Time-dependent Ginzburg-Landau theory for atomic Fermi gases near the BCS-BEC crossover

M. Machida
CCSE, Japan Atomic Energy Research Institute, 6-9-3 Higashi-Ueno, Taito-ku, Tokyo 110-0015, Japan and CREST(JST) 4-1-8 Honcho, Kawaguchi, Saitama 332-0012, Japan

T. Koyama
IMR, Tohoku University, 2-1-1 Katahira Aoba-ku, Sendai 980-8577, Japan

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We construct a time-dependent Ginzburg-Landau (TDGL) theory for the superfluid atomic Fermi-gases near the Feshbach resonance from the fermion-boson model on the basis of the functional integral formalism. We show that the GL coefficients in the TDGL equation are complex numbers except in the BEC limit. The complex TDGL equation describes both damping and propagating dynamics, which leads to very rich superfluid dynamics near the Feshbach resonance. We predict multiple plane-wave modes corresponding to the in-phase and out-of-phase like oscillations between fermionic and bosonic superfluids and a galaxylike spiral pattern for a vortex.

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I. INTRODUCTION

The superfluidity in the ultra-cold atomic Fermi gases [1–4] provides a useful testbed for studying high-temperature superconductivity in strongly correlated fermionic systems, since the large value of the normalized critical temperature $T_c/E_F \sim 0.2$ indicates that these atomic gases can be classified as extremely high-T_c superfluids. These superfluid systems also exhibit a crossover from the weak-coupling BCS state to the strong-coupling BEC one [5–7], depending on the strength of the pairing that is controllable by means of the field-dependent Feshbach resonance. In this paper we develop a Ginzburg-Landau (GL) theory for the superfluid atomic Fermi gases showing the BCS-BEC crossover.

The Ginzburg-Landau (GL) theory has played an important role in the history of superconductivity research, because it has captured almost every unique feature that the superfluid exhibits macroscopically [8], though its mathematical framework is simple. Thus, it is reasonable to expect that the GL theory will also be a useful theoretical tool for studying macroscopically the dynamics of the superfluid atomic Fermi gases. To derive GL equations from a microscopic model in these systems is, therefore, one of the important issues. In this paper, we construct a two-component time-dependent Ginzburg-Landau (TDGL) theory applicable to the superfluid atomic Fermi gases in which BCS pairs and tightly bound diatomic condensate coexist.

In the superfluid atomic Fermi gases near the Feshbach resonance the strong attractive interaction is realized between fermion atoms, which can cause the BCS-BEC crossover [5–7]. To our knowledge the GL theory has not yet been fully studied in the BCS-BEC crossover regime except for a few pioneering works [9,10] and recent related ones [11,12] in the single-component fermion systems (single channel model). In this paper, following the procedure similar to Refs. [9,10] we construct the GL theory of the fermion-boson model (double-channel model) [5–7], which is regarded as one of the microscopic models for atomic Fermi gases near the Feshbach resonance, and derive the TDGL equations valid near T_c. The GL effective action in the fermion-boson model is expressed in terms of a functional of two kinds of order parameters, namely the ones for the BCS-like Cooper pairs and for the condensed diatomic molecules [5–7]. From the effective action, one can derive the coupled TDGL equations. In this paper we show that the coupled TDGL equations have a remarkable feature, that is, the GL coefficients, which are functions of the coupling constants, the chemical potential, and the threshold energy of the Feshbach resonance, are generally complex numbers in these systems. The dynamics of the superfluid atomic Fermi gases near the Feshbach resonance, then, can be described by the "coupled complex TDGL equation" with complex coefficients.

The complex TDGL equation has so far been extensively studied, since it can describe a large variety of nonequilibrium dynamics observed in physical systems [13]. For example, the equation has solutions giving spatially extended quasistable patterns such as sink and source solutions (spirals in 2D and filaments in 3D) and also dynamical structures corresponding to spatiotemporal chaos [13]. In general, the complex TDGL equation contains two kinds of competing dynamics, i.e., the conserved dynamics and the damped one, and the competition leads to very rich nonequilibrium phenomena. The competing dynamics in the atomic Fermi gases near the Feshbach resonance physically arises from the difference in the dynamics of the condensed molecular bosons and that of the BCS pairing fermions; that is, the condensed bosons obey primarily the propagating dynamics of the Gross-Pitaevskii type [14], whereas the pairing fermions follow the damping one like that of an electron-pair in a superconductor near T_c [15,16]. Thus, one can expect that the nonequilibrium dynamics in the superfluid atomic Fermi gases reveals a very rich variety in the quenching processes. However, we note that such rich dynamics cannot be described by the single-channel model since the TDGL equation derived from the single-channel model near T_c is only the damped one, except for the BEC regime. This is a marked difference between the single and the double-channel models. We point out that the difference becomes more remarkable in narrow Feshbach resonance, in which the molecular boson population is widely non-negligible.

