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Order parameter and Josephson effect of nonuniform molecular Bose-Einstein condensates

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For two-component Fermi gases at zero temperature, through a general derivation based on the BCS wave function, our research shows that the nonuniform molecular Bose-Einstein condensate has off-diagonal long-range order and can be described well by an order parameter when the size of molecules is much smaller than the size of the whole system. We also give the equation of state and nonlinear evolution equation for the order parameter of nonuniform molecular condensates, which is similar to the Gross-Pitaevskii equation of atomic Bose condensate. The nonlinear evolution equation is applied to consider the Josephson effect for two weakly linked molecular condensates in the presence of nonuniform magnetic field. We find clear particle-number oscillation where spatially dependent binding energy of molecules plays an important role.

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I. INTRODUCTION

With remarkable experimental realizations of molecular Bose-Einstein condensates (BECs) [1] for two-component Fermi gases near a Feshbach resonant magnetic field [2], molecular BECs, and the physics of BCS-BEC crossover have attracted more and more theoretical and experimental studies from different physical disciplines [3]. Theoretically, there are a lot of studies [4–19] relevant to ultracold two-component Fermi gases such as resonance superfluid [7,8], and universal behavior with divergent scattering length [16–18], etc. Experimentally, a lot of pioneering works were done such as the condensate fraction [20], collective excitation [21], pairing gap [22], heat capacity [23], Efimov state [24], etc. For two-component Fermi gases, below a critical magnetic field, there is repulsive interaction between atoms in different internal states, and the atoms are converted into stable molecules due to the mechanism of magnetic-field Feshbach resonance. Similarly to the atomic condensate, below a transition temperature, stable molecular BECs can be created in thermal equilibrium.

Most recently, stable vortex lattices for two-component Fermi gases were observed in a beautiful experiment [25]. This experiment gives strong evidence that molecular BEC has superfluid behavior and can be described by a macroscopic wave function or an order parameter. About fifty years ago, the idea of off-diagonal long-range order (ODLRO) [26] was proposed to give the general criterion of a Bose condensate. The ODLRO describes the essential quantum feature of a Bose condensate because it has no classical analog [26,27]. For a Bose condensate, if the one-particle density matrix can be factorized, there is an ODLRO for the condensate, and the condensate can be regarded as a macroscopic quantum object which has stable spatial coherence property. On the side of molecular BECs for two-component Fermi gases, it is necessary to carefully consider the ODLRO and order parameter of the system because a bosonic molecule consists of two fermionic atoms.

Presently, the theoretical studies on the ODLRO and order parameter are mainly carried out for uniform two-component Fermi gases [29–31]. In this paper, we give a general study on the ODLRO and order parameter of nonuniform molecular BECs for two-component Fermi gases. The analytic derivations show clearly that molecular BECs have ODLRO and can be described by an order parameter. On the other hand, we also give the equation of state (or analytic expression of the overall energy) for nonuniform two-component Fermi gases in the deep BEC regime, which is consistent with a previous conjecture [32]. The special case for uniform system of our results agrees with the low-order term of the equation of state based on Monte Carlo method [33] and the result obtained from a new many-body wave function [35]. From the expression of the overall energy, we give the nonlinear evolution equation for the order parameter of molecular BECs, and consider the role of spatially dependent binding energy in the Josephson effect for two weakly linked molecular BECs in a double-well potential.

The paper is organized as follows. In Sec. II, we consider the ODLRO and order parameter for nonuniform molecular BEC based on the BCS wave function. In Sec. III, we give the nonlinear evolution equation for the order parameter of molecular BECs. In Sec. IV, based on the nonlinear evolution equation, we consider the Josephson effect for two weakly linked molecular BECs in the presence of a nonuniform magnetic field near the Feshbach resonance. A brief summary and discussion is given in Sec. V.

II. ODLRO AND ORDER PARAMETER OF NONUNIFORM MOLECULAR BECS

We first consider the off-diagonal long-range order (ODLRO) and order parameter of molecular BECs based on the BCS wave function. On the repulsive side of interatomic interaction, the BCS wave function has been studied using the Monte Carlo method for uniform trapping potential [28,29,33]. Recently, the BCS wave function was also used to consider the two-component Fermi gases at finite temperature and in the regime of unitarity limit [34], and interesting results about condensate fraction and critical temperature were given. The BCS wave function can be applied to the regimes of molecular BECs and unitarity limit. In the regime of fermionic superfluid, however, the Jastrow-Slater wave function is more appropriate because the ground-state energy based on this wave function is smaller than that based on the BCS wave function. On the side of molecular BECs, here we give general analytic derivations for nonuniform two-component Fermi gases by using the BCS wave function. For an equal number of fermionic atoms in two different internal states, at zero temperature, the BCS wave function takes the following form:

