Fragmentation of Bose-Einstein condensates

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We present the theory of bosonic systems with multiple condensates, providing a unified description of various model systems that are found in the literature. We discuss how degeneracies, interactions, and symmetries conspire to give rise to this unusual behavior. We show that as degeneracies multiply, so do the varieties of fragmentation, eventually leading to strongly correlated states with no trace of condensation.

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I. BOSE-EINSTEIN CONDENSATION, FRAGMENTED CONDENSATES, AND STRONGLY CORRELATED STATES

Bose-Einstein condensation (BEC) is a very robust phenomenon. Because of Bose statistics, noninteracting bosons seek out the lowest single-particle energy state and (below a critical temperature T_c) condense into it, even though many almost degenerate states may be nearby. At temperature $T=0$, the condensate contains all the particles in the system [1]. Although this remarkable phenomenon was originally predicted by Einstein for noninteracting systems [2], it became understood, starting with the work of London [3], that it occurs in strongly interacting systems such as ^{4}He. While interactions remove particles from the condensate into other states, the energy gained by macroscopically occupying the lowest energy state (or another state, such as a vortex) is sufficiently great that the interactions in a Bose-Einstein-condensed system manage only to deplete a fraction of the condensate, but not destroy it [4]. This is the case for liquid ^{4}He [5] as well as dilute gases of bosonic alkali-metal atoms [6].

In certain situations, however, a system does not condense into a *single* condensate [7–19]. In this paper we explore the physics of condensation when the ground state can contain several condensates—situations of *fragmented* condensation. This possibility arises naturally from the very concept of BEC—i.e., that the *nondegenerate* ground state of a single-particle Hamiltonian h_0 is macroscopically occupied. How, instead, does condensation take place when the ground state of h_0 is degenerate, with two or more states competing simultaneously for condensation? What happens if the ground-state degeneracy G is not just of order unity, but of order N, the number of particles? And what happens if G becomes much greater than N, or even approaches infinity? How do the bosons distribute themselves in these competing levels? In all these cases, interaction effects play an important role in determining the structure of the many-body ground state. Different types of interactions produce different fluctuations (such as those of phase, number, or spin) and lead to different classes of ground states.

Exact degeneracy is difficult to achieve, since small external fields or weak tunneling effects are generically present. Such energy-splitting effects lead to a nondegenerate ground state and therefore favor the formation of a single condensate. Only when interaction effects dominate over energy splittings can a single condensate break up.

Considerations of the effect of ground-state degeneracies are not merely theoretical exercises. Rather, such degeneracies occur in a wide range of current experiments in cold atoms. The cases where the degeneracy G is of order unity are related to bosons with internal degrees of freedom. Examples include a pseudospin-1/2 Bose gas made up of two spin states ($|F=1,m=-1\rangle$ and $|F=2,m=1\rangle$) of ^{87}Rb [20] and a spin-1 or spin-2 Bose gas such as ^{23}Na or ^{87}Rb in an optical trap [21]. In the former case, even though the two spin states of ^{87}Rb are separated by a hyperfine splitting of order GHz, they can be brought to near degeneracy, with $G=2$, by applying an external rf field. In the spin-1 Bose gas, the three spin states are degenerate at zero external magnetic field and $G=3$. The case of $G\sim N^{1/2}$ is encountered in one-dimensional geometries [22], where the density of states has a power law singularity at low energies. The case of $G\sim N$ is realized for bosons in optical lattices [23] with G sites each having a few bosons; in the limit of zero tunneling, G equivalent sites compete for bosons. The case $G \gg N$ is realized in rotating Bose gases with very large angular momentum L in a transverse harmonic trap [24,25]. As L increases, the rotation frequency Ω of the atom cloud approaches the frequency of the transverse harmonic trap, ω_T, causing the single-particle states to organize into Landau orbitals, which become infinitely degenerate as $\Omega \to \omega_T$. The great diversity of phenomena in these experiments is a manifestation of the physics of Bose-Einstein condensation for varying degrees of degeneracy.

As we shall see, when the degeneracy is low ($G\sim 1$), a single condensate can break up into G condensates. Following Nozières and Saint James [8,9], we refer to such a state as a *fragmented condensate* (defined more precisely below). We caution, however, that there are many sorts of fragmented states. Even degeneracy as small as $G=2$ can give rise to fragmented condensates with distinct properties which depend on whether interaction effects favor phase or number fluctuations. Systems with larger degeneracies usually have a larger number of relevant interaction parameters and are more easily influenced by external fields. As a result, they have a greater variety of fragmented states. For degeneracies

G comparable to or much greater than the particle number, the system does not have enough particles to establish separate condensates. Instead, interaction effects tend to distribute bosons among different degenerate single-particle states in a coherent way, establishing correlations between them. In such *strongly correlated* systems, interaction effects obliterate all traces of a conventional singly condensed state.

We stress at the outset that while interaction effects can cause fragmentation in the presence of degenerate single-particle states, the presence of near degeneracies does not force the condensate to fragment. In many cases the ground state for a *macroscopic* Bose system is a conventional single condensate. Creating a fragmented state typically requires carefully tuning the parameters of the system, and whether such a state can occur in practice is very much dependent on the system. In optical lattices and double-well systems, where the tunneling between wells can be tuned arbitrarily finely, a fragmented state can easily be achieved. Yet in other systems such as a spin-1 Bose gas in a single trap or a rotating Bose gas, the parameter range allowing the existence of fragmented states scales like $1/N$, making it difficult to realize these ground states unless the number of particles is reduced to $\sim 10^3$ or fewer.

In this paper, we focus on fragmentation in the ground state of Bose systems. We consider a number of canonical examples (double-well systems, spin-1 Bose gases, and rotating Bose gases), which have increasing degeneracy in the single-particle Hamiltonian. These examples illustrate the origin of fragmentation, the variety of fragmented states, and their key properties. We will not discuss here fragmentation in optical lattices or in dynamical processes, for they are a sufficiently large subject to require separate discussions. In isolation, all of the examples that we discuss have appeared in the literature and all of their properties have been well established. By bringing these examples together, however, we present a global picture of multiply condensed systems. We show that although there are unifying features shared by all fragmented condensates, there are also striking differences. In summary, we see is that in all of these examples fragmentation is due to an interplay of degeneracies and interactions. In many, but not all, of these examples the fragmented state can be understood as a quantum mechanical average over symmetry broken singly condensed states. As the degree of degeneracies rises, so too does the complexity of the fragmented states. The properties of these complicated states are by no means generic, and one needs to know the structure of higher-order correlation functions to describe their behavior. To begin, we first define condensate fragmentation more precisely and then discuss general properties of certain classes of fragmented states.

II. DEFINITIONS OF CONDENSATION AND FRAGMENTATION

The concept of Bose-Einstein condensation was generalized to interacting systems by Penrose and Onsager [5,26] in the 1950s by defining condensation in terms of the single-particle density matrix

$$\rho^{(1)}(\mathbf{r},\mathbf{r}') = \langle \psi^\dagger(\mathbf{r}')\psi(\mathbf{r})\rangle, \tag{1}$$

where $\psi^\dagger(\mathbf{r})$ creates a scalar boson at position $\mathbf{r}$ and $\langle\cdots\rangle$ is the thermal average at temperature T. Since $\rho^{(1)}$ is a Hermitian matrix with indices $\mathbf{r}$ and $\mathbf{r}'$, it can be diagonalized as

$$\rho^{(1)}(\mathbf{r},\mathbf{r}') = \sum_i N_i(T) f_i(\mathbf{r}')^* f_i(\mathbf{r}), \tag{2}$$

where the $N_i(T)$ are the eigenvalues and $f_i(\mathbf{r})$ the orthonormal eigenfunctions of $\rho^{(1)}$; $\int f_i^*(\mathbf{r}) f_j(\mathbf{r}) d(\mathbf{r}) = \delta_{ij}$. Setting $\mathbf{r} = \mathbf{r}'$ and integrating over $\mathbf{r}$ we have $\Sigma_i N_i(T) = N$, where N is the number of particles. We label the eigenvalues in descending order—i.e., $N_0 \geqslant N_1 \geqslant N_2 \geqslant \cdots$. Equation (2) implies that if one measures the number of bosons in the single-particle state f_i, one finds N_i. This does not mean that the wave function of the many-particle interacting system is a product of such single-particle eigenstates. Unless there are special reasons (such as strict symmetry constraints—e.g., translational invariance), the eigenfunctions $\{f_i\}$ need not be the same as the single-particle eigenstates $\{u_i\}$ of the single-particle Hamiltonian of the noninteracting system. Two relevant examples are a homogeneous system of particles in free space and a system in a harmonic trap. In the former case, where the momentum $\mathbf{k}$ is a good quantum number, we have $i = \mathbf{k}$ and $\{f_i\}$ and $\{u_i\}$ are the same plane-wave momentum eigenstates. In an inhomogeneous trapped system, there is no simple relation between the $\{f_i\}$ and the $\{u_j\}$ [27].

The usual situation of Bose-Einstein condensation corresponds to the one eigenvalue N_0 being of order N, while other eigenvalues are of order unity—i.e.,

$$\rho^{(1)}(\mathbf{r},\mathbf{r}') = N_0(T) f_0(\mathbf{r}')^* f_0(\mathbf{r}) + \cdots \tag{3}$$

$$\equiv \Psi^{(0)*}(\mathbf{r}')\Psi^{(0)}(\mathbf{r}) + \cdots \tag{4}$$

or, simply,

$$\rho^{(1)} = N_0 |\Psi^{(0)}\rangle\langle\Psi^{(0)}| + \cdots , \tag{5}$$

where $\Psi^{(0)}(\mathbf{r}) \equiv \sqrt{N_0} f_0(\mathbf{r})$ and $(\cdots)$ denotes terms with eigenvalues $N_i \sim O(1)$. Since the macroscopic term in Eq. (4) is identical to the density matrix of the pure single-particle quantum state, $|\Psi^{(0)}\rangle$, the function $\Psi^{(0)}(\mathbf{r})$ is often referred to as the "macroscopic wave function" of the system. Systems in which $\rho^{(1)}$ has only one macroscopic eigenvalue, Eq. (4), have *single condensates*. The advantage of the Penrose-Onsager characterization of BEC, Eq. (4), is that it applies to both interacting and noninteracting systems, since it makes no reference to dynamics. Penrose and Onsager also demonstrated the remarkable fact that Eq. (4) holds for a Jastrow function, which is a reasonable approximation to the ground state of a system of hard-core bosons, therefore substantiating Eq. (4) as a general property of interacting Bose systems.

The Penrose-Onsager characterization can be easily generalized to bosons with internal degrees of freedom, labeled by an index μ. With field operator $\psi_\mu^\dagger(\mathbf{r})$, the single-particle density matrix is

$$\rho^{(1)}(\mathbf{r},\mu;\mathbf{r}',\mu') = \Psi_{\mu'}^{(0)*}(\mathbf{r}')\Psi_\mu^{(0)}(\mathbf{r}) + \cdots . \tag{6}$$

In a conventional noncondensed system, such as a zero temperature gas of noninteracting fermions or a high-temperature gas of bosons, all the occupation numbers are small: $N_i \sim O(N^0)$. A conventional singly condensed system has one large eigenvalue $N_0 \sim O(N)$, with all other eigenvalues small, $N_{i>0} \sim O(N^0)$. A fragmented system has $q>1$ large eigenvalues, $N_{i<q} \sim O(N^1)$. There is clearly a range of other possiblilites such as having an extremely large number $q \sim O(N^{1/2})$ eigenvalues, each of which are of size $N_{i<q} \sim O(N^{1/2})$. This latter case occurs in an interacting system of one dimensional bosons and is associated with a phase-incoherent *quasicondensate* [22,28].