Though the conventional TDGL equation in weak-coupling BCS superconductors [15,16] includes basically the temporal parts that lead to both damping and propagating motions of the order parameter [18], the equation is primarily the damping one with a tiny propagating fraction which is $\frac{T_c}{E_f}$ times less than the damped one [18]. This result comes from the fact that the propagating component originates from the particle-hole asymmetry in fermionic excitations, which is usually very small in BCS superconductors [18]. Thus, whether the propagating component exists or not in superconductors has been experimentally a subtle issue, though it has been regarded as being observable by flux-flow Hall effects in type II superconductors [19,20]. On the other hand, in the atomic Fermi gases near the Feshbach resonance, as shown in this paper, the propagating component plays an important role, since the condensed molecular bosons whose dynamics is primarily the conserved one intrinsically exist. In this paper, we clarify the dynamics of a mixture of gases composed of BCS-like fermion pairs and condensed molecular bosons near the Feshbach resonance, and discuss how its dynamics changes in the BCS-BEC crossover.

In neutral superfluid atomic gases the collective plane-wave motion known as the Goldstone mode exists. In the BCS superfluid gases this mode is located in the energy range below twice the superfluid energy-gap $2\Delta(T)$, as shown by Ohashi and Griffin in their explicit calculations for the fermion-boson model [17]. However, the TDGL theory close to T_c, which describes the low-energy dynamics up to linear in frequency ω, does not include the Goldstone mode in its solutions [15,16]. This is because the Goldstone mode is not well-defined at temperatures near T_c since the superfluid gap almost vanishes. Thus, one understands that the TDGL theory near T_c from the single-channel model describes only the dynamics of the Cooper-pair fluctuations, i.e., the process where Cooper pairs decay into the fermionic continuum which leads to the damped dynamics for the order parameter. On the other hand, in the fermion-boson model other processes, i.e., a Cooper pair changing into a diatomic molecule and its inverse one are incorporated. These processes take place without dissipation and their relaxation time is roughly given by $\tau_B \sim \hbar/g$, g being the coupling constant in the Feshbach resonance, as given in Sec. II. By noting that the relaxation time of the BCS pair fluctuations decaying into the fermionic continuum τ_F becomes very long near T_c (due to critical slowing down), one finds that there is a temperature region in which τ_F is larger than τ_B that always exists. In this region the pair fluctuation changes to a BEC molecule before decaying into two fermions. Thus, one can expect that the dynamics of the order parameters close to T_c is dominated by the dissipationless resonance modes. As shown in this paper, the dynamics of the superfluid order parameters including these processes can be described by the coupled complex TDGL equations.

II. FERMION-BOSON MODEL AND TDGL ACTION

Let us begin with the fermion-boson model. The Hamiltonian is given by [5–7]

$$
\begin{aligned}
H_{BF} = \int d\mathbf{r}\Bigg\{ & \psi_\sigma^\dagger(\mathbf{r},t)\left(-\frac{1}{2m}\nabla^2-\mu\right)\psi_\sigma(\mathbf{r},t) \\
& - U\psi_\uparrow^\dagger(\mathbf{r},t)\psi_\downarrow^\dagger(\mathbf{r},t)\psi_\downarrow(\mathbf{r},t)\psi_\uparrow(\mathbf{r},t) \\
& + \varphi_B^\dagger(\mathbf{r},t)\left(-\frac{1}{4m}\nabla^2+2\nu-2\mu\right)\varphi_B(\mathbf{r},t) \\
& + g[\varphi_B^\dagger(\mathbf{r},t)\psi_\downarrow(\mathbf{r},t)\psi_\uparrow(\mathbf{r},t)+\varphi_B(\mathbf{r},t)\psi_\uparrow^\dagger(\mathbf{r},t)\psi_\downarrow^\dagger(\mathbf{r},t)]\Bigg\},
\end{aligned} \tag{1}
$$

