Perturbation theory for isotropic velocity-dependent potentials: Scattering case

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The time-independent Schrödinger equation with an isotropic velocity-dependent potential is considered. Treating the velocity-dependent interaction as a small perturbation, we develop analytical formulas for the changes in the scattering phase shifts and wave functions. It is shown that only the zeroth-order solution and the perturbing potential are needed to determine the phase-shift and wave-function corrections. No prior knowledge of the unperturbed scattering-states continuum is required. In order to test the validity of our approach, we applied it to an exactly solvable model for nucleon-nucleon scattering. The results of the perturbation formalism compare quite well with those of the exactly solvable model. The developed formalism can be applied in problems concerning pion-nucleon, nucleon-nucleon, and electron-atom scattering. It may also be useful in studying the scattering of electrons in semiconductor heterostructures.

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I. INTRODUCTION

Velocity-dependent potentials have been used in different interesting fields of physics. In nuclear physics a model assuming the nucleon-nucleon interaction to be velocity dependent reproduced the ^1S, ^1D, and ^1G singlet-even phase shifts [1]. Further, the predominantly p-wave nature of the pion-nucleon scattering was correctly predicted using a velocity-dependent potential [2]. In the field of atomic physics, the scattering of electrons from atomic oxygen and neon was studied in the framework of an analytic velocity-dependent potential [3].

The Schrödinger equation with a velocity-dependent interaction can be recast in a form that describes a particle endowed with a position-dependent effective mass (PDEM). The Schrödinger equation for a PDEM has been the subject of many interesting recent works for its increasing relevance in describing the dynamics of electrons in semiconductor heterostructures such as, compositionally graded crystals [4], quantum dots [5], and liquid crystals [6]. For example, electron scattering on disordered double-barrier heterostructures has been considered in Ref. [7]. Such studies are important in designing semiconductor electronic devices.

Furthermore, interesting theoretical works exploited the application of supersymmetric quantum mechanics to a position-variable mass [8]. Later works [9,10] extended the application of supersymmetry to the Schrödinger equation for a PDEM by taking into account the different ordering ambiguities resulting from the noncommutativity of the variable mass with the momentum operator. In addition, in the quantum many-body problem, the concept of an effective mass is relevant in connection with the energy-density-functional (EDF) approach. Here, the nonlocal terms of the associated potential can often be interpreted as a three-dimensional position-dependent effective mass [11].

Clearly, the Schrödinger equation with a velocity-dependent potential is of vital importance. However, the physically interesting problems where such an equation admits exact solutions are very few and hence approximation schemes become important. Consequently, in this work we shall consider the scattering of a particle moving in a velocity-dependent potential, which shall be treated as a small perturbation. Exact analytical formulas for the changes in the scattering phase shifts and wave functions will be derived.

In an earlier work [12] we developed perturbation formulas for the bound-state energy and wave-function corrections when an isotropic velocity-dependent perturbing interaction is introduced. Furthermore, the effect of different-ordering ambiguities on the bound-state perturbation formalism has been investigated in Ref. [13].

II. PERTURBATION THEORY

Let us consider the following potential which consists of two parts; a local potential $V(r)$ and a velocity-dependent one $\tilde{V}(r,p)$:

$$\hat{V}(r,p) = V(r) + \tilde{V}(r,p) = V(r) + \frac{\hbar^2}{2m}\nabla \cdot f(r)\nabla$$
$$= V(r) + \frac{\hbar^2}{2m}\{f(r)\nabla^2 + \nabla f(r) \cdot \nabla\}. \quad (1)$$

Both $V(r)$ and $f(r)$ are isotropic functions of the radial variable r, assumed to have a common range $r=b$ beyond which each identically vanishes. The corresponding s-wave Schrödinger equation is

$$\left[\frac{-\hbar^2}{2m}\{[1-f(r)]\nabla^2 - \nabla f(r) \cdot \nabla\} + V(r)\right]\psi(\vec{r}) = E\psi(\vec{r}), \quad (2)$$

where $\psi(\vec{r})$ is a three-dimensional wave-function. Setting $f(r)=\lambda\rho(r)$ and using the reduced wave function $v(r)=rR(r)$, the above equation reduces to