$$\Psi_{\mathrm{BCS}} = \frac{c}{\sqrt{N_\downarrow !}} \mathcal{A}[\phi_a(\mathbf{r}_{11'}) \ldots \phi_a(\mathbf{r}_{ii'}) \ldots \phi_a(\mathbf{r}_{N_\uparrow N_\downarrow}) \times \phi_0(\mathbf{R}_{11'}) \ldots \phi_0(\mathbf{R}_{jj'}) \ldots \phi_0(\mathbf{R}_{N_\uparrow N_\downarrow})], \tag{1}$$

where $\mathcal{A}=\Sigma_P(-1)^p P$ is the antisymmerizer about $\mathbf{r}_{1'},\ldots,\mathbf{r}_{i'},\ldots,\mathbf{r}_{N_\downarrow}$. Σ_P is extended over the $N_\downarrow!$ possible permutations. Here $i,j,\ldots$, and $i',j',\ldots$, label the atoms with internal states $|\uparrow\rangle$ and $|\downarrow\rangle$, respectively. $N_\uparrow$ and $N_\downarrow$ are, respectively, the number of atoms in the internal states $|\uparrow\rangle$ and $|\downarrow\rangle$. We consider here the case of $N_\uparrow = N_\downarrow$. $\mathbf{r}_{ii'}=\mathbf{r}_i-\mathbf{r}_{i'}$ and $\mathbf{R}_{ii'}=(\mathbf{r}_i+\mathbf{r}_{i'})/2$ are, respectively, the relative coordinate and coordinate of the center of mass of two paired atoms with coordinate $\mathbf{r}_i$ and $\mathbf{r}_{i'}$. ϕ_a is the wave function of the bound-state solution of two paired atoms. ϕ_0 is the wave function for the spatial freedom of molecules which is crucial when a nonuniform trapping potential is considered. ϕ_a and ϕ_0 are both normalized to 1.

In Eq. (1), c is a normalization constant which is determined by

$$\int |\Psi_{\mathrm{BCS}}|^2 \Pi_{i=1}^{N_\uparrow} d\mathbf{r}_i \Pi_{j=1}^{N_\downarrow} d\mathbf{r}_{j'} = 1 . \tag{2}$$

After straightforward calculations, we have

$$c^2\left[1 - \frac{N_\downarrow(N_\downarrow-1)\chi_1}{2} + \frac{N_\downarrow(N_\downarrow-1)(N_\downarrow-2)\chi_2}{3} + (\text{higher-order terms})\right] = 1 . \tag{3}$$

Here χ_1 and χ_2 are

$$\chi_1 \approx \int |\phi_0(\mathbf{r})|^4 d\mathbf{r} \int \phi_a^*(-\mathbf{r}_{1'})\phi_a^*(\mathbf{r}_2-\mathbf{r}_{2'})\phi_a(-\mathbf{r}_{2'}) \times \phi_a(\mathbf{r}_2-\mathbf{r}_{1'}) d\mathbf{r}_{1'} d\mathbf{r}_2 d\mathbf{r}_{2'} \tag{4}$$

and

$$\chi_2 \approx \int |\phi_0(\mathbf{r})|^6 d\mathbf{r} \int \phi_a^*(-\mathbf{r}_{1'})\phi_a^*(\mathbf{r}_2-\mathbf{r}_{2'})\phi_a^*(\mathbf{r}_3-\mathbf{r}_{3'}) \times \phi_a(-\mathbf{r}_{2'})\phi_a(\mathbf{r}_2-\mathbf{r}_{3'})\phi_a(\mathbf{r}_3-\mathbf{r}_{1'}) d\mathbf{r}_{1'} d\mathbf{r}_2 d\mathbf{r}_{2'} d\mathbf{r}_3 d\mathbf{r}_{3'} . \tag{5}$$

Here χ_1 and χ_2 represent, respectively, the four-body and six-body correlations. In χ_1 and χ_2, we have used the condition that the size l_m of a molecule is much smaller than the overall size L of the system. In the final expression of the ODLRO, order parameter and overall energy, the coefficient c^2 is cancelled out with an error on the order of $1/N_\downarrow$. Thus, we omit here the cumbersome expression for the higher-order terms in Eq. (3).

We define the ODLRO of molecular BECs as follows:

$$\rho(\mathbf{x},\mathbf{x}';\mathbf{y},\mathbf{y}') = \int \Psi^*_{\mathrm{BCS}}(\mathbf{x},\mathbf{x}';\mathbf{r}_2,\mathbf{r}_2';\ldots;\mathbf{r}_{N_\uparrow},\mathbf{r}_{N_\downarrow}) \times \Psi_{\mathrm{BCS}}(\mathbf{y},\mathbf{y}';\mathbf{r}_2,\mathbf{r}_2';\ldots;\mathbf{r}_{N_\uparrow},\mathbf{r}_{N_\downarrow}) \times \Pi_{i=2}^{N_\uparrow} d\mathbf{r}_i \Pi_{j=2}^{N_\downarrow} d\mathbf{r}_{j'} . \tag{6}$$