A. Simple example of fragmentation and its relation to coherent states

Before examining the origin of fragmentation in detail, let us consider a basic example of fragmentation: the Nozières model [9]. Consider a system of N bosons each of which has available two internal states 1 and 2. As we consider in detail later, this model can also be used to describe atoms in a double-well potential. The Hamiltonian of Nozières' model consists solely of an interaction between atoms in the two states,

$$H = \frac{g}{2} a_1^\dagger a_2^\dagger a_2 a_1 = \frac{g}{2} n_1 n_2, \tag{7}$$

where the $a_i^\dagger$ create bosons in state $i=1,2$ and $n_i = a_i^\dagger a_i$ is the number of particles in i. The interaction between the particles can be either repulsive ($g>0$) or attractive ($g<0$). The eigenstates have a definite number of particles in each well $|N_1, N_2\rangle$, with $N=N_1+N_2$, and energy

$$E = \frac{g}{2} N_1 N_2. \tag{8}$$

Clearly, for $g>0$, the ground state is twofold degenerate, with $N_1=N$, $N_2=0$ or $N_1=0$, $N_2=N$; these states have single condensates whose density matrices have eigenvalues 0 and N. On the other hand, for $g<0$, the state with $N_1=N_2=N/2$ has the lowest energy. This *Fock* state,

$$|F\rangle = \frac{a_1^{\dagger N/2} a_2^{\dagger N/2}}{(N/2)!} |0\rangle \tag{9}$$

has a fragmented condensate; the corresponding single-particle density matrix

$$\rho^{(1)} = \langle a_\mu^\dagger a_\nu \rangle = \frac{N}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{10}$$

has two macroscopic eigenvalues.

One can contrast this fragmented state to the coherent state

$$|\phi_N\rangle = \frac{1}{\sqrt{2^N N!}} (e^{-i\phi/2} a_1^\dagger + e^{i\phi/2} a_2^\dagger)^N |0\rangle \tag{11}$$

in which N bosons are condensed into the single-particle state $(e^{-i\phi/2} a_1^\dagger + e^{i\phi/2} a_2^\dagger)/\sqrt{2}$. The coherent state is an example of a single condensate, where the single-particle density matrix $\rho^{(1)} = \langle a_\mu^\dagger a_\nu \rangle$ is

$$\rho^{(1)} = \frac{N}{2} \begin{pmatrix} 1 & e^{i\phi} \\ e^{-i\phi} & 1 \end{pmatrix} \tag{12}$$

$$= \frac{N}{2} \begin{pmatrix} e^{-i\phi/2} \\ e^{i\phi/2} \end{pmatrix}^* (e^{-i\phi/2} \; e^{i\phi/2}). \tag{13}$$

The difference between Eqs. (13) and (10) is the absence of the off-diagonal term $\langle a_1^\dagger a_2 \rangle$ in the latter, which represents the loss of phase coherence in state (pseudospin) space.

B. Relation between Fock and coherent states

The Fock state (9) is an average over all coherent phase states $|\phi_N\rangle$, as we see from the relation

$$|F\rangle = \frac{2^{N/2}}{\sqrt{N!}} (N/2)! \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} |\phi_N\rangle \tag{14}$$

$$\approx \left(\frac{\pi N}{2} \right)^{1/4} \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} |\phi_N\rangle; \tag{15}$$

the latter relation holds for $N \gg 1$. As we discuss in the next section, this connection is very useful for understanding the origin of various ground states. An important implication of this relation is that *for a macroscopic system, the expectation value of any p-body operator*, $\mathcal{O}_p \sim a_{\mu_1}^\dagger a_{\mu_2}^\dagger \cdots a_{\mu_p}^\dagger a_{\nu_p} \cdots a_{\nu_2} a_{\nu_1}$, *in the Fock state is indistinguishable from that in an ensemble of coherent phase states* $|\phi\rangle$, *as long as* $p \ll N$,

$$\langle F|\mathcal{O}_p|F\rangle = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \langle \phi_N|\mathcal{O}_p|\phi_N\rangle. \tag{16}$$

This equation, which we shall prove momentarily, shows that by measuring quantities associated with few-body operators, one cannot distinguish a Fock state from an ensemble of coherent states with random phases [29,30]. An illustration of this effect is the interference of two condensates initially well separated from each other. Prior to any measurement process, the system is in a Fock state, Eq. (9), since there is no phase relation between the two condensates. Experimentally, in any single-shot measurement (a photo of the interfering region), one finds an interference pattern consisting of parallel fringes whose location is specified by a phase ϕ, as if the two far away condensates actually had a well-defined relative phase with this value [31]. The value of ϕ, however, varies randomly from shot to shot, so that if one averages over all the measurements, the interference fringes average out, as described simply by Eq. (16) [32].

To prove Eq. (16), we use Eq. (15) to write

$$\langle F|\mathcal{O}_p|F\rangle = \sqrt{\frac{\pi N}{2}} \int_{-\pi}^{\pi} \frac{d\phi'}{2\pi} \frac{d\phi}{2\pi} \langle \phi_N'|\mathcal{O}_p|\phi_N\rangle; \tag{17}$$

it is then sufficient to show that $\langle \phi_N'|\hat{\mathcal{O}}_p|\phi_N\rangle$ vanishes unless the phases ϕ and ϕ' are very close to each other. We note that

$$\langle \phi_N' | \phi_N \rangle = \cos^N((\phi - \phi')/2) \approx e^{-N(\phi-\phi')^2/8} \approx \sqrt{8\pi/N}\delta(\phi - \phi'), \tag{18}$$

for $N \gg 1$. For p-body operators of the form $\hat{O}_p = a_1^{\dagger q'} a_2^{\dagger p-q'} a_2^{p-q} a_1^{q}$,

$$\langle \phi_N' | \hat{O}_p | \phi_N \rangle = \frac{N!}{2^p (N-p)!} e^{i[\phi(p/2-q)-\phi'(p/2-q')]} \langle \phi_{N-p}' | \phi_{N-p} \rangle. \tag{19}$$

For $N \gg p$, we see from Eq. (18) that $\langle \phi_N' | \hat{O}_p | \phi_N \rangle = \sqrt{8\pi/N} \langle \phi_N | \hat{O}_p | \phi_N \rangle \delta(\phi - \phi') + O(1/N)$, from which Eq. (16) follows.

III. CHARACTERISTIC EXAMPLES OF FRAGMENTATION

We now develop three different examples that illustrate the origin of fragmentation. These examples are chosen to illustrate the increasingly complex behavior of fragmentation when the number of degenerate single-particle states increases.

A. Scalar bosons in a double well

After the model of Sec. II A, the simplest model with a fragmented ground state is that of bosons in a double-well potential with tunneling between the wells. Unlike in the previous example, this model produces two distinct types of fragmented states. We label the wells by $i=1,2$; we assume that there is only one relevant state in each well and that particles within a given well have an interaction U, which can be either repulsive ($U>0$) or attractive ($U<0$). We take the Hamiltonian to be

$$H = -t(a_1^\dagger a_2 + a_2^\dagger a_1) + \frac{U}{2}[n_1(n_1 - 1) + n_2(n_2 - 1)], \tag{20}$$

where the $a_i^\dagger$ creates a boson in well i and $n_i = a_i^\dagger a_i$ is the number of particles in well i. The first term describes tunneling between the wells via a tunneling matrix element t (which we assume to be real and positive). The form $Un_i(n_i - 1)/2 = Ua_i^\dagger a_i^\dagger a_i a_i/2$ is the usual contact interaction $(g/2)\int \psi^\dagger \psi^\dagger \psi \psi$ reduced to the single mode in each well. For a fixed number of particles, $n_1 + n_2 = N$, the interaction term can be written simply as

$$\hat{U} = \frac{U}{4}[(n_1 - n_2)^2 + N^2 - 2N]. \tag{21}$$

This model, simple as it is, has wide applicability to many physical situations: atoms in a double-well potential [33], internal hyperfine states coupled by electromagnetic fields [20,21], atoms in a rotating toroidal trap [34], or wave packets in an optical lattice [35].

In solving this model it is useful to write Eq. (20) in the Wigner-Schwinger pseudospin representation [36]. We introduce the operators

$$J_x = (a_1^\dagger a_2 + a_2^\dagger a_1)/2, \quad J_y = (a_1^\dagger a_2 - a_2^\dagger a_1)/2i, \quad J_z = (a_1^\dagger a_1 - a_2^\dagger a_2)/2, \tag{22}$$

which obey the angular momentum commutation relation $[J_i, J_j] = i\epsilon_{ijk} J_k$, $i = x, y, z$, and satisfy

$$\mathbf{J}^2 = J_x^2 + J_y^2 + J_z^2 = \frac{N}{2}\left(\frac{N}{2} + 1\right). \tag{23}$$

The Hamiltonian (20) can be written in terms of $\mathbf{J}$ as

$$H = -2tJ_x + U(J_z^2 + \mathbf{J}^2 - N). \tag{24}$$

1. Mean-field solution

As we shall see, the Hamiltonian (20) can be solved exactly. Nonetheless, the mean-field solutions illustrate much of the physics of the true ground state. They also allow one to see the kind of fluctuations about the mean-field state that lead to condensate fragmentation.

The mean-field states are of the form of (pseudo)spinor condensates

$$|\theta, \phi\rangle = \frac{1}{\sqrt{N!}}(ua_1^\dagger + va_2^\dagger)^N |0\rangle, \tag{25}$$

where $u = e^{-i\phi/2}\cos(\theta/2)$ and $v = e^{i\phi/2}\sin(\theta/2)$. The matrix elements of the density matrix in this state are $\langle a_1^\dagger a_1 \rangle = \langle n_1 \rangle = N\cos^2(\theta/2)$, $\langle a_2^\dagger a_2 \rangle = \langle n_2 \rangle = N\sin^2(\theta/2)$, and $\langle a_1^\dagger a_2 \rangle = N\sin(\theta/2)\cos(\theta/2)e^{i\phi}$. The angles θ and ϕ therefore characterize the density and phase difference between the bosons in the two wells. In pseudospin language, the state (25) describes a ferromagnet with total spin, $\langle \mathbf{J} \rangle = (N/2)\hat{\mathbf{n}}$, where $\hat{\mathbf{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ is the unit vector with polar angles (θ, ϕ). According to Eq. (24), its energy is

$$E(\theta, \phi) = \langle \theta, \phi | H | \theta, \phi \rangle = -tN\cos\phi\sin\theta + U\left(\frac{N^2}{4}(\cos^2\theta + 1) - \frac{N}{2}\right). \tag{26}$$

For a repulsive interaction, $U>0$, $E(\theta, \phi)$ is minimum at $\phi=0$, $\theta=\pi/2$—or $\hat{\mathbf{n}}=\hat{\mathbf{x}}$. The mean-field approach therefore selects the noninteracting ground state $|C\rangle = (a_1^\dagger + a_2^\dagger)^N |0\rangle / \sqrt{2^N N!} = |\hat{\mathbf{x}}\rangle$ as optimal. Condensates in the neighborhood of $|C\rangle$ have energy

$$E(\pi/2 + \epsilon, \phi) = E(\pi/2, 0) + \frac{1}{2}tN\phi^2 + \frac{N}{4}(2t + UN)\epsilon^2 + \cdots . \tag{27}$$

From this result one can begin to see problems with the mean-field solution: as $t \to 0$ with fixed U, the energy of phase fluctuations (ϕ) vanishes; therefore quantum fluctuations begin to mix in many nearly degenerate phase states, $|\pi/2, \phi\rangle$.