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$$[1-\lambda\rho(r)]v''(r)-\left[v'(r)-\frac{v(r)}{r}\right]\lambda\rho'(r)=\frac{2m}{\hbar^2}[V(r)-E]v(r), \tag{3}$$

where the prime denotes derivation with respect to the radial variable r. The parameter λ signifies the strength of the perturbing velocity-dependent interaction, $\tilde{V}(r,p)$, and falls in the range $0\leq\lambda\leq 1$. In the asymptotic region $r\geq b$, where the local and velocity-dependent parts of the potential defined in Eq. (1) vanish identically, the solution of the above equation is

$$v(r)\rightarrow\frac{1}{k}\sin(kr+\delta),\quad r\geq b, \tag{4}$$

where δ is the perturbed s-wave phase shift and $k^2=2mE/\hbar^2$.

The lower limit $\lambda=0$ corresponds to the case where the perturbation is switched off. In such a case $\tilde{V}(r,p)=0$ and hence Eq. (3) reduces to the unperturbed Schrödinger equation, namely,

$$v_0''(r)=\frac{2m}{\hbar^2}[V(r)-E]v_0(r), \tag{5}$$

where $v_0(r)$ is the unperturbed wave function which, in the asymptotic region, takes the form

$$v_0(r)\rightarrow\frac{1}{k}\sin(kr+\delta_0),\quad r\geq b. \tag{6}$$

Both v_0 and the corresponding unperturbed s-wave phase-shift δ_0, resulting from scattering off the local potential $V(r)$ only, are assumed to be known.

In Ref. [14] a type of perturbation theory was developed for the bound-state energy and wave-function corrections due to the introduction of a small perturbing local potential only. Following this work, let us consider the wave-function expansion

$$v(r)=v_0(r)+\lambda v_1(r)+\lambda^2 v_2(r)+\cdots, \tag{7}$$

and we propose the following phase-shift expansion:

$$\delta=\delta_0+\lambda\delta_1+\lambda^2\delta_2+\cdots, \tag{8}$$

where v_n and δ_n $(n\geq 1)$ are the wave-function and phase-shift corrections, respectively. The wave functions $v(r)$ and $v_0(r)$ of Eqs. (3) and (5) must vanish at the origin. Hence by considering Eq. (7) we impose the boundary condition

$$v_n(0)=0,\quad n\geq 0. \tag{9}$$

Inserting the above expansion for $v(r)$ in Eq. (3) one finds that the coefficient of λ^n, where $n\geq 1$, satisfies the following equation:

$$v_n''-\rho v_{n-1}''-\left[v_{n-1}'-\frac{v_{n-1}}{r}\right]\rho'=\frac{v_0''v_n}{v_0}, \tag{10}$$

where we have substituted for $2m[V(r)-E]/\hbar^2$ using the unperturbed Schrödinger equation given in Eq. (5). Multiplying Eq. (10) throughout by v_0 and rearranging leads to

$$v_n''v_0-v_0''v_n=v_0\frac{d}{dr}(\rho v_{n-1}')-\frac{v_0v_{n-1}}{r}\rho'. \tag{11}$$

A. First-order phase-shift correction

We start by setting $n=1$ in the last equation, which leads to

$$v_1''v_0-v_0''v_1=v_0\frac{d}{dr}(\rho v_0')-\frac{v_0^2}{r}\rho'. \tag{12}$$

Integrating the above from the origin to r and noting the boundary condition in Eq. (9) results in

$$v_1'v_0-v_0'v_1=\int_0^r v_0\frac{d}{dr'}(\rho v_0')dr'-\int_0^r\frac{v_0^2}{r'}\rho'dr'. \tag{13}$$