where $\psi_\sigma(\mathbf{r},t)$ and $\varphi_B(\mathbf{r},t)$ are the field operators of the fermionic atoms with pseudospin $\sigma=\uparrow,\downarrow$ and the quasimolecular bosons, respectively, μ is the chemical potential, and 2ν is the threshold energy of the Feshbach resonance. The last term with the coupling constant g in Eq. (1) describes the process arising from the Feshbach resonance, in which a boson is created from two fermionic atoms and vice versa [5–7]. The grand canonical partition function in this system is expressed in terms of the imaginary-time path integral as follows:

$$
Z=\prod\int \mathcal{D}\bar{\psi}_\sigma\mathcal{D}\psi_\sigma\mathcal{D}\bar{\varphi}_B\mathcal{D}\varphi_B e^{-S[\bar{\psi}_\sigma,\psi_\sigma,\bar{\varphi}_B,\varphi_B]}, \tag{2}
$$

with $S[\psi_\sigma,\bar{\psi}_\sigma,\varphi_B,\bar{\varphi}_B]$ being the Euclidean action defined as

$$
S=\int_0^\beta d\tau\int d\mathbf{r}\left[\bar{\psi}_\sigma\frac{\partial}{\partial\tau}\psi_\sigma+\bar{\varphi}_B\frac{\partial}{\partial\tau}\varphi_B+H_{BF}\right]. \tag{3}
$$

By use of the Hubbard-Stratonovich transformation the four-fermion term in Eq. (2) can be eliminated and the effective action is rewritten in terms of the auxiliary bosonic field, i.e., the gap function for the pairing fermions Δ and the molecular boson field φ_B as

$$
\begin{aligned}
S(\Delta,\varphi_B)=\int_0^\beta d\tau\int d\mathbf{r}\Bigg\{ & \frac{1}{U}|\Delta(\mathbf{r},\tau)|^2-\mathrm{Tr}\ln\mathbf{G}^{-1}-\bar{\varphi}_B(\mathbf{r},\tau) \\
& \times\left[-\partial_\tau+\frac{1}{4m}\nabla^2-(2\mu-2\nu)\right]\varphi_B(\mathbf{r},\tau)\Bigg\},
\end{aligned} \tag{4}
$$

where $\mathbf{G}^{-1}$ is the inverse Nambu propagator defined as

$$
\mathbf{G}^{-1}(\mathbf{r}\tau,\mathbf{r}'\tau')=\begin{pmatrix} -\partial_\tau+\frac{1}{2m}(\nabla)^2+\mu & \Delta+g\varphi_B \\ \bar{\Delta}+g\bar{\varphi}_B & -\partial_\tau-\frac{1}{2m}(\nabla)^2-\mu \end{pmatrix} \times\delta(\mathbf{r}-\mathbf{r}')\delta(\tau-\tau'). \tag{5}
$$

Note that the off-diagonal component in $\mathbf{G}^{-1}$ is given by the sum of the fermion-pair field Δ and the condensed boson field φ_B. Near T_c the effective action can be expanded in power of $\Delta+g\varphi_B$ up to the fourth order as

$$S_{eff}(\Delta + g\varphi_B) = S_{eff}(0) - \sum_{\mathbf{q},i\omega_l} \pi(\mathbf{q},i\omega_l)|\Delta(\mathbf{q},i\omega_l) + g\varphi_B(\mathbf{q},i\omega_l)|^2 + \frac{1}{2}b|\Delta(\mathbf{q},i\omega_l) + g\varphi_B(\mathbf{q},i\omega_l)|^4, \tag{6}$$

where we utilized the Fourier representation, i.e., $\mathbf{q}$ and ω_l are, respectively, the wave number and the bosonic Matsubara frequency. Here, we note $\Delta \sim g\varphi_B$ for both the broad ($g \gg E_F$) and narrow ($g < E_F$) Feshbach resonance cases as shown in Ref. [21]. This fact guarantees the validitiy of the expansion in powers of $\Delta + g\varphi_B$ and yields the fluctuation dynamics that is different from the single-channel model [9–12] for both the resonance cases as shown below. The function $\pi(\mathbf{q},i\omega_l)$ in Eq. (6) represents the correlation function of the Cooper-pair field [7,17],