When $l_m \ll L$ and $N_\downarrow \gg 1$, our calculations give

$$\rho(\mathbf{x},\mathbf{x}';\mathbf{y},\mathbf{y}') \approx N_\downarrow \phi_0^*\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right)\phi_0\left(\frac{\mathbf{y}+\mathbf{y}'}{2}\right)\phi_a^*(\mathbf{x}-\mathbf{x}') \times \phi_a(\mathbf{y}-\mathbf{y}') . \tag{7}$$

It is natural to define further the following ODLRO $f(\mathbf{x},\mathbf{y})$ which is determined by

$$f^*(\mathbf{x},\mathbf{y})f(\mathbf{x},\mathbf{y}) = \int \rho^*(\mathbf{x},\mathbf{x}';\mathbf{y},\mathbf{y}')\rho(\mathbf{x},\mathbf{x}';\mathbf{y},\mathbf{y}') d\mathbf{x}' d\mathbf{y}' = N_\downarrow^2 |\phi_0(\mathbf{x})\phi_0(\mathbf{y})|^2 . \tag{8}$$

We see that $f(\mathbf{x},\mathbf{y}) = N_\downarrow \phi_0^*(\mathbf{x})\phi_0(\mathbf{y})$ shows the ODLRO of molecular BECs when the internal freedom of molecules is omitted. The factorability of $f(\mathbf{x},\mathbf{y})$ shows that there is off-diagonal long-range order for molecular BECs when the fermionic atoms in different internal states are in pairs. $\Psi_m = \sqrt{N_\downarrow}\phi_0$ is then the order parameter of the molecular BEC, and $n_m = N_\downarrow |\phi_0|^2$ is the density distribution of the molecular BEC. These properties about molecular BECs are natural results of the BCS wave function, because $\phi_a(\mathbf{r}_{ii'})$ and antisymmerizer $\mathcal{A}$ in Eq. (1) mean the pairing of all the fermionic atoms (in different internal states) in the same way.

For $l_m \ll L$, the factor $N_\uparrow$ in $f(\mathbf{x},\mathbf{y})$ shows that the condensate fraction at zero temperature can approach 100%. This result for uniform case is consistent with the result based on the diffusion Monte Carlo method [29] and recent works in Refs. [30,31]. Because Jastrow-Slater wave function is more appropriate to consider the fundamental properties of fermionic superfluid, even when the size of atomic Cooper pairs is much smaller than L, we cannot get the result that the condensate fraction approaches 100%. As shown in Refs. [29–31] based on the Jastrow-Slater wave function for fermionic superfluid, the condensate fraction is much smaller than 1 in the regime of fermionic superfluid. In the regime of unitarity limit, because the BCS wave function can give quite good description for two-component Fermi gases, we expect that there is quite large condensate fraction at zero temperature [29–31,34].

III. NONLINEAR EVOLUTION EQUATION OF MOLECULAR BECS

When the trapping potential and magnetic field inducing the Feshbach resonance are considered, the two-component Fermi gases in which all the fermionic atoms are paired into molecules are described by the following Hamiltonian:

$$\hat{H} = \hat{H}_1 + \hat{H}_2 \tag{9}$$

with

$$\hat{H}_1 = -\frac{\hbar^2}{2m}\left(\sum_{i=1}^{N_\uparrow}\nabla^2_{\mathbf{r}_i} + \sum_{i'=1}^{N_\downarrow}\nabla^2_{\mathbf{r}_{i'}}\right) + \sum_{i,i'} v(\mathbf{r}_{ii'}) \tag{10}$$

and

$$\hat{H}_2 = \frac{1}{N_\downarrow}\sum_{i,i'} V_{\text{ext}}(\mathbf{R}_{ii'}) - \frac{1}{N_\downarrow}\sum_{i,i'} \mathbf{B}\cdot\mu_m(\mathbf{r}_{ii'}). \tag{11}$$

Here m denotes the mass of fermionic atom. $v(\mathbf{r}_{ii'})$ is the two-body potential between two fermionic atoms in different internal states, while $V_{\text{ext}}(\mathbf{R}_{ii'})$ is the trapping potential for the molecule. $\mathbf{B}$ and μ_m are, respectively, the magnetic field and molecular magnetic moment.

The overall energy of the system is then

$$E = \langle\Psi_{\text{BCS}}|\hat{H}|\Psi_{\text{BCS}}\rangle. \tag{12}$$

Considering the permutation symmetry of the Hamiltonian and the permutation antisymmetry for any two atoms with the same internal state in the BCS wave function, we have

$$\begin{aligned}\langle\Psi_{\text{BCS}}|\hat{H}_1|\Psi_{\text{BCS}}\rangle = \langle\Psi_{\text{BCS}}|\Bigg\{&\frac{N_\downarrow}{2}\Bigg[-\frac{\hbar^2}{2m}(\nabla^2_{\mathbf{r}_1} + \nabla^2_{\mathbf{r}_2} + \nabla^2_{\mathbf{r}_{1'}} \\ &+ \nabla^2_{\mathbf{r}_{2'}}) + v(\mathbf{r}_{11'}) + v(\mathbf{r}_{22'})\Bigg] \\ &+ \frac{N_\downarrow(N_\downarrow - 1)}{2}[v(\mathbf{r}_{12'}) + v(\mathbf{r}_{1'2})]\Bigg\}|\Psi_{\text{BCS}}\rangle.\end{aligned} \tag{13}$$