The mean-field solution for attractive U is very different from that of repulsive U. The solution depends on whether $|U|N < 2t$ or $|U|N > 2t$. In the former case, $|C\rangle$ is locally stable since the energy of fluctuations, described by Eq. (27), remains positive definite. However, the stiffness constant for density fluctuations (ϵ) is lower than that for phase fluctuations (ϕ). Thus, as $|U|$ increases, density fluctuations become dominant. Note, however, that the condition $|U|N<2t$ only occurs for a bounded number of particles and under ordinary circumstances is not expected to be achievable in macroscopic systems. On the other hand, when $|U|N>2t$, we see from Eq. (26) that the optimal states satisfy $\sin\theta=2t/|U|N$. There are two degenerate solutions: $\theta=\theta_0=\arcsin(2t/|U|N)$ and $\pi-\theta_0$. For $|UN|/2t\gg1$, these two states approach $\sim a_1^{\dagger N}|0\rangle/\sqrt{N!}$ and $\sim a_2^{\dagger N}|0\rangle/\sqrt{N!}$, corresponding to all particles being in well 1 or 2, respectively.

2. Exact ground states

We now construct the exact ground state of the Hamiltonian (20) and see how interactions can cause condensate fragmentation. At the same time, we can see how different types of interaction cause different types of fragmented states.

Noninteracting case. Let us first consider the simplest case of noninteracting bosons, with Hamiltonian $H=-t(a_1^\dagger a_2+\text{H.c.})$. The single-particle eigenstates are the symmetric state $(a_1+a_2)/\sqrt{2}$ and antisymmetric state $(a_1-a_2)/\sqrt{2}$ with energy $-t$ and t, respectively. For a system of N bosons, the ground state is

$$|C\rangle=\frac{1}{\sqrt{2^N N!}}(a_1^\dagger+a_2^\dagger)^N|0\rangle, \tag{28}$$

with energy $-tN$. The single-particle density matrix of this state is

$$\langle a_\mu^\dagger a_\nu\rangle_C=\frac{N}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}=N\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right), \tag{29}$$

which has a single macroscopic eigenvalue $\lambda=N$. The ground state is therefore a single condensate with condensate wave function $\zeta_\mu^T=\sqrt{N/2}(1,1)$ (the superscript T stands for transpose).

Since the ground state $|C\rangle$ is a linear combination of number states $|n_1,n_2\rangle=a_1^{\dagger n_1}a_2^{\dagger n_2}|0\rangle/\sqrt{n_1!n_2!}$, the number of particles in each well fluctuates.

We calculate the number fluctuations of the coherent state $|C\rangle$ by writing it in the number basis. For even N, we have

$$|C\rangle=\sum_{\ell=-N/2}^{N/2}\Psi_\ell^{(0)}|\ell\rangle, \tag{30}$$

where $|\ell\rangle\equiv|\frac{N}{2}+\ell,\frac{N}{2}-\ell\rangle$ and

$$\Psi_\ell^{(0)}=\left(\frac{N!}{2^N\left(\frac{1}{2}N+\ell\right)!\left(\frac{1}{2}N-\ell\right)!}\right)^{1/2}\approx\frac{e^{-\ell^2/N}}{(\pi N/2)^{1/4}}. \tag{31}$$

The number fluctuations are then

$$\langle\Delta N_1^2\rangle=\langle(N_1-\langle N_1\rangle)^2\rangle\approx\int d\ell\frac{\ell^2e^{-2\ell^2/N}}{\sqrt{\pi N/2}}=N/4, \tag{32}$$

which, despite the approximations made in this derivation, coincides with the exact result.

Interacting case. The many-body physics of this double-well system is completely tractable. While one can calculate the properties the ground state numerically to arbitrary accuracy, we derive below all the essential features of the ground state by studying the effect of interactions on the noninteracting ground state—i.e., the coherent state $|C\rangle$. We shall see that depending on whether the interactions are repulsive or attractive, the coherent state can be turned into one of two distinct fragmented states: a Fock-like state, where the number of particles in each well is fixed; or a Schrödinger-cat-like state, where the system is in a superposition of having a macroscopic excess of particles in each of the two wells.

We first look at the Schrödinger equation of this system. Writing the ground state in the number basis

$$|\Psi\rangle=\sum_{\ell=-N/2}^{N/2}\Psi_\ell|\ell\rangle, \tag{33}$$

we can write the Schrödinger equation $H|\Psi\rangle=E|\Psi\rangle$, where H is given by Eq. (20), as

$$E\Psi_\ell=-t_{\ell+1}\Psi_{\ell+1}-t_\ell\Psi_{\ell-1}+U\ell^2\Psi_\ell, \tag{34}$$

$$t_\ell=t\sqrt{(N/2+\ell)(N/2-\ell+1)}. \tag{35}$$

The many-body problem then reduces to a one-dimensional tight-binding model in a harmonic potential. The special feature of this model is that the tunneling matrix element t_ℓ is highly nonuniform [37]: $t_\ell\sim N/2$ for $\ell\approx0$ and $t_\ell\sim\sqrt{N/2}$ for $\ell\approx\pm N/2$. This nonuniformity is a consequence of bosonic enhancement, $a^\dagger|N\rangle=\sqrt{N+1}|N+1\rangle$ and $a|N\rangle=\sqrt{N}|N-1\rangle$, which increases the matrix element by a factor of $\sqrt{N}$ when removing a particle from a system with N bosons. As a result, $\langle\ell+1|a_1^\dagger a_2|\ell\rangle$ is maximum when both wells have an equal number of bosons (i.e., $\ell\sim0$) and drops rapidly when the difference in boson numbers between the wells begins to increase (i.e., $\ell\gg1$). A consequence is that hopping favors wave functions Ψ_ℓ having large amplitudes near $\ell=0$. For example, in the noninteracting case the wave function, Eq. (31), is a sharply peaked Gaussian at $\ell=0$.

The interaction term $U(n_1-n_2)^2/4$ leads to a harmonic potential in Eq. (34). Repulsive interactions ($U>0$) suppress number fluctuations, meaning that the Gaussian distribution [Eq. (31)] of the coherent state will be squeezed into an even narrower distribution. In the limit of zero number fluctuation,

$$\langle(\Delta n_1)^2\rangle=\langle(\Delta n_2)^2\rangle=0, \tag{36}$$

the system becomes the Fock state

$$|F\rangle=\frac{a_1^{\dagger N/2}a_2^{\dagger N/2}}{(N/2)!}|0\rangle, \tag{37}$$

which is clearly fragmented, since it is made up of two independent condensates. This fragmentation shows up in the single-particle density matrix,

$$\langle a_\mu^\dagger a_\nu \rangle = \frac{N}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{38}$$

$$= \frac{N}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}(1\ 0) + \frac{N}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}(0\ 1), \tag{39}$$

which has two macroscopic eigenvalues, corresponding to independent condensation in each well. The evolution from the coherent state $|C\rangle$ to the Fock state $|F\rangle$ can be captured by the family of states

$$\Psi_\ell(\sigma) = \frac{e^{-\ell^2/\sigma^2}}{(\pi\sigma^2/2)^{1/4}}. \tag{40}$$

As $1/\sigma^2$ varies from $1/N$ to much greater than unity, the initial coherent state $|C\rangle$ becomes more and more Fock like. Indeed, exact numerical solution of Eq. (34) shows that Eq. (40) is an accurate description of the evolution from a coherent state towards a Fock state. During the collapsing process when the wave function Ψ_ℓ still extends over many number states [but fewer than $(N/2)^{-1/2}$], one can take the continuum limit of Eq. (34), which reduces to the equation for a particle in a harmonic oscillator potential. From its Gaussian ground-state wave function, we extract $\sigma^{-2}=(2/N)(1+UN/t)^{1/2}$.

Using this continuum approximation, we calculate the off-diagonal matrix element,

$$\langle a_1^\dagger a_2 \rangle = \sum_\ell \sqrt{(N/2-\ell)(N/2+\ell+1)}\Psi_{\ell+1}\Psi_\ell \tag{41}$$

$$\approx (N/2)e^{-1/(2\sigma^2)}, \tag{42}$$

which leads to a single-particle density matrix

$$\langle a_\mu^\dagger a_\nu \rangle = \frac{N}{2}\begin{pmatrix} 1 & e^{-1/2\sigma^2} \\ e^{-1/2\sigma^2} & 1 \end{pmatrix}. \tag{43}$$

The eigenvalues are $\lambda_1=\frac{N}{2}(1+e^{-1/\sigma^2})$ and $\lambda_2=\frac{N}{2}(1-e^{-1/\sigma^2})$. The relative number fluctuations are

$$\langle (\Delta n_1)^2 \rangle = \sigma^2/2. \tag{44}$$

As $1/\sigma^2$ varies from $1/N$ to a number much larger than unity, the eigenvalues (λ_1,λ_2) vary from $(N,0)$ to $(N/2,N/2)$ and $\sqrt{\langle(\Delta n_1)^2\rangle}$ varies from $\sqrt{N}$ to 0.

This transition of the coherent state into a Fock state, described by the family of Eq. (40), is due to increasing phase fluctuations, as discussed in Sec. III A 1. The phase fluctuation effect can be seen by writing Eq. (40) in terms of phase states,

$$|\phi\rangle = \frac{1}{\sqrt{2^N N!}}(e^{i\phi/2}a_1^\dagger + e^{-i\phi/2}a_2^\dagger)^N|0\rangle = \sum_\ell e^{i\phi\ell}\Psi_\ell^{(0)}|\ell\rangle, \tag{45}$$

where $\Psi_\ell^{(0)}$ are the coefficients given by Eq. (31). The family of Eq. (40) then becomes

$$|\Psi(\sigma)\rangle = \frac{(\pi\bar{\sigma}^2)^{1/2}}{(\sigma^2/N)^{1/4}}\int_{-\pi}^{\pi}\frac{d\phi}{2\pi}e^{-\bar{\sigma}^2\phi^2/4}|\phi\rangle, \tag{46}$$

where $\bar{\sigma}^{-2}=\sigma^{-2}-N^{-1}$. As σ^{-2} varies from $\sigma^{-2}\sim N^{-1}$ (coherent state) to $\sigma^{-2}\gg 1$, the Gaussian in Eq. (46) changes from a delta function $\delta(\phi)$ to a uniform distribution, thereby driving the coherent state towards a Fock state.