The left-hand side may be evaluated in the asymptotic region $r\geq b$. In this region the behavior of v_0 is given by Eq. (6). However, the asymptotic form of v_1 can be established by introducing the phase-shift expansion of Eq. (8) into Eq. (4) and expanding the right-hand side to second order in λ. Substituting the resulting expansion for $v(r)$ on the left-hand side of Eq. (7) and equating the coefficients of equal powers of λ it is straight forward to show that

$$v_1\rightarrow\frac{\delta_1}{k}\cos(kr+\delta_0),\quad r\geq b \tag{14}$$

and

$$v_2\rightarrow\frac{1}{k}\left\{\delta_2\cos(kr+\delta_0)-\frac{1}{2}\delta_1^2\sin(kr+\delta_0)\right\},\quad r\geq b. \tag{15}$$

Inserting the asymptotic forms of v_0 and v_1 on the left-hand side of Eq. (13) and simplifying leads to the first-order phase-shift correction namely,

$$\delta_1=k\int_0^\infty\frac{v_0^2}{r}\rho'dr-k\int_0^\infty v_0\frac{d}{dr}(\rho v_0')dr. \tag{16}$$

Performing the second integral on the right-hand side by parts simplifies the above to

$$\delta_1=k\int_0^\infty\frac{v_0^2}{r}\rho'dr+k\int_0^\infty v_0'\rho v_0'dr. \tag{17}$$

It is worth noting that the phase-shift correction δ_1 depends not only on the form of $\rho(r)$ but also on its first derivative with respect to r. This feature seems to be specific to velocity-dependent potentials and will be seen in all the higher-order phase-shift corrections.

B. First-order wave-function correction

In order to determine the first-order wave-function correction we rewrite Eq. (13) in the form

$$\frac{d}{dr}\left(\frac{v_1}{v_0}\right)=\frac{1}{v_0^2}\int_0^r\left[v_0\frac{d}{dr'}(\rho v_0')-\frac{v_0^2}{r'}\rho'\right]dr'. \tag{18}$$

Integrating the above from b to r results in

$$v_1 = v_0 \int_b^r \frac{dr'}{v_0^2} \int_0^{r'} \left[v_0 \frac{d}{dr''}(\rho v_0') - \frac{v_0^2}{r''}\rho' \right] dr'' + C_1 v_0, \tag{19}$$

where C_1 is an integration constant whose value depends on b, the common range of the local and velocity-dependent parts of the potential $\hat{V}(r,p)$ given in Eq. (1). The value of C_1 may be determined by requiring v_1 to assume the asymptotic form given in Eq. (14) in the external region $r \geq b$. Consequently, the final expression for v_1 is

$$v_1 = v_0 \int_b^r \frac{dr'}{v_0^2} \int_0^{r'} \left[v_0 \frac{d}{dr''}(\rho v_0') - \frac{v_0^2}{r''}\rho' \right] dr'' + \delta_1 \cot(kb + \delta_0) v_0. \tag{20}$$

C. Second-order phase-shift correction

To determine the expression for the second-order phase-shift correction we start by setting $n=2$ in Eq. (11), which results in

$$v_2'' v_0 - v_0'' v_2 = v_0 \frac{d}{dr}(\rho v_1') - \frac{v_0 v_1}{r}\rho'. \tag{21}$$

Integrating the above from 0 to r and making use of the boundary condition in Eq. (9) gives

$$v_2' v_0 - v_0' v_2 = \int_0^r v_0 \frac{d}{dr'}(\rho v_1') dr' - \int_0^r \frac{v_0 v_1}{r'}\rho' dr'. \tag{22}$$

For $r \geq b$, evaluating for the left-hand side using Eqs. (6) and (15) results in the following expression for the second-order phase correction:

$$\delta_2 = k \int_0^\infty \frac{v_0 v_1}{r}\rho' dr + k \int_0^\infty v_0' \rho v_1' dr. \tag{23}$$

D. Second-order wave-function correction

Let us start by rewriting Eq. (22) in the form

$$\frac{d}{dr}\left(\frac{v_2}{v_0}\right) = \frac{1}{v_0^2} \int_0^r \left[v_0 \frac{d}{dr'}(\rho v_1') - \frac{v_0 v_1}{r'}\rho' \right] dr'. \tag{24}$$