Introducing the relative coordinate and the coordinate of the center of mass, the above equation can also be written as

$$\begin{aligned}\langle\Psi_{\text{BCS}}|\hat{H}_1|\Psi_{\text{BCS}}\rangle = \langle\Psi_{\text{BCS}}|\Bigg\{&\frac{N_\downarrow}{2}\Bigg[-\frac{\hbar^2}{2M}(\nabla^2_{\mathbf{R}_{11'}} + \nabla^2_{\mathbf{R}_{22'}}) \\ &- \frac{\hbar^2}{2\mu}(\nabla^2_{\mathbf{r}_{11'}} + \nabla^2_{\mathbf{r}_{22'}}) + v(\mathbf{r}_{11'}) + v(\mathbf{r}_{22'})\Bigg] \\ &+ \frac{N_\downarrow(N_\downarrow - 1)}{2}[v(\mathbf{r}_{12'}) + v(\mathbf{r}_{1'2})]\Bigg\}|\Psi_{\text{BCS}}\rangle.\end{aligned} \tag{14}$$

Here M is the molecular mass, while the reduced mass is $\mu = m/2$. It is obvious that this equation can be simplified as

$$\begin{aligned}\langle\Psi_{\text{BCS}}|\hat{H}_1|\Psi_{\text{BCS}}\rangle = \langle\Psi_{\text{BCS}}|\Bigg\{&N_\downarrow\Bigg[-\frac{\hbar^2}{2\mu}\nabla^2_{\mathbf{r}_{11'}} + v(\mathbf{r}_{11'})\Bigg] \\ &- \frac{\hbar^2 N_\downarrow}{2M}\nabla^2_{\mathbf{R}_{11'}} + \frac{N_\downarrow(N_\downarrow - 1)}{2}[v(\mathbf{r}_{12'}) \\ &+ v(\mathbf{r}_{1'2})]\Bigg\}|\Psi_{\text{BCS}}\rangle.\end{aligned} \tag{15}$$

We see that in the brace of the above equation, the first term represents the molecular binding energy, while the second term represents the kinetic energy of molecules. In the last term in the brace $v(\mathbf{r}_{12'}) + v(\mathbf{r}_{1'2})$ denotes the interacting potential between two molecules. When $l_m < \bar{l}$ with $\bar{l}$ being the mean distance between particles, $v(\mathbf{r}_{12'}) + v(\mathbf{r}_{1'2})$ can be regarded as $g_m\delta^{(3)}(\mathbf{R}_{11'} - \mathbf{R}_{22'})$ according to the pseudopotential method [36]. Here $g_m = 4\pi\hbar^2 a_m/M$ with a_m being the scattering length between molecules.

After straightforward calculations, for $l_m \ll L$ and $N_\downarrow \gg 1$, we get

$$\begin{aligned}E = N_\downarrow\varepsilon_b + N_\downarrow\int \phi_0^*\left(-\frac{\hbar^2}{2M}\nabla^2 + V_{\text{ext}} - \mathbf{B}\cdot\mu_m\right)\phi_0 dV \\ + \frac{N_\downarrow(N_\downarrow - 1)g_m}{2}\int|\phi_0|^4 dV.\end{aligned} \tag{16}$$

The last term represents the interaction energy between molecules. Because $N_\downarrow$ is the number of molecules N_m, we see that this term is similar to that of atomic condensate, especially the identical factor $1/2$. Together with the ODLRO and order parameter in the last section, these calculations show clearly the validity of the concept of molecular BECs for two-component Fermi gases in the deep BEC regime. This equation of state is consistent with a previous work in Ref. [32].

In the above formal derivation, we give general studies on the overall energy, and do not involve the concrete calculations about the two-body interaction potential and scattering length. In Ref. [37], it is shown based on four-body collisions that $a_m \approx 0.6a$ with a being the scattering length between fermionic atoms in different internal state, and the binding energy $\varepsilon_b = -\hbar^2/ma^2$. When this important result about the molecular scattering length and binding energy is used, the overall energy for the special case of uniform system given by Eq. (16) agrees with the result of Monte Carlo method [33] in the deep BEC regime.

Based on the overall energy given by Eq. (16) and the ordinary action principle, we get the following nonlinear evolution equation about the order parameter Ψ_m:

$$i\hbar\frac{\partial\Psi_m}{\partial t} = \left(-\frac{\hbar^2}{2M}\nabla^2 + V_{\text{ext}} + \varepsilon_b - \mathbf{B}\cdot\mu_m\right)\Psi_m + g_m|\Psi_m|^2\Psi_m. \tag{17}$$

We see that this equation is similar to that of atomic condensate. We stress that the valid condition of this equation is $l_m < \bar{l}$, and this equation can be applied for nonuniform trapping potential and nonuniform magnetic field once this condition is satisfied for the whole system.