Attractive interaction. When $U<0$, the potential energy $U(n_1-n_2)^2/4$ in Eq. (21) favors a large number difference between the two wells, in particular the states $|\ell=N/2\rangle=|N,0\rangle$ and $|\ell=-N/2\rangle=|0,N\rangle$. It therefore acts in the opposite direction as hopping, which favors a Gaussian distribution of number states around $|\ell=0\rangle$. The effect of the interaction is then to split the Gaussian peak of the coherence state, Eq. (31), into two peaks, a process which can be described by the family of states [38]

$$\Psi_\ell(a) = C(e^{-(\ell-a)^2/2\sigma'^2} + e^{-(\ell+a)^2/2\sigma'^2}), \tag{47}$$

where $2a$ is the separation between the peaks, σ' is the width of the peaks, and C is the normalization constant. As a varies from 0 toward $N/2$ and σ' shrinks at the same time from $1/\sqrt{N}$ to 0, the state $|\Psi(a)\rangle=\Sigma_\ell\Psi_\ell|\ell\rangle$ evolves from the coherent state $|C\rangle$ to a Schrödinger-cat state

$$|\text{Cat}\rangle = \frac{1}{\sqrt{2}}(|N,0\rangle + |0,N\rangle). \tag{48}$$

The Schrödinger-cat state is fragmented in the sense that its single-particle density matrix has two large eigenvalues

$$\langle a_\mu^\dagger a_\nu \rangle = \frac{N}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{49}$$

identical to that of the Fock state. On the other hand, contrary to the Fock state, it has a huge number fluctuations,

$$\langle(\Delta n_1)^2\rangle = \langle(\Delta n_2)^2\rangle = N^2/4. \tag{50}$$

Details of how a and σ' depend on the ratio U/t are found in [38], where it is also shown that the family of Eq. (47) accurately represents the numerical solution of Eq. (34). This double-well example brings out the important point that fragmented condensates cannot be characterized by the single-particle density matrix alone. Higher-order correlation functions such as number fluctuations are needed. This example also shows how a coherent state can be brought into a Fock state (or Schrödinger-cat state) through the phase (or number) fluctuations caused by repulsive (or attractive) interaction.

B. Spin-1 Bose gas

The spin-1 Bose gas, which is only marginally more complicated, provides an excellent illustration of the role of symmetry in condensate fragmentation. Here the degeneracies that give rise to fragmentation are due to a symmetry: rotational invariance in spin space. We will see that in the presence of local antiferromagnetic interactions all low-energy singly condensed states break this symmetry. The true ground state, which is a quantum superposition of all members of this degenerate manifold, is a fragmented condensate.

We consider a spin-1 Bose gas with $2N$ particles in the single-mode approximation, where each spin state has the same spatial wave function. Several recent experiments have been carried out in this limit [39–41] (note that several of these experiments also explored regimes where the single-mode approximation was not valid [42]). For conceptual clarity we consider the extreme case where the trap is so deep that all particles reside in the lowest harmonic state, trivially leading to the single-mode limit. In this extreme situation there are no particle fluctuations: the only relevant interaction is scattering between different spin states. We denote the creation operator of a boson with spin projection $\mu=(1,0,-1)$ in the lowest harmonic state by $a_\mu^\dagger$; the number of bosons with spin μ is $N_\mu=a_\mu^\dagger a_\mu$. The most general form of the scattering Hamiltonian which is rotational invariant in spin space is [43]

$$H = c\mathbf{S}^2, \tag{51}$$

where $\mathbf{S}=\Sigma_{\mu\nu}a_\mu^\dagger \mathbf{S}_{\mu\nu}a_\nu$, with $S^i_{\mu\nu}$ the spin-1 matrices $(i=x,y,z)$ and c the interaction constant. We consider the case $c>0$. More complicated Hamiltonians, with more interaction terms, are allowed for atoms with higher spins [44,45]. The variety of fragmented states proliferates rapidly as the atomic spin increases.

To illustrate how condensate fragmentation is affected by external perturbations, we include the linear Zeeman effect, with Hamiltonian

$$H_Z = -pS_z = -p(N_1 - N_{-1}), \tag{52}$$

where p is the Zeeman energy proportional to the external magnetic field B. To explain current experiments one also needs to include the quadratic Zeeman effect: the atomic energy levels of atoms are not linear in B due to hyperfine interaction between electron spins and nuclear spins. We ignore these nonlinearities as they are irrelevant for describing fragmentation.

For later discussion, we also include a term of the form

$$H_G = \epsilon(a_1^\dagger a_{-1} + a_{-1}^\dagger a_1), \tag{53}$$

which mixes spin states 1 and -1, where ϵ is a constant. Terms of this form can be generated by magnetic field gradients [13], in which case ϵ is proportional to the square of the field gradient. Since both H and H_Z conserve S_z, the density matrix $\langle a_\mu^\dagger a_\nu\rangle$ for the ground state is diagonal in the presence of these terms alone and the system is generally fragmented unless the density matrix happens to have only one macroscopic eigenvalue. The effect of H_G is to mix the 1 and -1 states, bringing the system into coherence, similar to the role of the tunneling term in the double-well system.

1. Mean-field approach

Like the double-well problem in the previous section, the Hamiltonian in Eq. (51) is exactly soluble. As previously, we first consider the mean-field solution so that we can relate the fragmented condensate to a linear combination of singly condensed states.

The general form of a singly condensed spinor Bose-Einstein condensate is

$$|\zeta\rangle_N = \frac{1}{\sqrt{N!}}\left(\sum_\mu \zeta_\mu a_\mu^\dagger\right)^N|0\rangle, \quad \sum_\mu |\zeta_\mu|^2 = 1. \tag{54}$$

Using the fact that $a_\beta a_\nu|\zeta\rangle_N=\sqrt{N(N-1)}\zeta_\beta\zeta_\nu|\zeta\rangle_{N-2}$, one readily finds that the number fluctuations are $\langle N_\nu^2\rangle-\langle N_\mu\rangle^2=\langle N_\nu\rangle=N|\zeta_\mu|^2$. Writing the Hamiltonian as

$$H = \mathbf{S}_{\mu\nu}\cdot\mathbf{S}_{\alpha\beta}a_\mu^\dagger a_\alpha^\dagger a_\beta a_\nu + 2\hat{N}, \tag{55}$$

we have, up to terms $O(N^{-1})$,

$$\langle H + H_Z\rangle_{|\zeta\rangle_N} = c\langle\mathbf{S}\rangle^2 - p\langle S_z\rangle + 2cN, \tag{56}$$

where $\langle\mathbf{S}\rangle=N\Sigma_{\mu,\nu}\zeta_\mu^*\mathbf{S}_{\mu\nu}\zeta_\nu$. It is easy to see that Eq. (56) is minimized by

$$\zeta_\phi^T = e^{i\gamma}\left(e^{-i\phi}\sqrt{\frac{N_1}{N}},0,e^{i\phi}\sqrt{\frac{N_{-1}}{N}}\right), \tag{57}$$

where, as before, T stands for transpose, ϕ is the relative phase between the 1 and -1 components, $N_1+N_{-1}=N$, and the difference $M=N_1-N_{-1}=\langle S_z\rangle$ is given by the nearest integer to $p/2c$. Note that $\langle S_x\rangle=\langle S_y\rangle=0$ and the entire family $\{\zeta_\phi\}$ is degenerate, reflecting the invariance of Eq. (56) under spin rotation about the z axis. In the absence of a magnetic field, $p\to0$, we have $\zeta_\phi^T\to e^{i\gamma}(e^{-i\phi},0,e^{i\phi})/\sqrt{2}$. However, since Eq. (56) becomes fully rotationally invariant at zero field, any arbitrary rotation of the state $(1,0,1)/\sqrt{2}$ is also an optimal spinor condensate. The entire degenerate family (referred to as the "polar" family in literature) is given by

$$\zeta_\phi^{(0)} = e^{i\gamma}\begin{pmatrix} -\frac{1}{\sqrt{2}}e^{-i\phi}\sin\theta \\ \cos\theta \\ \frac{1}{\sqrt{2}}e^{i\phi}\sin\theta \end{pmatrix}, \tag{58}$$

with energy

$$\langle H\rangle_{\text{polar}} = 2cN. \tag{59}$$

Before discussing the exact ground states, we introduce the "Cartesian" operators

$$A_x = -\frac{a_1 + a_{-1}}{\sqrt{2}}, \quad A_y = \frac{a_1 - a_{-1}}{i\sqrt{2}}, \quad A_z = a_0. \tag{60}$$

The important properties of these operators is that that under a spin rotation $a_\mu\to U_{\mu\nu}a_\nu$, where $U=\exp(-i\vec{\theta}\cdot\mathbf{S})$, $\mathbf{A}$ rotates like a Cartesian vector—i.e., $A_i\to R(\vec{\theta})_{ij}A_j$, where $R_{ij}(\vec{\theta})$ is a rotational matrix in Cartesian space (xyz). It is easily verified that $[A_i,A_j^\dagger]=\delta_{ij}$ and

$$\mathbf{S} = -i\mathbf{A}^\dagger\times\mathbf{A}, \quad N = \mathbf{A}^\dagger\cdot\mathbf{A}, \tag{61}$$

$$H = c[N^2 - \Theta^\dagger\Theta], \quad \Theta = \mathbf{A}^2. \tag{62}$$

and the operator that creates a singlet pair is

$$\Theta^\dagger = \mathbf{A}^{\dagger 2} = -2a_1^\dagger a_{-1}^\dagger + a_0^{\dagger 2}. \quad (63)$$

The mean-field state, Eq. (54), can now be written as

$$|\vec{\alpha}\rangle = (\vec{\alpha}\cdot\mathbf{A}^\dagger)^N|0\rangle/\sqrt{N!}, \quad \vec{\alpha}^*\cdot\vec{\alpha} = 1, \quad (64)$$

which has average spin

$$\langle\mathbf{S}\rangle = -iN\vec{\alpha}^*\times\vec{\alpha}. \quad (65)$$

The optimal mean-field state is determined by minimizing

$$\langle\vec{\alpha}|H|\vec{\alpha}\rangle = cN^2[1-|\vec{\alpha}\cdot\vec{\alpha}|^2] - ipN\hat{\mathbf{z}}\cdot\vec{\alpha}^*\times\vec{\alpha}. \quad (66)$$

In zero field ($p=0$), the optimal mean-field state is $\vec{\alpha}=\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a *real* unit vector. The polar family mentioned above can now be conveniently represented by all possible directions of $\hat{\mathbf{n}}$, and the polar state in Eq. (58) corresponds to

$$|\hat{\mathbf{n}}\rangle = \frac{(\hat{\mathbf{n}}\cdot\mathbf{A}^\dagger)^N}{\sqrt{N!}}|0\rangle. \quad (67)$$

2. Exact ground state in a uniform magnetic field

In the absence of a magnetic field, $H=c\mathbf{S}^2$. Since H is proportional to the angular momentum operator, the ground state for $c>0$ (with an even number of bosons, N) is a singlet. The singlet state of a single-mode spin-1 Bose gas is unique [44], and hence any manifestly rotationally invariant state must be the ground state. One can therefore obtain the ground state by forming singlet pairs, which for an even number of particles yields [12,13,46]

$$|S=0\rangle \propto (\mathbf{A}^\dagger\cdot\mathbf{A}^\dagger)^{N/2}|0\rangle. \quad (68)$$

Since this state is rotationally invariant, its single-particle density matrix $\langle a_\mu^\dagger a_\nu\rangle$ is proportional to the identity matrix (Shur's theorem). With the constraint $N=\Sigma_\mu a_\mu^\dagger a_\mu$, we then have

$$\langle a_\mu^\dagger a_\nu\rangle_{|S=0\rangle} = \frac{N}{3}\delta_{\mu\nu}. \quad (69)$$

The condensate is therefore fragmented into three large pieces. The energy of the ground state is exactly zero, whereas that of the mean-field state is $2cN$ [as shown in Eq. (59)].