Integrating from b to r gives

$$v_2 = v_0 \int_b^r \frac{dr'}{v_0^2} \int_0^{r'} \left[v_0 \frac{d}{dr''}(\rho v_1') - \frac{v_0 v_1}{r''}\rho' \right] dr'' + \left[\delta_2 \cot(kb + \delta_0) - \frac{\delta_1^2}{2} \right] v_0, \tag{25}$$

where the integration constant multiplying v_0 has been deduced by requiring v_2 to assume the asymptotic form of Eq. (15) for $r \geq b$.

E. Higher-order corrections

Higher-order corrections can be obtained to any given order in the perturbing potential by employing the same technique already used to derive the first- and second-order phase-shift and wave-function corrections. For example, the third-order phase-shift and wave-function corrections are

$$\delta_3 = k \int_0^\infty \frac{v_0 v_2}{r}\rho' dr + k \int_0^\infty v_0' \rho v_2' dr + \frac{1}{6}\delta_1^3, \tag{26}$$

$$v_3 = v_0 \int_b^r \frac{dr'}{v_0^2} \int_0^{r'} \left[v_0 \frac{d}{dr''}(\rho v_2') - \frac{v_0 v_2}{r''}\rho' \right] dr'' + C_3 v_0, \tag{27}$$

where the constant of integration is given by

$$C_3 = \delta_1 \delta_2 + \tfrac{1}{6}(\delta_1^3 - 6\delta_3)\cot(kb + \delta_0). \tag{28}$$

III. EXACTLY SOLVABLE MODEL: NUCLEON-NUCLEON SCATTERING

The validity of the perturbation approach developed in Sec. II can be established by comparing its predictions with the results of an exactly solvable model. As a concrete example, we shall apply our perturbation formalism to a simple model of nucleon-nucleon scattering proposed in Ref. [1]. The authors assumed the nucleons to interact via a potential, similar to that presented in Eq. (1), consisting of a local part as well as an isotropic velocity-dependent one. They solved the corresponding s-wave Schrödinger equation exactly and obtained an exact analytical expression, which they used to calculate the ^1S scattering phase-shift values exactly.

To this end let us consider the potential proposed in Ref. [1], which is a specific form of the potential presented in Eq. (1),

$$\hat{V}(r,p) = -V_0 J_1(r) + \frac{\hbar^2}{M} \boldsymbol{\nabla} \cdot \rho(r) \boldsymbol{\nabla}, \tag{29}$$

where

$$J_1(r) = 1 - U(r-b), \quad \rho(r) = \beta[1 - U(r-b)], \tag{30}$$

and the step function is defined as

$$U(r-b) = 0, \quad r < b; \quad = 1, \quad r \geq b. \tag{31}$$

The values of the potential parameters as reported in Ref. [1] are

$$V_0 = 16.9 \text{ MeV}, \quad \beta = -0.21, \quad b = 2.4 \text{ fm}, \tag{32}$$

and M is the nucleon mass. Clearly, the local and velocity-dependent parts of the potential have square well shapes with a common range $r=b$. In terms of the reduced wave function $v(r)=rR(r)$, the corresponding s-wave time-independent Schrödinger equation is exactly solvable and has the form

$$[1 - \beta + \beta U(r-b)]v''(r) + \left[k^2 + \frac{MV_0}{\hbar^2}[1 - U(r-b)] \right] v(r) = -\beta \left[v'(r) - \frac{v(r)}{r} \right] \delta(r-b), \tag{33}$$

where $k^2 = ME/\hbar^2$, and E is the energy available in the center

TABLE I. Exact ^1S phase shifts in the absence (δ_0) and presence (δ) of the perturbing potential. Columns 4–6 give the phase-shift corrections using our perturbation formalism.