$$\pi(\mathbf{q},i\omega_l) = \sum_{\mathbf{k}} \frac{1 - f(\xi_{\mathbf{k}+\mathbf{q}/2}) - f(\xi_{\mathbf{k}-\mathbf{q}/2})}{\xi_{\mathbf{k}+\mathbf{q}/2} + \xi_{\mathbf{k}-\mathbf{q}/2} - i\omega_l}, \tag{7}$$

with $\xi_{\mathbf{k}} \equiv \epsilon_{\mathbf{k}} - \mu$. Since the spatiotemporal variation of the bosonic field $\Delta + g\varphi_B$ is weak near T_c, one can expand Eq. (7) as

$$\pi(\mathbf{q},i\omega_l) = a + \frac{c|\mathbf{q}|^2}{4m} + di\omega_l + \cdots. \tag{8}$$

The coefficients a, b, c, and d correspond to the GL coefficients in the present TDGL theory, though the definition a is a little bit different from the conventional one [22]. In this paper we focus only on the coefficients a and d [23], which dominate the transition temperature T_c and the temporal dynamics of the superfluid.

Let us first study the coefficient a. Setting $\mathbf{q}=0$ and $\omega_l = 0$ in Eq. (7), we obtain the relation, $a \equiv \pi(0,0) = \Sigma_{\mathbf{k}}[\tanh(\beta\xi_{\mathbf{k}}/2)/2\xi_{\mathbf{k}}]$. In the static case the linearized GL equation for the uniform systems follows the effective action containing the coefficient a alone,

$$S_{eff}[2] = \left[\frac{1}{U} - a\right]|\Delta|^2 - a(\Delta^* g\varphi_B + \text{c.c.}) + [(2\nu - 2\mu) - ag^2]|\varphi_B|^2. \tag{9}$$

Then, from the saddle-point condition for Δ and φ_B, i.e., $\delta S_{eff}[2]/\delta\Delta^* = 0$ and $\delta S_{eff}[2]/\delta\varphi_B^* = 0$, we obtain the linearized gap equation to determine T_c,

$$1 = \left[U + \frac{g^2}{2\nu - 2\mu}\right]\sum_{\mathbf{k}} \frac{\tanh\frac{\beta\xi_{\mathbf{k}}}{2}}{2\xi_{\mathbf{k}}}. \tag{10}$$

This equation indicates that the BCS coupling constant U is renormalized as $U \to U_{eff}\{\equiv U + [g^2/(2\nu - 2\mu)]\}$. The gap equation (10) is the same as the mean-field one given in Refs. [7] and [17]. In addition, using the expansion (6) for the action in Eq. (4), one can integrate out Δ and φ_B in the partition function, which yields the thermodynamic potential Ω near T_c as $\Omega = \Omega_0 - \beta^{-1}\Sigma_{\mathbf{q},i\omega_n} \ln[1 - U_{eff}\pi(\mathbf{q},i\omega_n)]$. The thermodynamic potential Ω is related to the number of fermion atoms N and the chemical potential by the relation $N = -\partial\Omega/\partial\mu$. We note that since the thermodynamic function also coincides with the one given in Refs. [7,10,17], our GL theory can reproduce the same ν dependence of the critical temperature in the BCS-BEC crossover region as that in Refs. [7,10,17]. Furthermore, by using the gap equation (10), one can obtain the explicit expression for a close to T_c as

$$a(T \sim T_c) = \frac{1}{U + \frac{g^2}{2\nu - 2\mu}} + \sum_k \frac{1}{2k_B T_c^2} \sec^2 \frac{\xi_k}{2k_B T_c}(T - T_c). \tag{11}$$

From this expression one may understand how a varies with the experimentally tunable parameter ν and the parameters μ and T_c. The coefficients b [24] and c are also functions of these parameters [10].