To relate the exact ground to the optimal mean-field state (the polar family $\{|\hat{\mathbf{n}}\rangle\}$), we note that

$$|S=0\rangle \propto \int \frac{d\hat{\mathbf{n}}}{4\pi}(\hat{\mathbf{n}}\cdot\mathbf{A}^\dagger)^N|0\rangle; \quad (70)$$

i.e., the exact ground state is an average over the family of optimal mean-field states. This is the analog of the symmetry averaging relation in Eq. (16). This averaging process represents the quantum fluctuations within the family of degenerate mean-field states; the effect of these fluctuations is to reduce the mean-field energy from $2cN$ to zero.

Next we consider the case of nonzero magnetic field $p\neq 0$. The Hamiltonian is $H+H_Z=c\mathbf{S}^2-pS_z$. The ground state is $|S,S_z=S\rangle$ or simply denoted as $|S,S\rangle$, where S is an integer closest to $p/2c$. The state $|S,S\rangle$ can be easily obtained by changing a singlet pair into a triplet pair, and we have

$$|S,S\rangle = \mathcal{D}a_1^{\dagger S}(\mathbf{A}^{\dagger 2})^{(N-S)/2}|0\rangle, \quad (71)$$

where $\mathcal{D}$ is a normalization constant and Bose statistics requires S to be even if N is even. Since the Hamiltonian $H+H_Z$ conserves S_z, the single-particle density matrix remains diagonal, $\langle a_\mu^\dagger a_\nu\rangle=N_\mu\delta_{\mu\nu}$. As shown in [12,13], the eigenvalues N_μ have a very interesting behavior as a function of S—namely,

$$N_1 = \frac{N(S+1)+S(S+2)}{2S+3}, \quad (72)$$

$$N_{-1} = \frac{(N-S)(S+1)}{2S+3}, \quad N_0 = \frac{N-S}{2S+3}. \quad (73)$$

If S is of order 1, all three eigenvalues $N_{\pm 1}, N_0$, are macroscopic. On the other hand, if S becomes macroscopic (i.e., S/N is less than but of order 1), both N_1 and N_{-1} remain macroscopic [with $N_1\to(N+S)/2$ and $N_{-1}\to(N-S)/2$] while N_0 becomes of order unity. This means as S increases, the $S_z=0$ component is completely depleted. Although the system is still fragmented, the number of fragmented pieces is reduced from 3 to 2, even for a tiny spin polarization. Further analysis [13] shows that as one increases S, the fluctuations $\langle\Delta N_1^2\rangle=\langle N_1^2\rangle-\langle N_1\rangle^2$ drop rapidly.

The reduction in the number of fragmented pieces mirrors the previously discussed reduction in the size of the space of degenerate mean-field states. In the absence of a magnetic field, polar states with arbitrary $\hat{\mathbf{n}}$ are degenerate, while a magnetic field in the $\hat{\mathbf{z}}$ direction favors those with $\hat{\mathbf{n}}$ pointing in the x-y plane.

To connect the exact polarized states to the mean-field states, we use the relation $\mathbf{A}^{\dagger 2}=-2a_1^\dagger a_{-1}^\dagger+a_0^{\dagger 2}$, Eq. (63), to write

$$(\mathbf{A}^{\dagger 2})^{(N-S)/2}|0\rangle = \sum_{p=0}^{(N-S)/2} D_p|p,N-S-2p,p\rangle, \quad (74)$$

where the exact form of D_p is not important for our discussion. If we now act on $(\mathbf{A}^{\dagger 2})^{(N-S)/2}|0\rangle$ by $a_1^\dagger$, the coefficients of the *large-p* states in Eq. (74) will be bosonically enhanced via $a_1^\dagger|N_1\rangle=\sqrt{(N_1+1)}|N_1+1\rangle$. Further acting on the same state by $a_1^\dagger$ eventually picks out the term with largest p, which is $|(N+S)/2,0,(N-S)/2\rangle$, in the sum Eq. (74). This process is equivalent to replacing $a_1^{\dagger S}(-2a_1^\dagger a_{-1}^\dagger+a_0^{\dagger 2})^{(N-S)/2}|0\rangle$ by $a_1^{\dagger S}(-2a_1^\dagger a_{-1}^\dagger)^{(N-S)/2}|0\rangle$. We then have

$$|S,S\rangle \propto a_1^{\dagger S}(\mathbf{A}^{\dagger 2})^{(N-S)/2}|0\rangle \approx a_1^{\dagger S}(-2a_1^\dagger a_{-1}^\dagger)^{(N-S)/2} \propto a_1^{\dagger(N+S)/2}a_{-1}^{\dagger(N-S)/2}|0\rangle. \quad (75)$$

From our previous analysis of the two-well system it is clear that this state is formed from an angular average of Eq. (57). The field gradient term (53) plays the role of tunneling in the two-well system and can drive the system from a fragmented to coherent state.

To complete the connection between these examples, we note that when the occupation of the $S_z=0$ state is negligible, we can set $a_0=0$ and only two spin states are required to describe the system. Under these conditions, the Hamiltonian for the spin-1 gas [Eqs. (51)–(53)] reduces to the two-well Hamiltonian (20) with $U=4C$, $t=-\epsilon$, and an asymmetry between the wells given by p. This analogy breaks down when $p\to 0$ and the $S_z=0$ state becomes occupied.

C. Fast rotating Bose gas

In the previous cases of pseudospin-1/2 and spin-1 Bose gases, the number of degenerate single-particle states is of order unity, far fewer than the number of bosons. Bose gases with large amounts of angular momentum have just the opposite behavior; the number of nearly degenerate states can be much larger than the number of particles. As we shall see, for relatively small angular momentum, fragmentation similar to that in pseudospin-1/2 and spin-1 Bose gases can occur. However, for very large angular momentum, it is energetically more favorable for a repulsive Bose gas to organize itself into a quantum Hall state, which possesses an extreme form of fragmentation in which all traces of conventional condensation disappear. Before discussing the fragmentations of rotating Bose gas, we first discuss how the single particle energy levels of a rotating Bose gas turn into Landau levels and achieve high degeneracy as the angular momentum of the system increases. For simplicity, we limit the discussion to two-dimensional (2D) systems.

1. Lowest Landau levels and the general properties of many-body wave functions in lowest Landau levels

The single-particle Hamiltonian for a particle in the rotating harmonic trap is

$$h_0-\Omega L_z=\frac{\mathbf{p}^2}{2M}+\frac{1}{2}M\omega_\perp^2 r^2-\Omega\hat{\mathbf{z}}\cdot\mathbf{r}\times\mathbf{p}, \qquad (76)$$

where $\omega_\perp$ is the trap frequency and Ω is the rotational frequency of the trap. Equation (76) can be written as

$$h_0-\Omega L_z=\frac{(\mathbf{p}-M\omega_\perp\hat{\mathbf{z}}\times\mathbf{r})^2}{2M}+\frac{1}{2}M(\omega_\perp^2-\Omega^2)r^2. \qquad (77)$$

If one rewrites $\mathbf{p}-M\omega_\perp\hat{\mathbf{z}}\times\mathbf{r}$ as $\mathbf{p}-e\mathbf{B}\times\mathbf{r}/2c$, with $eB/Mc=2\omega_\perp$, one sees that Eq. (77) is identical to the Hamiltonian of an electron in a magnetic field $B\hat{\mathbf{z}}$ in a harmonic potential with reduced frequency $\omega_\perp^2-\Omega^2$. To diagonalize the single-particle Hamiltonian, Eq. (76), we begin by noting that the 2D simple harmonic oscillator $H_0=\mathbf{p}^2/2M+M\omega_\perp^2 r^2/2$ is diagonalized as $h_0=\hbar\omega_\perp(a_x^\dagger a_x+a_y^\dagger a_y+1)$, where $a_x=(d/\hbar)p_x-ix/d$, $a_y=(d/\hbar)p_y-iy/d$, $d^2=\hbar/m\omega_\perp$, and $L_z=-i\hbar(a_x^\dagger a_y-a_y^\dagger a_x)$. Defining $a_\pm=(a_x\pm ia_y)/\sqrt{2}$, we have

$$L_z=-\hbar(a_+^\dagger a_+-a_-^\dagger a_-) \qquad (78)$$

and

$$h_0-\Omega L_z=\hbar(\omega_\perp+\Omega)a_+^\dagger a_++\hbar(\omega_\perp-\Omega)a_-^\dagger a_-+\hbar\omega_\perp. \qquad (79)$$

The eigenstates are therefore

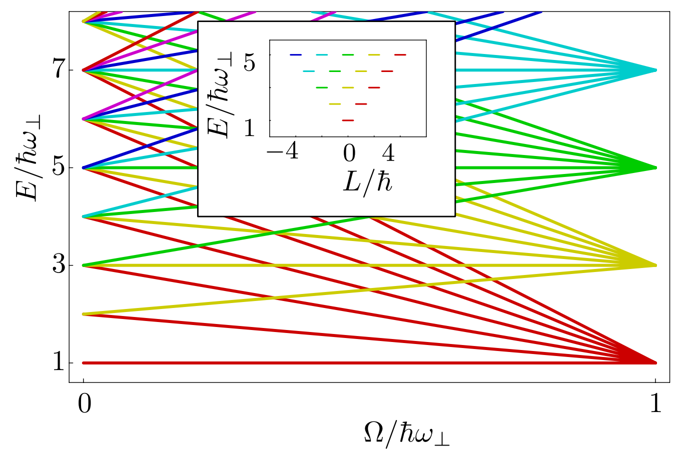

FIG. 1. (Color online) Energy eigenvalues of a two-dimensional harmonic oscillator as a function of angular velocity Ω. For clarity only seven values of m are shown, so that the infinite degeneracies at $\Omega=\omega_\perp$ appear to be only sevenfold. The inset shows the energy states at $\Omega=0$ as a function of their angular momentum L. (Colors: each Landau level is colored with a distinct hue.)

$$|n,m\rangle=\frac{a_+^{\dagger n}}{\sqrt{n!}}\frac{a_-^{\dagger m}}{\sqrt{m!}}|0\rangle, \qquad (80)$$

with eigenvalues

$$E_{n,m}=\hbar(\omega_\perp+\Omega)n+\hbar(\omega_\perp-\Omega)m+\hbar\omega_\perp, \qquad (81)$$

where $n,m=0,1,2,\ldots$. The energy levels, shown in Fig. 1, are organized into Landau levels, labeled by n, separated by $\delta E_1=\hbar(\omega_\perp+\Omega)$. States within a given Landau level are labeled by m with spacing $\delta E_2=\hbar(\omega_\perp-\Omega)$. At criticality, $\Omega=\omega_\perp$, each Landau level becomes infinitely degenerate.

The many-body Hamiltonian, including a contact interaction g, is

$$H=\sum_{i=1}^{N}(h_i-\Omega L_{zi})+g\sum_{i>j}\delta(\mathbf{r}_i-\mathbf{r}_j). \qquad (82)$$

As $\Omega\to\omega_\perp$, mixing is the strongest among the states in the same Landau level. To simplify matters, we consider only extremely weak interactions, $gn\ll 2\hbar\Omega$. This limit is naturally reached when Ω is close to $\omega_\perp$, where the centrifugal potential largely cancels the trapping potential and the cloud becomes large and dilute. For such weak interactions, only the lowest Landau level is populated. The eigenfunctions $u_{n=0,m}(\mathbf{r})$ in the lowest Landau level,

$$u_m(\mathbf{r})=\frac{(z/d)^m}{\sqrt{\pi d^2 m!}}e^{-|z|^2/2d^2}\equiv\langle\mathbf{r}|m\rangle,\quad z\equiv x+iy, \qquad (83)$$

are angular momentum eigenstates with $L=m$ and a spatial peak at $\sqrt{\langle m|r^2|m\rangle}=d\sqrt{2m+1}$.