	Exact phase shifts		Phase shift corrections			Perturbed phase shifts
E_{lab} MeV	δ_0 $\beta=0$	δ $\beta=-0.21$	δ_1	δ_2	δ_3	$\delta^p=\Sigma_{i=1}^3\delta_i$
20	0.949	0.871	−0.0992	0.0270	−0.0071	0.870
100	0.426	0.217	−0.2660	0.0764	−0.0170	0.219
180	0.276	−0.082	−0.4271	0.0664	0.0054	−0.079
260	0.256	−0.187	−0.4756	0.0279	−0.0002	−0.192
340	0.253	−0.197	−0.4790	0.0497	−0.0303	−0.207

of mass of the two interacting nucleons. The delta function on the right-hand side arises due to the sharp edge at $r=b$. Although the exact solutions must be continuous at the edge of the square well, their derivatives are not continuous at this point. In terms of the radial wave function $R(r)=v(r)/r$, the following boundary condition must be satisfied

$$(1-\beta)R'(b^-)=R'(b^+), \tag{34}$$

where $R'(b^-)$ and $R'(b^+)$ are the derivatives of the radial wave functions for r less and greater than b, respectively. This condition can be easily established by integrating Eq. (33) from $b-\epsilon$ to $b+\epsilon$, then taking the limit as ϵ tends to zero. By applying the boundary conditions to the exact solutions of the Schrödinger equation in Eq. (33) Razavy *et al.* obtained

$$k\cot(kb+\delta)=(1-\beta)K\cot(Kb)+\frac{\beta}{b}, \tag{35}$$

where

$$K^2=\frac{k^2}{(1-\beta)}+\frac{MV_0}{\hbar^2(1-\beta)}. \tag{36}$$

Using Eq. (32), the authors substituted for the potential parameters V_0, β, and b in Eq. (35) and obtained the exact values of the perturbed ^1S phase shifts δ, which are listed in the third column of Table I. Similarly, by switching off the perturbing velocity-dependent potential (which corresponds to setting $\beta=0$) we used the same equation to calculate the exact unperturbed phase-shift values δ_0, which are shown in the second column of Table I.

A. Results of the perturbation formalism

For the above interaction model of a two-nucleon scattering system, the perturbation formulas, developed in this work, produced exact analytical expressions for the changes in the phase shifts and wave functions. For example, using Eq. (17), the analytical expression obtained for the first-order phase-shift correction is

$$\delta_1=\frac{\beta}{k}\left(-\frac{1}{b}+\frac{K_0^2 b}{2\sin^2(K_0 b)}\right)\sin^2(kb+\delta_0)+\frac{\beta}{4}\sin(2kb+2\delta_0), \tag{37}$$

where

$$K_0^2=k^2+\frac{MV_0}{\hbar^2}. \tag{38}$$

However, the analytical expressions for the phase-shift corrections get progressively more involved as the perturbation order increases. Therefore, instead of giving the analytical forms of δ_2 and δ_3, we list their values together with that of δ_1 in columns 4–6 of Table I. In the last column we show the perturbed phase-shift values δ^p calculated up to and including the third-order correction. By comparing the perturbed values with the exactly determined ones, shown in the third column, it is clear that the perturbation approach has produced quite accurate results.

Using the simple potential in Eq. (29), it has also been possible to obtain analytical expressions for the wave-function corrections. In the internal region $r\leq b$, solving the differential Equation (12) directly, or using the derived formula in Eq. (20), results in the following analytical expression for the first-order wave-function correction:

$$v_1(r)=\frac{1}{2}A\beta K_0 r\cos(K_0 r)+\left(\frac{\delta_1\cos(kb+\delta_0)}{k\sin(K_0 b)}-\frac{1}{2}A\beta K_0 b\cot(K_0 b)\right)\times\sin(K_0 r), \tag{39}$$

where

$$A=\frac{\sin(kb+\delta_0)}{k\sin(K_0 b)}. \tag{40}$$

In the asymptotic region, $r\geq b$, the form of $v_1(r)$ is given by Eq. (14). Rather than giving the more complicated expression for the second-order wave-function correction, we present Fig. 1, which shows the unperturbed wave function $v_0(r)$ in addition to the first- and second-order wave-function corrections $v_1(r)$ and $v_2(r)$. All the plots correspond to a laboratory kinetic energy $E_{\text{Lab}}=2E=100$ MeV. Clearly, the