Let us next study the coefficient d. The explicit expression for the third term on the right-hand side of Eq. (8) is given after the analytical continuation $i\omega_l \to \omega + i\delta$ as

$$d\omega \sim [\pi(0,i\omega_l) - \pi(0,0)]|_{i\omega_l \to \omega + i\delta} = \sum_{\mathbf{k}} \frac{-[1 - 2f(\xi_{\mathbf{k}})]\omega}{2(\omega + i\delta - 2\xi_{\mathbf{k}})\xi_{\mathbf{k}}}. \tag{12}$$

Note that d is generally complex, i.e.,

$$d = -P\sum_{\mathbf{k}} \frac{\tanh(\beta\xi_{\mathbf{k}}/2)}{2\omega\xi_{\mathbf{k}} - 4\xi_{\mathbf{k}}^2} + i\pi\sum_{\mathbf{k}} \delta(\omega - 2\xi_{\mathbf{k}})\frac{\tanh(\beta\xi_{\mathbf{k}}/2)}{2\xi_{\mathbf{k}}}, \tag{13}$$

where P denotes the Cauchy principal value. Suppose $\Delta \ll \omega \ll \mu$. In this region, Eq. (13) yields the approximate expression as

$$d \sim \sum_{\mathbf{k}} \frac{\tanh(\beta\xi_{\mathbf{k}}/2)}{4\xi_{\mathbf{k}}^2} + i\frac{\pi}{2\sqrt{2}}N(\epsilon_F)\frac{1}{\omega\sqrt{\epsilon_F}} \times (2\mu + \omega)^{1/2} \tanh\left(\frac{\omega}{4T_c}\right)\theta(\omega + 2\mu), \tag{14}$$

where ε_F is the Fermi energy and $\theta(\omega)$ is the Heaviside step function. In obtaining the imaginary part in Eq. (14) we performed the k integral, assuming the dispersion relation $\xi_k = k^2/2m - \mu$. We notice that $[\tanh(\beta\xi_{\mathbf{k}}2)/4\xi_{\mathbf{k}}^2]$ changes signs at $\epsilon_k = \mu$ and, as a result, the real part of d vanishes if $\mu \gg \Delta > 0$. In the BCS limit, this condition is well satisfied. In fact, the real part is estimated to be less than the imaginary part by the factor $\frac{T_c}{\epsilon_F}$ [18]. Thus, d can be considered to be purely imaginary in the BCS limit. In this case the conventional TDGL equation becomes the damped one. Next, consider the BEC region. In this region, since the chemical potential is negative, $\mu < 0$, the imaginary part of d vanishes if $\omega < 2|\mu|$, as seen in Eq. (14). Then, the real part of d dominates the time dependence of the TDGL equation, that is, we have a conserved TDGL equation like the GP equation [9,10]. Noting the above results, we summarize the approximate expression of d in the two limiting cases as follows:

$$d \to i\pi N(\epsilon_F)/8T_c[1 - i(2T_c/\pi\epsilon_F)] \text{ (BCS)}$$

$$d \to N(\epsilon_F)/8\sqrt{\epsilon_F|\mu|} \text{ (BEC)}. \tag{15}$$

In the BCS-BEC crossover region both the real and imaginary parts of d have finite values, since the breaking of the particle-hole symmetry is significant and the ratio T_c/ϵ_F is not small ($T_c/\epsilon_F \sim 0.2$). Thus, one can conclude that the GL coefficient d for atomic Fermi gases is a complex number in the BCS-BEC crossover region. As seen in the following, the complex d causes peculiar dynamics in the superfluid atomic Fermi gases.

III. TDGL EQUATIONS AND DYNAMICAL SOLUTIONS

Having the coefficients a and d derived in the above and b [24] and c, whose explicit expressions are not shown here, we can derive the coupled TDGL equations for Δ and φ_B, using the saddle-point approximation, i.e., $\partial S_{eff}/\partial\Delta^*=0$ and $\partial S_{eff}/\partial\varphi_B^*=0$, as follows:

$$-id\frac{\partial\Delta}{\partial t} = \left[-\frac{dg^2+1}{U}+a\right]\Delta + g[a+d(2\nu-2\mu)]\varphi_B + \frac{c}{4m}\nabla^2\Delta + \frac{g}{4m}(c-d)\nabla^2\varphi_B - b|\Delta+g\varphi_B|^2(\Delta+g\varphi_B) \tag{16}$$

$$i\frac{\partial\varphi_B}{\partial t} = -\frac{g}{U}\Delta + (2\nu-2\mu)\varphi_B - \frac{1}{4m}\nabla^2\varphi_B. \tag{17}$$