The many-body wave function for a systems of N bosons is then

$$\Psi(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \sum_{[m]} C_{[m]} u_{m_1}(\mathbf{r}_1) \cdots u_{m_N}(\mathbf{r}_N), \qquad (84)$$

where $\{m\}$ denotes the set of non-negative integers $(m_1, m_2, \ldots, m_N)$. Since, apart from the Gaussian factor, the single-particle wave functions are of the form $u_m \propto z^m$, the many-body wave function is

$$\Psi(\mathbf{r}_1, \ldots, \mathbf{r}_N) = f(z_1, \ldots, z_N)\exp\left(-\sum_{i=1}^{N} |z_i|^2/2d^2\right), \quad (85)$$

where f is an analytic function which is symmetric in z_i. In particular, a Bose-condensed state corresponds to

$$\Psi_w(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \prod_{i=1}^{N} w(\mathbf{r}_i), \quad w(\mathbf{r}) = \sum_m \alpha_m u_m(\mathbf{r}). \quad (86)$$

If Ψ is also an eigenstate of the total angular momentum $L_z = L$, then f in Eq. (85) must be a homogeneous symmetric polynominal of $\{z_i\}$ of degree L. Moreover, the single particle density matrix in the angular momentum basis must be diagonal—i.e.,

$$\langle a_m^\dagger a_n \rangle = \delta_{mn} \langle a_m^\dagger a_m \rangle. \qquad (87)$$

Thus a Bose-Einstein-condensed state [Eq. (86)] that is also an angular momentum eigenstate must be fully condensed into the single-particles state u_m—i.e., $w(\mathbf{r}) = u_m(\mathbf{r})$. The total angular momentum must then be $L = mN\hbar$. Such a coherent state, however, offers no flexibility to redistribute particles to lower the energy and is therefore unlikely to be the ground state at fixed angular momentum or fixed rotational frequency Ω, except for certain special cases in which parameters of the system are carefully tuned. A general Bose-Einstein-condensed state, Eq. (86), would be one where $w(\mathbf{r})$ contains more than a single angular momentum state u_m. Such states break rotational symmetry and are not angular momentum eigenstates. Furthermore, due to rotational symmetry, any rotation of w [i.e., $w(r, \phi) \to w(r, \phi + \theta)$] will have the same energy and angular momentum $L = \langle \Psi_w | L_z | \Psi_w \rangle$. One can therefore form an angular momentum eigenstate with angular momentum L by forming the average

$$\Phi_L(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \int d\theta\, e^{iL\theta} \Psi(\{r_i, \phi_i + \theta\}). \qquad (88)$$

Such averaging always lowers the energy of the system and generically results in a fragmented state. Specific examples follow. Note that with small modification these arguments also apply in the general case when interactions are not weak.

2. Ground state of the rotating Bose gas with attractive interactions

For the rotating Bose gas with attractive interaction, Wilkin, Gunn, and Smith [10] pointed out the lowest energy state with nonzero angular momentum is one with all angular momentum carried by the center of mass and has the form

$$\Psi(\mathbf{r}_1, \ldots, \mathbf{r}_N) = K(Z/d)^L \exp\left(-\sum_{i=1}^{N} |z_i|^2/2d^2\right), \qquad (89)$$

where Z is the center of mass and K is a normalization constant,

$$Z = \frac{1}{N}\sum_{i=1}^{N} z_i, \quad K = \frac{1}{\sqrt{L!(\pi d^2)^N}}. \qquad (90)$$

It is straightforward to show that the single-particle density matrix of this state is

$$\langle a_m^\dagger a_n \rangle = \delta_{mn} N \left(\frac{L!(N-1)^{L-m}}{m!(L-m)!N^L} \right). \qquad (91)$$

The ground-state is fragmented because Eq. (91) has a distribution of large eigenvalues. Later, Pethick and Pitaevskii [47] pointed out that this state is of the form

$$\Psi(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \Psi_{\text{c.m.}}(\mathbf{R})\, \phi_{rel}(\{\vec{\rho}_i\}), \qquad (92)$$

where $\Psi_{\text{c.m.}}(\mathbf{R}) = (Z/d)^L e^{-|Z|^2/2d^2}$ is the wave function of the center of mass $\mathbf{R} = \sum_{i=1}^{N} \mathbf{r}_i / N$ and $\phi_{rel}(\{\vec{\rho}_i\}) = K \exp^{(-\sum_{i=1}^{N} |\vec{\rho}_i|^2/2d^2)}$ is a product of single-particle states, where the particle coordinates are $\{\vec{\rho}_i = \mathbf{r}_i - \mathbf{R}\}$. Given this structure, it is natural to refer to this state as being singly condensed in the center-of-mass frame. Here, we show that this exact ground state can also be written as a symmetry average of broken symmetry states of the form of Eq. (88), as in many of the previous examples.

Attractive interactions favor particles clumping together. In homogeneous systems, such clumping leads to collapse. In the fast rotating limit, however, the analyticity of the wave function in the lowest Landau level and the Gaussian factor impose strong constraints on the degree of localization possible. The most localized state is $u_0(\mathbf{r})$, a Gaussian of a width given by the trap length d. A similar localized wave packet at location $a \equiv a_x + i a_y$ is

$$\phi_a(\mathbf{r}) = e^{(2a^* z - |z|^2 - |a|^2)/2d^2} / \sqrt{\pi} d \qquad (93)$$

$$= e^{-|z-a|^2/2d^2} e^{(a^* z - a z^*)/2d^2} / \sqrt{\pi} d, \qquad (94)$$

which gives

$$|\phi_a(\mathbf{r})|^2 = e^{-|z-a|^2/2d^2} / \pi d^2. \qquad (95)$$

A many-body coherent state formed from these single-particle states is

$$\Psi(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \prod_{i=1}^{N} \phi_a(\mathbf{r}_i). \qquad (96)$$

This state carries angular momentum $\hbar L = \langle \Psi | L_z | \Psi \rangle = \langle \Psi | \int d\mathbf{r}\, \psi^\dagger(\mathbf{r}) \mathbf{r} \times (\hbar \nabla / i) \psi(\mathbf{r}) | \Psi \rangle$, with

$$L = N \int d\mathbf{r} |\phi_a(\mathbf{r})|^2 ([r/d]^2 - 1) = N |a/d|^2. \qquad (97)$$

The single-particle density matrix is

$$\langle \psi^{\dagger}(\mathbf{r}')\psi(\mathbf{r})\rangle = N\phi_a^*(\mathbf{r}')\phi_a(\mathbf{r}), \tag{98}$$

in which

$$\langle a_m^{\dagger}a_n\rangle = \pi d^2 N u_m^*(\mathbf{a})u_n(\mathbf{a}) = N\left(\frac{L}{N}\right)^m \frac{e^{-L/N}}{m!}. \tag{99}$$

From this degenerate set of coherent states we construct an eigenstate of angular momentum by taking the superposition

$$\frac{1}{d^{L+2}N^L\sqrt{L!}}\int \frac{d^2a}{2\pi}e^{-|a|^2/2d^2}a^L|a\rangle = \frac{Z^L \exp\left(-\sum_{i=1}^{N}|z_i|^2/2d^2\right)}{\sqrt{L!(\pi d^2)^N}}, \tag{100}$$

where $Z=\frac{1}{N}\sum_{i=1}^{N}z_i/d$.

3. Fragmentation in a rotating Bose gas with repulsive interactions

Similar fragmentation is found in the repulsive case, which for $L \ll N\hbar$ is described by a vortex lattice. A general mean-field state is of the form Eq. (86), with $w(\mathbf{r})=f(z)e^{-|z|^2/2d^2}$,

$$f(z) = \prod_a (z-a), \tag{101}$$

where the zeros a are the locations of the vortices [48]. The angular momentum carried by this state can be obtained by noting that within the lowest Landau level,

$$\langle L_z\rangle = \hbar\langle[(r/d)^2 - 1]\rangle, \tag{102}$$

which increases as the density of vortices increases. Following the arguments which we used for the attractive case, we are once again led to a fragmented condensate (see [49] for numerical studies).

The rotating Bose gas with repulsive interactions is, however, much richer than that with attractive interactions, since the mean-field picture breaks down in a fundamental way at large values of angular momentum. One sees this breakdown by first noting that with increase of the angular momentum, the vortex density increases and the particle density decreases [48,50,51]. Eventually, the density of vortices is comparable to the density of particles and one can significantly improve the energy of the system by correlating the positions of the vortices with the positions of the particles. These correlations cannot be captured by any simple manipulation of the mean-field states; the general description of the system with large angular momentum is quite complicated [24,25].

Despite this complexity, there is a limit in which we can find the exact ground state. Imagine that the system is rotating with Ω sufficiently close to $\omega_\perp$ that the energy spacing within the lowest Landau level may be treated as a perturbation. One would find the ground state by first minimizing the interaction energy in the lowest Landau level and then perturbatively including the level spacing. For short-ranged interactions the energy is minimized by any state for which the wave function vanishes whenever two particles come together. Degenerate perturbation theory then says that the lowest energy state of all these wave functions is the one with lowest angular momentum. In the lowest Landau level, the lowest-angular-momentum bosonic wave function which vanishes when two particles touch is the $\nu=1/2$ Laughlin state

$$\Psi([\mathbf{r}]) = \prod_{i>j}(z_i - z_j)^2 e^{-|z_i|^2/2d^2}. \tag{103}$$

For an infinite system, one readily sees that

$$\langle a_m^{\dagger}a_n\rangle = (1/2)\delta_{mn} \tag{104}$$

for all m. Not only are there no eigenvalues of order N, but they are all less than unity. All traces of the conventional Bose-Einstein condensation are obliterated.

IV. DISCUSSION

A. Salient features

The examples shown in Sec. III share a number of common features. Many of the fragmentation processes we have discussed, such as those in Eqs. (46), (70), and (100), can be described by a family of quantum states that are weighted averages of broken symmetry states over the space of broken-symmetry, typically of the form

$$|\Psi(\lambda)\rangle = \int d\chi\, W(\chi,\lambda)|\chi\rangle_c, \tag{105}$$

where $|\chi\rangle_c$ is a coherent state with broken-symmetry parameter χ (e.g., a spin direction), W is a distribution function in the space of broken symmetry, and λ is the parameter that controls the fragmentation of the system. If, as λ changes—say, from 0 to 1—W changes from a distribution sharply peaked at χ_0 to a completely uniform distribution, Eq. (105) will evolve from the coherent state χ_0 to a fragmented state. Such changes in the distribution function reflect a growing fluctuation about the initial coherent state χ_0 singled out by a tiny symmetry-breaking field in the coherent regime.

For example, in the case of repulsive Bose gas in a double well, χ is the relative phase between the condensates in the two wells and $\chi_0=0$. In the spin-1 Bose gas, χ is the vector $\hat{\mathbf{n}}$ and χ_0 is a direction normal to an infinitesimal external magnetic field. In the rotating attractive Bose gas, χ is the location in space of the coherent state and χ_0 is the equilibrium location determined by, e.g., a weak potential that breaks rotational symmetry. Exactly how the fluctuations about χ_0 grow depends on the specific dynamics of the system. Large fluctuations about χ_0 directly reflect the competition between different degenerate states for Bose-Einstein condensation. This competition is the ultimate cause of fragmented structure, as discussed in the Introduction.