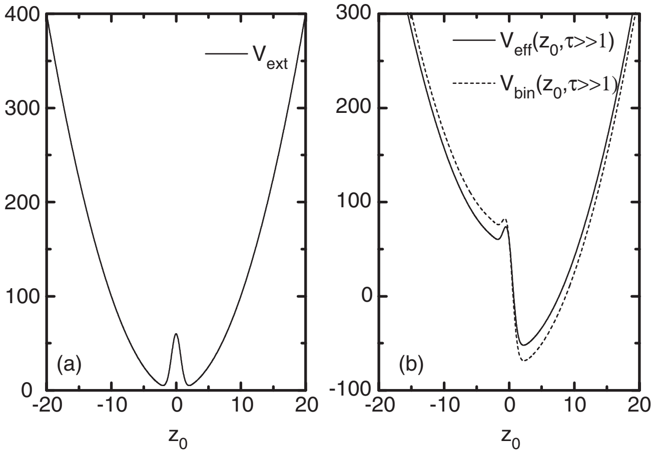

FIG. 1. (a) shows the double-well potential V_{ext}, while Fig. 1(b) shows the effective potential V_{eff} (solid line) and V_{bin} (dashed line) when the potential $\alpha \mathbf{e}_x \cdot \mu_m$ due to a nonuniform magnetic field is not considered. Here V_{ext}, V_{eff}, and V_{bin} are in units of E_0, while the spatial coordinate is in units of l_z. We see that the spatially dependent binding energy can change significantly the effective potential, while the term $\alpha \mathbf{e}_x \cdot \mu_m$ plays a minor role.

IV. JOSEPHSON EFFECT OF TWO WEAKLY LINKED MOLECULAR CONDENSATES

For two weakly linked atomic condensates, the Josephson effect has been studied both theoretically and experimentally (see Ref. [38], and references therein). Here we consider the Josephson effect of two weakly linked molecular condensates confined in a double-well potential based on the evolution equation (17).

We first give a general study based on the widely used two-mode approximation. Assume that

$$\Psi_m = \Gamma_1(t)\Phi_1(x) + \Gamma_2(t)\Phi_2(x). \tag{18}$$

Here $\Gamma_{1,2}(t) = \sqrt{N_{1,2}(t)}e^{i\theta_{1,2}(t)}$ with N_1 and N_2 being the number of molecules in each well. Based on Eq. (17), it is straightforward to get

$$\dot{z} = -\sqrt{1-z^2}\sin\phi,$$

$$\dot{\phi} = \Lambda z + \frac{z}{\sqrt{1-z^2}}\cos\phi + \Delta E. \tag{19}$$

Here $z = (N_1 - N_2)/N_m$ and $\phi = \theta_2 - \theta_1$. In addition

$$\Delta E = \frac{E_1^0 - E_2^0}{2K} + \frac{(U_1 - U_2)N_m}{4K},$$

$$\Lambda = \frac{(U_1 + U_2)N_m}{4K}, \tag{20}$$

where

$$E_{1,2}^0 = \int \left[\frac{\hbar^2}{2m}|\boldsymbol{\nabla}\Phi_{1,2}|^2 + (V_{\text{ext}} + \varepsilon_b - \mathbf{B}\cdot\mu_m)|\Phi_{1,2}|^2\right]dV,$$

$$U_{1,2} = \int g_m|\Phi_{1,2}|^4 dV,$$

$$K = -\int \left[\frac{\hbar^2}{2m}\boldsymbol{\nabla}\Phi_1 \cdot \boldsymbol{\nabla}\Phi_2 + (V_{\text{ext}} + \varepsilon_b - \mathbf{B}\cdot\mu_m)\Phi_1\Phi_2\right]dV. \tag{21}$$

Although the formal solution for the Josephson effect given by Eq. (19) is similar to atomic condensate [39], the coefficients given by Eq. (21) are quite different by noting especially the binding energy ε_b which is spatially dependent in the presence of a nonuniform magnetic field near B_0.