We find that the coefficients of the first, second, and fourth terms on the right-hand side of Eq. (16) are generally complex numbers except in the BEC limit, in which d is real. In the case of $g=0$ Eqs. (16) and (17) are decoupled and reduced to the conventional TDGL equation and the linearized GP equation as follows:

$$-id\frac{\partial\Delta}{\partial t} = -\left[\frac{1}{U}-a\right]\Delta + \frac{c}{4m}\nabla^2\Delta - b|\Delta|^2\Delta \tag{18}$$

$$i\frac{\partial\varphi_B}{\partial t} = (2\nu-2\mu)\varphi_B - \frac{1}{4m}\nabla^2\varphi_B. \tag{19}$$

In the BCS limit in which d can be regarded as pure imaginary, Eq. (18) describes only the diffusive motion of Δ, since the coefficients on the right-hand side of Eq. (18) are all real numbers. However, we note that in the presence of the coupling between the two condensates, i.e., $g\neq 0$, the GL coefficients becomes complex even in the BCS limit, as seen in Eq. (16). This result indicates that the dynamics of the superfluid Fermi gases is dramatically changed by the coupling with the bosonic condensate even in the BCS limit. This is a drastic difference between the single channel and the present model, which should be confirmed by experiments.

Let us now briefly discuss the dynamics brought about by Eqs. (16) and (17). Detail numerical studies based on these equations will be published elsewhere. First, we point out that Eqs. (16) and (17) have a plane-wave solution, $\Delta(\mathbf{r},t) = A_{\mathbf{k}}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)}$ and $\varphi_B(\mathbf{r},t)=B_{\mathbf{k}}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)}$, with the frequencies

$$\omega_{\mathbf{k}} \equiv \omega_{\mathbf{k}}[E_{\mathbf{k}}^b, f(\alpha_r)] = \frac{E_{\mathbf{k}}^b + f(\alpha_r) \pm \sqrt{[E_{\mathbf{k}}^b+f(\alpha_r)]^2 - 4\left[f(\alpha_r)E_{\mathbf{k}}^b - \frac{g^2}{U^2}\alpha_r\right]}}{2}, \tag{20}$$

where $f(\alpha_r)=\alpha_r(\frac{1}{U}-a)+\frac{c\alpha_r}{4m}|\mathbf{k}|^2$, $E_{\mathbf{k}}^b=\frac{g^2}{U}+(2\nu-2\mu)+\frac{\mathbf{k}^2}{4m}$, and $\alpha_r=\frac{d_r}{d_r^2+d_i^2}$, with d_r and d_i being, respectively, the real and imaginary parts of d. In the case of $g \ll U$, we reach two simple solutions with $\omega_{\mathbf{k}}^{\text{in}}=f(\alpha_r)$ and $\omega_{\mathbf{k}}^{\text{out}}=E_{\mathbf{k}}^b$, where $\omega_{\mathbf{k}}^{\text{in}}$ ($\omega_{\mathbf{k}}^{\text{out}}$) corresponds to the eigenfrequency of the in-phase (out-of-phase) motion of the mixture. In the BCS region, where d is pure imaginary, i.e., $\alpha_r=0$, we obtain more simple solutions with $\omega_{\mathbf{k}}^{\text{in}}=0$ and $\omega_{\mathbf{k}}^{\text{out}}=E_{\mathbf{k}}^b$ from Eq. (20). The eigenvector with the finite eigenfrequency $\omega_{\mathbf{k}}=E_{\mathbf{k}}^b$ satisfies the relation, $A_{\mathbf{k}}=-gB_{\mathbf{k}}$, which describes the damping-free out-of-phase mode in a mixture of the diatomic molecule and BCS-like fermion-pair condensates. Microscopically, such an oscillation arises from the decay process without energy loss in which fermion Cooper-pair fluctuations converts into diatomic molecules, and vice versa. Close to T_c, since the relaxation time of the BCS-pair fluctuations becomes very long, the pair fluctuation can change to a BEC molecule in addition to the simple decay into two fermions. This is in contrast to the consequence of the mode analysis for the TDGL equation [Eq. (18)] derived from the single-channel model, in which the channel between the pair fluctuation and the BEC molecule does not exit and only the simple decay process dominates the dynamics in the BCS regime. The existence of the mode described above is a crucial difference between the single channel and the present models. Also, we note that the eigenfrequency $\omega_{\mathbf{k}}$ lowers as the threshold energy 2ν decreases, which indicates that the conserved dynamics is easier to see when one approaches the BCS-BEC crossover region. This is also an important consequence from a viewpoint of the experimental observation.