Another important feature of states like those in Eq. (105) is that, as a consequence of bosonic enhancement, the range of the control parameter λ over which the system switches from a coherent to a fragmented state shrinks with particle number N. For a repulsive Bose gas in a double well, as discussed in Sec. III A 2, the transition from a fragmented to a coherent takes place around $t/U > 1/N$ [38]. A similar situation occurs in the spin-1 Bose gas, where the singlet (fragmented) state gives way to a coherent state for field gradient $G > 1/N$ [13]. Since the window for fragmentation vanishes as N increases, fragmented condensates will not be realized in condensed matter systems, with $N \sim 10^{23}$ particles. On the other hand, in mesocopic systems like quantum gases with typically $N \sim 10^{6}$ particles, fragmented states can exist in parameter ranges accessible to experiment.

Although we have paid special attention to the class of states in Eq. (105), we must emphasize that other fragmentation processes do not lead to states that are most naturally expressed in this manner—e.g., the Schrödinger-cat family in Eq. (47). One can have distinct fragmented states, such as the Fock state and the Schrödinger-cat state, with entirely different properties but identical single-particle density matrices. Such different states are therefore indistinguishable within the Penrose-Onsager scheme. To tell them apart, it is necessary to examine second-order correlation functions such as number fluctuations. One needs to examine even higher-order correlation functions to distinguish the fragmented states of more complicated systems, such as those of $F=2$ Bose gases of ^{87}Rb and the recently realized spin-3 Bose gas of ^{52}Cr [52]. In general, one can expect a large variety of fragmented states, differing from each other by high-order correlation functions.

The relative number fluctuations of a fragmented state (such as those in the pseudospin-1/2 and spin-1 Bose gases discussed in Sec. III) are a measure of the stability of the state. Huge fluctuations, such as in the Schrödinger-cat state and the singlet state of the spin-1 Bose gas, indicate that the system is easily damaged by external perturbations. Consider, for example, a perturbation of the form $H' = \eta c^{\dagger} a_1 + \text{H.c.}$, where a_1 is a boson in one of the wells in the double-well example or a boson in the $S_z=1$ spin state of a spin-1 Bose gas, $c^{\dagger}$ adds a particle in a different atomic state of the same boson (e.g., a plane-wave state in the background gas), and η is a very small parameter. The perturbation H' acting on the Fock state $a_1^{\dagger N/2} a_2^{\dagger N/2}|0\rangle$ simply changes it to another Fock state $c^{\dagger} a_1^{\dagger N/2-1} a_2^{\dagger N/2}|0\rangle$, which also has zero number fluctuations $\langle N_1^2 - \langle N_1\rangle^2\rangle = 0$. In contrast, H' acting on the Schrödinger-cat state $(a_1^{\dagger N} + a_2^{\dagger N})|0\rangle$ collapses it into the state $c^{\dagger} a_1^{\dagger N-1}|0\rangle$, immediately reducing the enormous number fluctuations of the Schrödinger-cat state to zero. More generally, if the cat state $|\Psi_{\text{Cat}}\rangle = |\Psi_1\rangle + |\Psi_2\rangle$ is a sum of two Gaussians in number space, as shown in Eq. (47), with one Gaussian ($|\Psi_1\rangle$) peaked at $|N_+, N_-\rangle$ and the other ($|\Psi_2\rangle$) peaked at $|N_-, N_+\rangle$, where $N_+ + N_- = N$, and $N_+ \gg N_-$, the action of H' on $|\Psi_{\text{Cat}}\rangle$ considerably enhances $|\Psi_1\rangle$ and suppresses $|\Psi_2\rangle$.

The large effect of the small perturbation H' is due to bosonic enhancement, which gives $H'|N_+, N_-\rangle = \sqrt{N_+} c^{\dagger}|N_+ - 1, N_-\rangle$ and $H'|N_-, N_+\rangle = \sqrt{N_-} c^{\dagger}|N_- - 1, N_+\rangle$. Since $N_+ \gg N_-$, the norm of $H'|N_-, N_+\rangle$ is considerably smaller than that of $H'|N_+, N_-\rangle$. As a result, $H'|\Psi_{\text{Cat}}\rangle \approx H'|\Psi_1\rangle \approx \sqrt{N_+} c^{\dagger}|N_+ - 1, N_-\rangle$, which is no longer a Schrödinger-cat state. In a spin-1 Bose gas, a similar action changes the singlet state $(2a_1^{\dagger} a_{-1}^{\dagger} - a_0^{\dagger 2})^{N/2}|0\rangle$ to one with much smaller relative number fluctuations.

Since Schrödinger-cat-like states can collapse into Fock states with the slightest perturbation and Fock states can easily be reassembled into a single condensate by any small amount of tunneling between different fragmented pieces, why should one bother with fragmented states? Is fragmentation relevant? The point, as mentioned before, is that even though fragmented states cannot be realized in macroscopic systems, the situation is different for mesoscopic systems like trapped quantum gases. The huge reduction in particle number considerably relaxes the constraint of formation of Fock states and Schrödinger-cat-like states, and fragmented ground states become realizable.

The phenomenon of fragmentation becomes even richer if it takes place in both real and spin space, such as with high-spin bosons in an optical lattices close to the Mott limit. The combined effect of spin degeneracy and spatial degeneracy (due to different isolated wells) produces a great variety of quantum phases as the spin of boson increases. In addition, the singlet ground state of a spin-1 Bose gas can also be viewed as a "resource" for singlets and may therefore be useful in developing quantum teleportation protocols in optical lattices with spin-1 bosons. Our understanding of the properties of fragmented states, and in particular their dynamics, is at such an early stage that it leaves open considerable room for inventive ideas, which is where the excitement lies.

We stress that we have described only the simplest types of fragmented states, and in systems with higher degrees of degeneracies one finds fragmented states which are not simply "Fock like" or "Schrödinger cat like." As discussed in Sec. III C 3, one can even find states, such as the $\nu=1/2$ Laughlin state, which contain no trace of condensation. In that example, there are no large eigenvalues singled out in the density matrix. Instead, all the eigenvalues are identical and of order unity.

B. Experimental consequences

At present, experiments on condensate fragmentation are limited primarily to double-well [33] and optical-lattice systems. The latter is the many well generalization of the double well case, and fragmentation there corresponds to the superfluid-insulator transition [23]. In all these experiments, interactions between bosons are repulsive, so the fragmented condensates are of the Fock type. There have not yet been any experiments on Bose gasses with attractive interaction in double wells, in which case the fragmented condensates would be Schrödinger cat like. Experimentally, Schrödinger-cat states can be easily distinguished from Fock states because measurement processes typically affect these states dramatically differently. For example, consider acting with the number operator for well 1, $\hat{N}_1 = a_1^{\dagger} a_1$, on the the ground state $|G\rangle$. In Fock states, $|G\rangle = |N/2, N/2\rangle = |F\rangle$ and the resulting state $\hat{N}_1|F\rangle = (N/2)|F\rangle$ remains the Fock state. In contrast, in a Schrödinger-cat state $|\text{Cat}\rangle = (|N,0\rangle + |0,N\rangle)/\sqrt{2}$, we have $\hat{N}_1|\text{Cat}\rangle = (N/\sqrt{2})|N,0\rangle$. The cat state is collapsed into a condensate on the left well. Were one to measure particle numbers in wells 1 and 2 in sequence, the outcome for the Fock state would be $N/2$ for each measurement since the Fock state $|F\rangle = |N/2, N/2\rangle$ is an eigenstate of $\hat{N}_1$ and $\hat{N}_2$; in a cat state, on the other hand, the measurement of $\hat{N}_1$ would yield either N or 0 with equal probability, since there is a 50-50 chance of realizing $|N,0\rangle$ or $|0,N\rangle$ in a measurement. However, once a nonzero outcome of N_1 is obtained, a subsequent measurement of $\hat{N}_2$ would yield zero identically.

While a Fock state can be easily distinguished from a Schrödinger-cat state, to distinguish a Fock state (for example, $|N/2,N/2\rangle$) from the corresponding coherent state $[(e^{-i\phi/2}a_1^\dagger+e^{i\phi/2}a_2^\dagger)^N|0,0\rangle/\sqrt{2^N N!}]$ is nontrivial. From a condensed matter viewpoint this result is counterintuitive, since one would have thought systems with phase coherence are fundamentally different from those without. For example, a coherent condensate in a double well will have Josephson oscillations (i.e., an oscillating particle current between the two wells) whereas a Fock state will not. While this is true, the situation in quantum gases is tricky because information on particle currents is not easily accessed in typical experiments. Instead, one studies coherence between condensates through their interference. The subtleties of such interference measurements were beautifully illustrated by Castin and Dalibard [30], who considered an idealized experiment where one sequentially removes particles from the system. They imagine sending each particle into a classical measurement apparatus which measures "the pseudospin S_x operator"; i.e., this apparatus reports that the particle is either in a symmetric or antisymmetric superposition of being in the right and left wells. As Castin and Dalibard argue, since there are quantum mechanical correlations in the many-body state, each time one measures the state of one particle, the many-body state is projected into one which is consistent with that measurement. They show that sequentially measuring a quantity which is sensitive to the relative phase (such as pseudospin S_x) projects the system into a state with phase coherence, even if it were initially incoherent.

The subtleties of interference experiments can also be understood without discussing projections. Imagine repeating Castin and Dalibard's thought experiment on several copies of a state. Each time that one runs the experiment, one sequentially removes two particles, measuring if each particle is in a symmetric or antisymmetric superposition of being in the right and left wells. Ensemble averaging allows one to measure the correlation functions $\langle b_\sigma^\dagger b_\sigma b_\tau^\dagger b_\tau\rangle$, where $\sigma,\tau=\pm$ and $b_\pm=(a_1\pm a_2)/\sqrt{2}$ is an annihilation operator of a particle in a superposition of being in each well. Naively, one would expect coherence to be encoded in $\langle|a_1^\dagger a_2|^2\rangle=(1/4)\langle|b_+^\dagger b_+ - b_-^\dagger b_-|\rangle$. However, as shown in Sec. II B, as long as the number of particles is large ($N\gg 2$) such expectation values are identical for an ensemble of coherent states and for the fragmented state.

A particularly dramatic manifestation of this equivalence is seen in actual experiments [31,53], where instead of removing particles from the wells, one turns off the traps and allows the cloud to expand ballistically. Naively one expects to see a matter-wave interference pattern in the overlap region if the wells are coherent and no interference pattern if they are incoherent. In fact, one always sees an interference pattern. Working in the Heisenberg picture, neglecting interactions during expansion, and neglecting the initial spatial extent of the cloud in each well, the field operator at time t is

$$\psi(\mathbf{r},t)=A[e^{i|\mathbf{r}-\mathbf{r}_1|^2/(2d_t^2)}a_1+e^{i|\mathbf{r}-\mathbf{r}_2|^2/(2d_t^2)}a_2], \tag{106}$$

where the a_j are zero-time annihilation operators in each well the $\mathbf{r}_j$ are the locations of the wells, $d_t^2=\hbar t/m$, and $A=(2\pi d^2)^{-3/2}/\sqrt{2}$. After expansion, a photograph is taken of the atomic cloud. An interference pattern in this image corresponds to modulations of the density-density correlation function $\rho_2(\mathbf{s})=\frac{1}{\Omega}\int d^3\mathbf{r}\langle\psi^\dagger(r+s)\psi(r+s)\psi^\dagger(r)\psi(r)\rangle$, where Ω is the volume of space. Carrying out the spatial average, one finds

$$\rho_2(\mathbf{s})=A^4\{\langle(a_1^\dagger a_1+a_2^\dagger a_2)^2\rangle+2\langle a_2^\dagger a_1^\dagger a_1 a_2\rangle\cos[\mathbf{s}\cdot(\mathbf{r}_1-\mathbf{r}_2)/d_t^2] + O(1/N)\}, \tag{107}$$

where N is the number of particles. To leading order in N this correlation function is identical for both Fock states and coherent states, implying that they both produce high-contrast interference. This result is an example of the Hanbury Brown-Twiss effect, whereby classical sources produce intensity interference patterns [36].