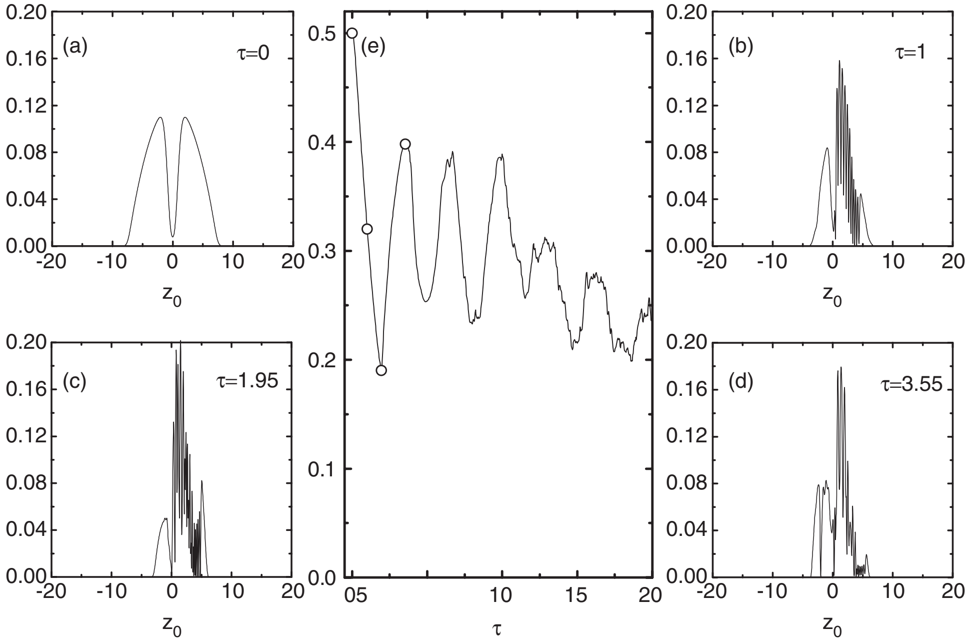

FIG. 2. Shown are the density distribution and particle-number oscillation with the development of time. A clear particle-number oscillation driven mainly by the spatially dependent binding energy is shown in this figure. The density distribution in Figs. 2(a)–2(d) corresponds to the circles in Fig. 2(e). Here the spatial coordinate is in units of l_z, while the time is in units of T_0.

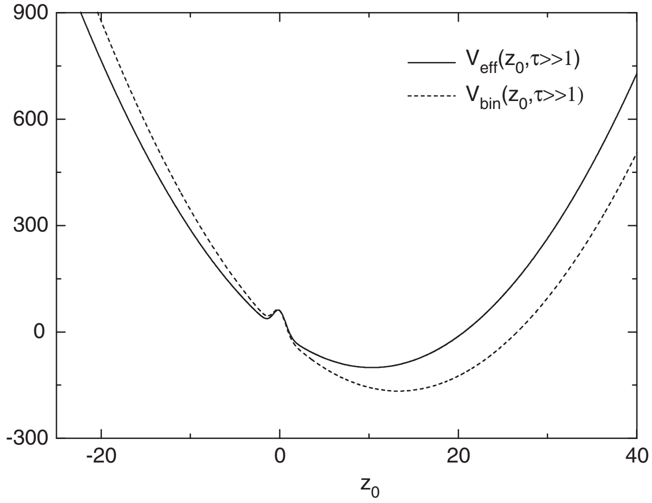

FIG. 3. Illustration of $V_{\rm eff}$ (solid line) and $V_{\rm bin}$ (dashed line) in the presence of the gradient magnetic field 2.44 G/m. Here $V_{\rm eff}$ and $V_{\rm bin}$ are in units of E_0, while the spatial coordinate is in units of l_z.

We calculate numerically the evolution equation (17) for two weakly linked molecular condensates confined in a double-well potential and in the presence of a nonuniform magnetic field. We consider the two-component fermionic gases of ^{6}Li with $|\uparrow\rangle=|F=1/2,m_F=1/2\rangle$ and $|\downarrow\rangle=|F=1/2,m_F=-1/2\rangle$. For a magnetic field inducing the Feshbach resonance, $a(B)=Aa_r[1-w/(B-B_0)]$ with a_r being the Bohr radius. We stress in the present work the role of the spatially dependent binding energy ε_b in the Josephson effect of two weakly linked molecular BECs. Thus, we consider the narrow resonance located at $B_0=543.25$ G. For this narrow resonance, $A=60$ and $w=0.1$ G [40]. The merit of the narrow resonance is that a specific nonuniform magnetic field can influence largely the binding energy, while the term $\mathbf{B}\cdot\mu_m$ is weakly spatially dependent. In this situation, the spatially dependent binding energy can play an important role in the Josephson effect.

The double-well potential is

$$V_{\rm ext}=\frac{1}{2}M[\omega_\perp^2(x^2+y^2)+\omega_z^2z^2]+Ue^{-z^2/\Delta_z^2-(x^2+y^2)/\Delta_\perp^2}, \tag{22}$$

with the first term being the harmonic potential due to an optical trap, while the second term being the central barrier due to a far-off blue-detuned laser beam located at the center of the harmonic potential [41]. Here U is proportional to the intensity of the laser beam.