Equations (16) and (17) have also a plane-wave solution corresponding to the damping-free small oscillation mode around the static solution in $T<T_c$, which has the form $\Delta=A_1+A_2(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)}$ and $\varphi_B=B_1+B_2(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)}$, where A_1 and B_1 are the real static solutions of Eqs. (16) and (17) and satisfy the relation $|A_2|,|B_2| \ll A_1, B_1$. The eigen frequencies of this plane-wave mode coincide with the expression given in Eq. (20), in which $f(\alpha_r)$ is replaced with $f(\alpha_r)+\alpha_r b(A_1+B_1)^2$. Then, in the BCS region with $\alpha_r=0$, the same out-of-phase oscillation as shown above appears. Furthermore, we can show that vortices, which are stable in conventional superconductors or superfluids, are changed to the rotating spiral ones by the mixing effect of the damping-free oscillatory motion. The single spiral vortex solution [13] in the present system has the general form

$$\Delta(r,\theta,t) = A(r)e^{i(\theta+S_A(r)+\omega t)} \tag{21}$$

$$\varphi_B(r,\theta,t) = B(r)e^{i(\theta+S_B(r)+\omega t)}. \quad (22)$$

This solution can connect asymptotically to the plane-wave solution given above in the limit $r\rightarrow\infty$. The spiral structure is found to arise from the asymptotic property. It should be noted that the spiral structure does not emerge in the diffusive TDGL and the GP equations whose solution is the well-known closed vortex [13].

Very recently, vortices and their lattices have been observed by Zwierlein *et al.*, [25] in atomic Fermi gases. The pictures displayed in their paper [25] are images of not the order parameter but the matter density. In such a measurement, it should be noted that the experimental confirmation of the spiral structure is not easy since a profile of the matter density does not directly reflect that of the order parameter. Thus, we suggest that the confirmation should be performed in not the weak-coupling BCS but the strong-coupling regime, since the matter density profile starts to follow that of the order parameter when going with the strong coupling one. In addition, the narrow Feshbach resonance is found to be clearly more favorable than the broad one for the spiral observation. Furthermore, we can comment on the lifetime of vortices reported in Ref. [25] based on the present theoretical result. The observed reduction of the lifetime in BCS regime can be clearly ascribed to the dependence of the relaxation time in the TDGL theory on the Feshbach-resonance threshold parameter. As one goes to the BCS side, the loosely bound Cooper pairs easily decay close to T_c and the damped feature becomes remarkable. In this case, both the single and the double-channel models can explain this observation result.

Finally, we point out that, since the real part of d acquires a sufficiently large value near the BCS-BEC crossover region, its dynamics is expected to be quite complicated in this region. We mention that the complex TDGL equation describes various types of extended spatial structures with sinks, sources, and unusual core structures, and also shows dynamical complexity, such as various types of spatiotemporal chaos and glassy states. In the superfluid atomic Fermi gases, one can, therefore, expect unusual spatiotemporal structures in the quenching process especially in the BCS-BEC crossover region, unlike in the single component BECs and BCS superfluids.

IV. SUMMARY AND CONCLUSION

In summary, we have derived the coupled TDGL equations for a mixture of the fermion-pair and the diatomic molecule condensates from the fermion-boson model. The obtained TDGL equations contain complex coefficients except in the BEC limit. The complex TDGL equation describes two competing dynamics, i.e., the conserved and nonconserved one. We have explicitly shown that the equations have a plane-wave solution even in the BCS limit. This result markedly contrasts to the fact that only the diffusive motion is possible in conventional BCS superconductors near T_c. We have also discussed that the dynamics of the superfluid atomic Fermi gases is very rich in the BCS-BEC crossover region.

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