The only macroscopic difference between the interference pattern from the Fock and coherent states is that for the coherent state the relative phase between the two wells sets the location of the fringes. In the Fock state the fringes appear at a random location. Recently Hadzibabic *et al.* [53] experimentally verified this prediction in a multiwell interference experiment.

An alternative way to determine the nature of the ground state in a double well would be to measure the number fluctuation $(\Delta N_1)^2=\langle(\hat{N}_1-\hat{N}_2)^2\rangle$, which will be of the order of N for a coherent state but of order unity for a Fock state. Fluctuations ΔN_1 below that of a coherent state $\sqrt{N}$ indicate that the system is more Fock like. Here one is trying to determine the Fano factor $F=(\Delta N_1)^2/N$ of the state, which is a measure of "squeezing."

Similar considerations can be made in thinking about a spin-1 Bose gas, where the system can evolve from a singlet state $|S=0\rangle\propto(\mathbf{A}^\dagger\cdot\mathbf{A}^\dagger)^{N/2}|0\rangle\propto\int d\hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{A}^\dagger)^N|0\rangle$ to a coherent (or polar) state $|\hat{\mathbf{n}}\rangle\propto(\hat{\mathbf{n}}\cdot\mathbf{A}^\dagger)^N|0\rangle$ as the external parameters (such as magnetic field gradient, etc.) vary. We again ask how one determines the nature of the ground state. A measurement of particle number would lead to "measurement induced coherence" [30] as in a fragmented condensate in a double well. That is, even if one starts with a singlet state $|S=0\rangle$, a typical measurement process will select a random direction $\hat{\mathbf{n}}_0$ in spin space, so that as further measurements are made, the system is changed more and more toward the polar state $|\hat{\mathbf{n}}_0\rangle$. As in the double-well case, to determine the nature of the ground state, one can study the higher-order correlation functions. The natural physical quantity is then the spin nematicity

$$\mathcal{N}_{ij}\equiv\left\langle\tfrac{1}{2}S_iS_j+\tfrac{1}{2}S_jS_i-\tfrac{1}{3}\delta_{ij}\mathbf{S}^2\right\rangle, \tag{108}$$

which can be accessed by light scattering [54]. In the singlet state, $\mathcal{N}_{ij}$ vanishes identically, whereas $\mathcal{N}_{ij}=N(\delta_{ij}/3-n_in_j)$ for the polar state. That is, as the system evolves from singlet polar, the nematic tensor $\mathcal{N}_{ij}$ grows from zero to a uniaxial tensor of order N.

C. Role of temperature

Up to this point, the examples of fragmentation given in this paper have focused on the case where interaction-driven quantum fluctuations break up the condensate. It is intuitively clear that thermal fluctuations can play a similar role. A trivial example is given by a noninteracting Josephson junction, governed by the Hamiltonian

$$H = - t(a^\dagger b + b^\dagger a). \tag{109}$$

This Hamiltonian is diagonal in the basis of symmetric and asymmetric states, for which the creation operators are $(a^\dagger + b^\dagger)/\sqrt{2}$ and $(a^\dagger - b^\dagger)/\sqrt{2}$, so the single-particle density matrix is diagonal in this basis. At zero temperature only the symmetric state is occupied—a single condensate. At very large temperature each of these states is equally occupied so the system is fragmented. For intermediate temperatures, the occupation of each of these states is

$$n_{\mathrm{a}}^{\mathrm{s}} = \frac{\sum_{m=-N/2}^{N/2} (N/2 \pm m) e^{2t\beta m}}{\sum_{m=-N/2}^{N/2} e^{2t\beta m}} = \frac{N}{2} \mp \left(\frac{N+1}{2} \coth[\beta t(N+1)] - \frac{1}{2}\coth(\beta t) \right). \tag{110}$$

The upper signs denote the symmetric state (s) and the lower the antisymmetric state (a). The crossover between the singly condensed state at $T=0$ and the fragmented state at $T \gg t$ is smooth. Extensions of this argument are relevant for spinor condensates in which one can, in principle, have a hierarchy of transition temperatures, where the $k=0$ mode becomes macroscopically occupied below some temperature T_c, but order in the spin-channel does not occur until a lower temperature [55].

A vortex lattice provides a qualitatively different example of how a finite temperature fragments a condensate. Imagine a bucket of ^{4}He rotating at frequency Ω. The ground state of the system contains a triangular array of vortices with n_{v} vortices per unit area. At finite temperature the vortex lattice is thermally excited, giving rise to a decay in the phase correlations across the sample. If we let $\psi(r)$ be the superfluid order parameter coarse-grained on a scale large compared to the vortex spacing, then according to [50,56], the correlation function $\langle \psi^*(r')\psi(r')\rangle$ decays as $|r-r'|^{-\eta}$ for large separations. The exponent $\eta = 1/(3\pi^2 \rho r_{\mathrm{v}} \Lambda^2)$ is proportional to the ratio of the distance between particles to the distance between vortices. Here the particle number density is ρ, the thermal wavelength is $\Lambda = \sqrt{2\pi/mk_bT}$, and the distance between vortices is r_{v}. This algebraic decay of correlations in real space corresponds to an algebraic decay in momentum space. For $\eta < 3$, the occupation of the $k=0$ mode scales as $N_0 \sim L^{3-\eta}$, and as as $k \to 0$, the occupations of the $k \neq 0$ modes scale as $N_k \sim k^{\eta-3}$. Thus the number of macroscopically occupied modes scales as L^η, and for carefully chosen L and η a fragmented state can result.

For a typical helium experiment $\eta < 10^{-8}$, and the depletion caused by this effect is negligible. Experiments on alkali-metal gases—e.g., [57,58]—create vortex lattices with $r_{\mathrm{v}} \sim 5\ \mu$m and a particle density $\rho \sim 10^{14}$ cm^{-3}, for which $\eta \sim 10^{-3}$, also a minor correction. Experiments with smaller vortex lattices in Paris [59] have a similar vortex spacing $r_{\mathrm{v}} \sim 2\ \mu$m and density $\rho \sim 10^{14}$ cm^{-3}, yielding a comparable value for η. Although vortex lattices in current experiments are not thermally fragmented, there does not appear to be any fundamental impediment to making η larger.

D. Other recent work

Not surprisingly, given the fascinating nature of fragmented states, several recent papers (all cited in the relevant sections of this paper) have been investigating models similar to those discussed here. As a guide to the reader, we briefly summarize these works, presenting them in roughly the same order as they appear in the main text.

The two-state system was explored by Nozières [9], who pointed out the existence of fragmentation in that system. The generic stability of a two-state fragmented condensate was considered by Rokhsar [16]. One realization of this model would be to place atoms in bona fide double-well potentials. Due to mode mixing, such a system is not identical to a simple two-mode model, and several authors have explored more realistic models. Spekkens and Sipe [17] used a variational approach to compare fragmented and singly condensed states in such potentials. More recently, Streltsov and Cederbaum [60], along with Streltsov, Cederbaum, and Moiseyev [61], produced similar results through a multimode mean-field theory. The evolution of the ground state from a Fock state to a coherent state and from a coherent state to a Schrödinger-cat state was studied by Ho and Ciobanu [38]. Experiments have recently produced a condensate confined in a double-well trap [33].

The fragmented nature of the ground state of the spin-1 Bose gas was first noted by Nozières and Saint James [8]. Later studies by Law, Pu, and Bigelow [46] investigated the ground-state properties and the spin dynamics in terms of the basis used in Eq. (74). Ho and Yip [13] showed that the fragmented singlet state has huge fluctuations; they also showed the relation between this fragmentation and spontaneous symmetry breaking. They explicitly showed how magnetic fields and field gradients drive the system into a singly condensed state. Similar considerations were addressed by Koashi and Ueda [12]. Javanainen discussed issues involving the measurement of the fragmented spin-1 ground state [18].

The rotating attractive gas was first studied by Wilkin *et al.* [10], who noted that the ground state was fragmented. The connection between this fragmentation and symmetry breaking was first discussed by Pethick and Pitaevskii [47].

There have been extensive studies of the properties of the rotating repulsive gas in the lowest Landau level—many of which have focused on the structure of the order in the single-particle density matrix. These works have mainly used a combination of exact diagonalization and variational techniques. A particularly relevant paper is the exact diagonalization study of Liu *et al.* [49], which focuses on the symmetry-breaking nature of the vortex states. A related paper by Jackson *et al.* [62] compares mean-field and exact wave functions at various values of the angular momentum. The Goldstone mode associated with vortex nucleation was discussed by Ueda and Nakajima [63].

Fragmentation occurs in "clumped" bosonic systems with attractive interactions. Ueda and Leggett used a two-mode approximation [34] to study fragmentation and soliton formation in a one-dimensional attractive Bose gas. A quite thorough comparison of mean-field theory and exact diagonalzation is found in the articles by Kanamoto *et al.* [64], as well as the closely related work of Kavoulakis [65] and the detailed studies of Alon *et al.* [66]. Montina and Arecchi [67] use a Monte Carlo scheme to investigate the degree of fragmentation of this system. In three dimensions, Elgaroy and Pethick [15] showed that a harmonically trapped gas of atoms with attractive interactions does not form a stable fragmented state.

Other systems with symmetry breaking and fragmentation include phase-separated two-component gases [68] and rotating gases during a phase-slip event [35]. Boson ground states where the condensate is broken into a macroscopic number of pieces include the Mott insulator [23], fractional quantum Hall states [24,25], and low-dimensional Bose gas [28,69]. Aspects of the dynamics of regaining phase coherence among condensates are discussed by Yi and Duan [70], a paper closely related to discussion of how the measurement process is influenced by fragmentation [29,30].

Finally we mention the lecture notes produced by Castin and Herzog [14], which lucidly introduces fragmentation and analyzes the case of spin-1 bosons and of the one-dimensional attractive gas. They included an extended discussion of the role of symmetry breaking.

V. SUMMARY

Bose-Einstein condensation is remarkably robust. It is therefore exciting to search for zero-temperature bosonic states which are not condensed and to understand the process that breaks up the condensate. Fragmentation, where the condensate breaks up into a few pieces, is the first step in this journey, which eventually ends at strongly correlated states possessing no trace of condensation.

We have described some of the canonical models which have fragmented ground states and extracted their properties. We see that there is a rich variety of fragmented ground states. Often fragmentation is associated with restoring a broken symmetry. Sometimes it is accompanied by order in higher-order correlation functions. The one unifying feature appears to be a combination of near-degeneracies and interactions. The higher the degree of degeneracy, the more fragmented the condensate may become.

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