For $t<0$, we assume that two weakly linked molecular condensates are confined in this double-well trap with a uniform magnetic field $\mathbf{B}=(B_0+\Delta B)\mathbf{e}_x$. For $t>0$, we change the magnetic field with the following form:

$$\mathbf{B}(\mathbf{r},t)=[B_0+\Delta B+\alpha(\mathbf{r},t)]\mathbf{e}_x. \tag{23}$$

In numerical calculations, it is useful to introduce the transformations $t=2\tau/\omega_z$ and $x=l_zx_0$, $y=l_zy_0$, $z=l_zz_0$ with $l_z=\sqrt{\hbar/M\omega_z}$. With these transformations, x_0, y_0, z_0, and τ become dimensionless. Introducing further the transformation $\Psi_m=\sqrt{N_m}\phi_d/l_z^{3/2}$, we get the following dimensionless evolution equation:

$$\begin{aligned}i\frac{\partial\phi_d}{\partial\tau}={}&-\nabla_0^2\phi_d+\Bigg[z_0^2+\frac{\omega_\perp^2}{\omega_z^2}(x_0^2+y_0^2)\\&+U_0e^{-z_0^2/(\Delta_z^0)^2-(x_0^2+y_0^2)/(\Delta_\perp^0)^2}\Bigg]\phi_d-\frac{2\mathbf{B}\cdot\mu_m}{\hbar\omega_z}\phi_d-\frac{4}{a_0^2}\phi_d\\&+g_0(z_0,\tau)|\phi_d|^2\phi_d,\end{aligned} \tag{24}$$

where $\nabla_0^2=\partial^2/\partial x_0^2+\partial^2/\partial y_0^2+\partial^2/\partial z_0^2$, $\Delta_z^0=\Delta_z/l_z$, $\Delta_\perp^0=\Delta_\perp/l_z$, $U_0=2U/\hbar\omega_z$, $a_0=a/l_z$. ϕ_d is normalized to 1. In addition, the dimensionless coupling constant $g_0=8\pi N_ma_m/l_z$. In the above equation, time and length are, respectively, in units of $T_0=2/\omega_z$ and l_z, while energy is in units of $E_0=\hbar\omega_z/2$. The parameters are chosen as $\omega_z=2\pi\times 50\ {\rm s}^{-1}$, $\omega_\perp=5\omega_z$, $N_m=10^3$, $\Delta B=-0.002$ G, $U_0=60$, $\Delta_z^0=1$, $\Delta_\perp^0=5\Delta_z^0$. Here,

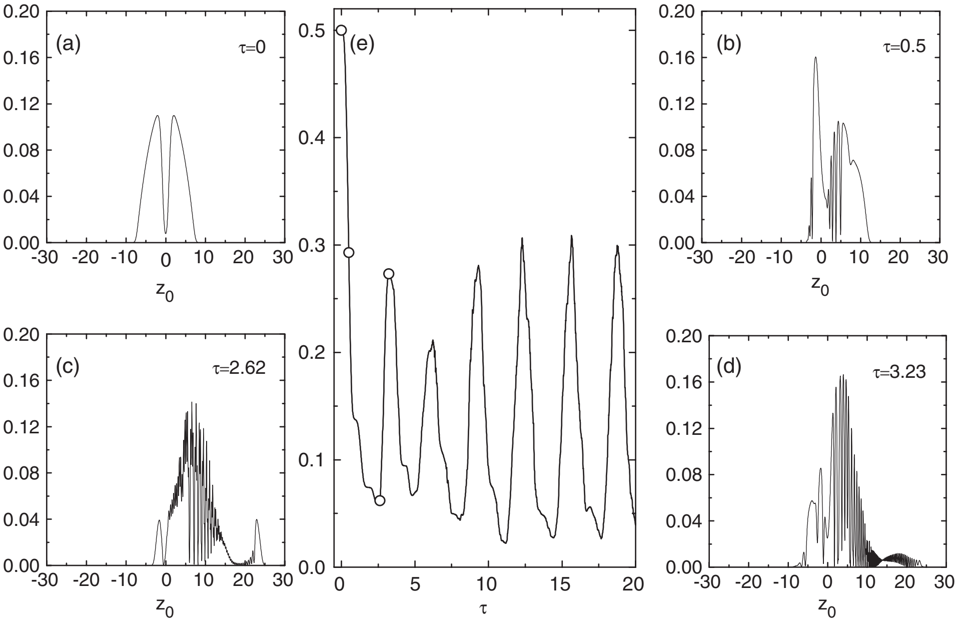

FIG. 4. In the presence of the gradient magnetic field, shown are the density distribution and particle-number oscillation with the development of time. The density distribution in Figs. 4(a)–4(d) corresponds to the circles in Fig. 4(e). Here the spatial coordinate is in units of l_z, while the time is in units of T_0.

$|\Delta B|$ is much smaller than the field width w, so that the analytic expression of the binding energy used in the present work may be applied approximately. In addition, $|\mu_m|=2\mu_B$ with μ_B being the Bohr magneton (see Ref. [42]). The spatially dependent magnetic field $\alpha(\mathbf{r},t)$ will be chosen as two different cases as shown in the following subsections.

