solitons and BEC's in periodic optical-lattice potentials.

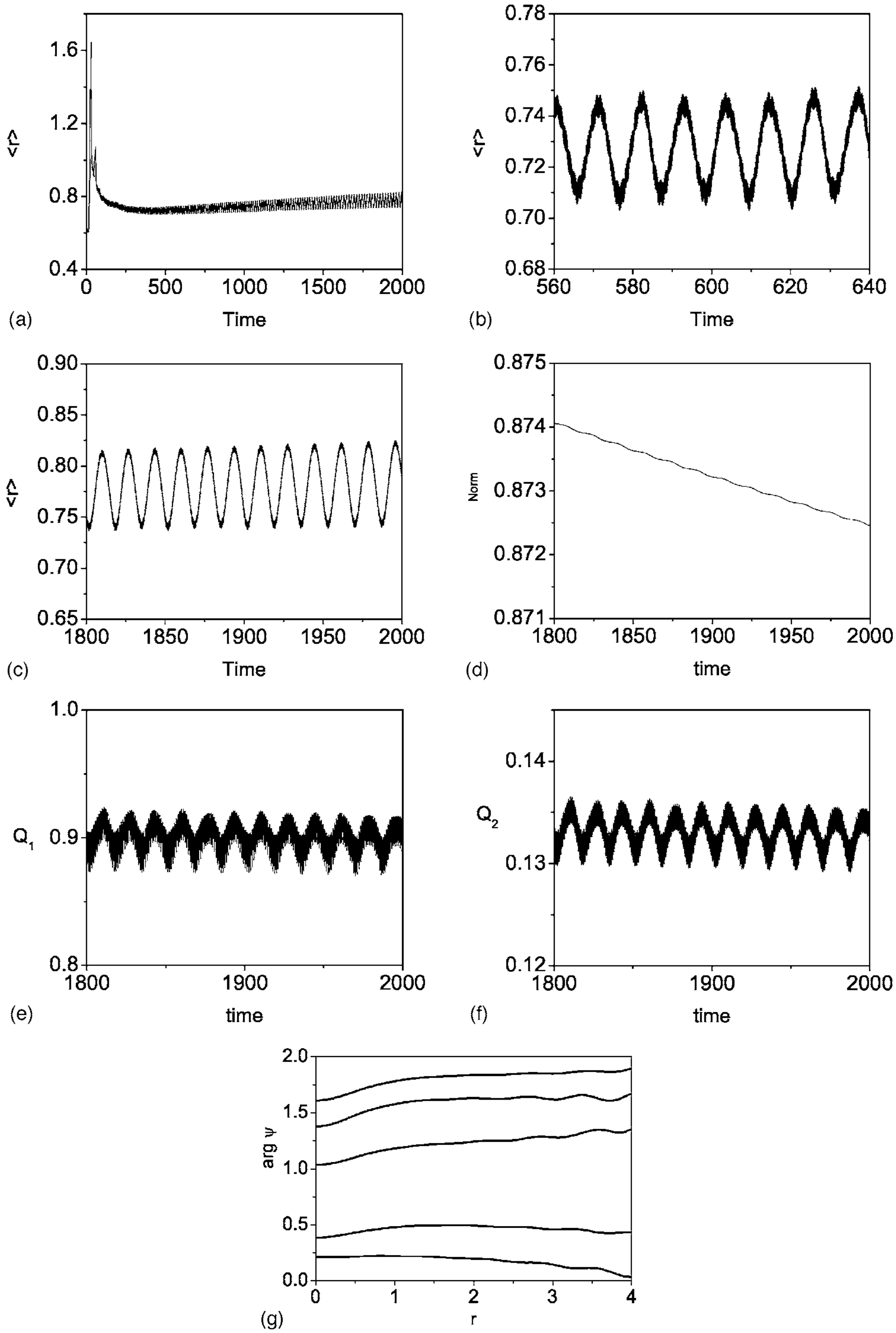

FIG. 6. Oscillations of the monopole moment of a quasistabilized solution. Parameters are $g_0 = -7.0$, $\Omega = 40$, and $g_1 = 8\pi$. Initial frequency of the parabolic trap was chosen to be $\omega(0) = 4.0$. (a) Time evolution of the monopole moment at very long time. (b) Detailed picture of the time evolution of the monopole moment of a quasistabilized solution about $t \approx 600$. (c) Detailed picture of the time evolution at $t = 1800-2000$. (d) Decaying norm of the solution. (e) Time evolution of the integral quantity Q_1 (calculated for the core part of the wave function). (f) Time evolution of Q_2. (g) Several snapshots of the phase factor of the quasistabilized solution (made at different moments). Note that the typical behavior of the phase factor is not quadratic with r at all.

Returning back to the BEC stabilization, we note that the two main difficulties should be resolved in the future: the nontrivial behavior of the argument of the stabilized wave function and the possibility to stop the leak of atoms from the tail of the solution.

Using several non-Gaussian variational functions, we were able to determine accurately one of the magnitudes characterizing the stabilization phenomena—the critical nonlinearity g_{cr}—but not other dynamical properties such as the frequency of slow oscillations.

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