A. Josephson effect driven by binding energy

To show the Josephson effect driven mainly by the binding energy, in this subsection, α is assumed as

$$\alpha(z_0,\tau)=\alpha_0(1-e^{-\gamma\tau})\frac{e^{\beta z_0}-e^{-\beta z_0}}{e^{\beta z_0}+e^{-\beta z_0}}. \tag{25}$$

In the numerical calculations, the parameters are $\gamma=1$ and $\beta=1$. In Fig. 1(a), we show the double-well potential V_{ext}, while we show in Fig. 1(b) the effective potential $V_{\text{eff}}(x_0=y_0=0,z_0)=V_{\text{ext}}-\alpha\mathbf{e}_x\cdot\mu_m+\varepsilon_b-\varepsilon_b(z_0=0)$ for $\alpha_0=3\times10^{-5}$ G and $\tau\gg1$ by solid line. We also give in this figure $V_{\text{bin}}(x_0=y_0=0,z_0)=V_{\text{ext}}+\varepsilon_b-\varepsilon_b(z_0=0)$ with the same parameters for V_{eff} by dashed line. It is shown clearly that the term $\alpha\mathbf{e}_x\cdot\mu_m$ plays a minor role in the effective potential. We see that the binding energy can change significantly the effective potential, and make the effective potential become nonsymmetric about $z_0=0$. It is natural that for nonzero α_0, there would be particle-number oscillation between two wells. For $\alpha_0=3\times10^{-5}$ G, $\Delta g_m/g_m<5\%$ (Δg_m is the difference of the coupling constant between two wells). Thus, the weakly spatially dependent coupling constant also plays a minor role in the Josephson effect.

In our numerical calculations, the initial ground state at $\tau=0$ is obtained based on Eq. (24) with an imaginary time propagation method. $n_0(z_0,\tau)=\int|\phi_d|^2dx_0dy_0$ is shown in Fig. 2(a) at $\tau=0$. For $\tau>0$, the evolution of ϕ_d is calculated by a split-step Fourier method. In Figs. 2(b)–2(d), the density distribution $n_0(z_0,\tau)$ is shown for different dimensionless time. In Fig. 2(e), we give the evolution of the proportion N_1/N_m of the number of molecules in the regime $z_0<0$, i.e., $N_1/N_m=\int_{-\infty}^{\infty}dx_0dy_0\int_{-\infty}^{0}dz_0|\phi_d|^2$, and clear particle-number oscillation is shown.

B. Josephson effect driven by a gradient magnetic field

At least in the near feature, the Josephson effect driven by a gradient magnetic field is feasible because the gradient magnetic field is a routine experimental technology. For this situation, however, in addition to the binding energy, the term $\mathbf{B}\cdot\mu_m$ and spatially dependent coupling constant g_m also play an important role in the Josephson effect. For the gradient magnetic field, α is

$$\alpha(z,\tau)=\beta(1-e^{-\gamma\tau})z, \tag{26}$$

with $\gamma=1$ and $\beta=2.44$ G/m. In Fig. 3, we give V_{eff} and V_{bin} for these parameters, which show that both the spatially dependent binding energy and $\alpha\mathbf{e}_x\cdot\mu_m$ play an important role in the Josephson effect. In Fig. 4(e), we give the numerical result of N_1/N_m with the development of time. In Figs. 4(a)–4(d), the density distribution $n_0(z_0,\tau)$ for different dimensionless time is shown. This sort of gradient magnetic field and $\Delta B=-0.002$ G can be realized in the present experimental manipulation of magnetic field with a pair of anti-Helmholtz coils and magnetic field shielding [43].

V. SUMMARY AND DISCUSSION

In summary, for ultracold two-component Fermi gases, we have shown that nonuniform molecular condensates have off-diagonal long-range order and can be described by an order parameter when the size of molecules is much smaller than the effective size of the whole system. In the deep BEC regime, we give the nonlinear evolution equation for nonuniform molecular condensates based on some formal derivations. Obviously, the general nonlinear evolution equation for nonuniform external potential and nonuniform magnetic field (or spatially dependent scattering length and binding energy) has a lot of interesting applications for the dynamic evolution of the ultracold two-component Fermi gases. As an application of this nonlinear evolution equation, we have studied the Josephson effect of two weakly linked molecular condensates after a nonuniform magnetic field is imposed. In our numerical results, clear particle-number oscillation is shown. At least for the case of gradient magnetic field, the theoretical predication may be tested in the present experimental technology. In previous work [44], our research shows that with an appropriate gradient magnetic field for ^{6}Li near the narrow resonance, ultracold two-component Fermi gases would be in a coexistence of the regimes of BCS, BEC, and unitarity limits. Combining with a double-well system, this coexistence gives us opportunities to investigate a new type of Josephson effect such as the Josephson effect between molecular BEC and gas in the unitarity limit (or BCS superfluid and gas in the unitarity limit, molecular BEC, and BCS superfluid). As a first step, here we have studied the Josephson effect driven by spatially dependent binding energy in the presence of a gradient magnetic field.

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