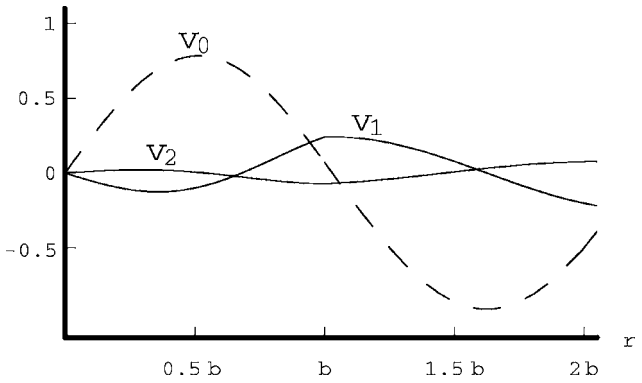

FIG. 1. The unperturbed wave-function $v_0(r)$. The first- and second-order wave-function corrections $v_1(r)$ and $v_2(r)$, respectively. All plotted at $E_{\mathrm{Lab}}=100$ MeV.

amplitudes of the wave-function corrections get progressively smaller as the perturbation order increases. This reflects the progressive decrease in the contributions to the perturbed phase shifts arising from higher-order corrections. Further, the discontinuity in the derivatives of the wave-function corrections at $r=b$ is clear.

The possibility of obtaining exact analytical expressions for the phase-shift and wave-function corrections depends on the form of the potential used. In general, one would expect that in cases where the velocity-dependent potential is more complicated, the perturbation formulas may not produce exact analytical solutions. In such situations, performing the integrals of the developed formulas numerically may become inevitable.

IV. COMPARISON WITH OTHER APPROACHES AND DISCUSSION

Other works have obtained phase shifts using perturbative or iterative approaches. However, unlike the case in this work, they considered a local potential only. For example, in the Born approximation [15] one has two coupled equations; one for $\sin(\delta)$ and the other for the radial-wave functions. The coupled equations may be solved by successive approximations in an iteration procedure. However, the process becomes cumbersome as one tries to go beyond the first-order approximation. In a more recent work [16] the authors started from the probability density equation and developed formulas for the changes in the phase shifts up to third order in the perturbing local potential. They also showed that only the zeroth-order probability density and the perturbing potential are needed to determine the phase-shift changes.

Another approach developed a hierarchical, logarithmic perturbation scheme for the scattering phase shifts in one dimension, which was then extended to a spherically symmetric three-dimensional system [17]. The phase-shift changes were expressed in terms of an effective potential in addition to known solutions of an unperturbed system, which appear in the denominator. In order to avoid the complications arising from the zeros of the physical wave functions belonging to the unperturbed problem, the authors took the logarithm not of the physical wave functions, but of an auxiliary complex wave function without nodes. Furthermore, using Levinson's theorem, they showed that when the unperturbed and exact-perturbed systems have the same number of bound states, the phase-shift series converges rapidly. In the following section we shall adopt the same approach in discussing the convergence of the perturbation series of Eq. (8).

In contrast with the above approaches we considered a velocity-dependent perturbing potential. Our derived formulas show that the phase-shift changes depend not only on the form of $\rho(r)$, but also on its derivative with respect to the radial variable r. Consequently, this dependence on $\rho'(r)$ seems to be a specific feature for velocity-dependent potentials. Although the wave-function corrections contain the term $1/v_0^2$, which is singular at the zeros of the unperturbed solutions, the changes in the wave functions remain finite as r approaches a given node of v_0. To prove this, let us consider the following general expression for the wave-function corrections, which can be obtained by inspecting Eqs. (20), (25), and (27):

$$v_n = v_0 \int_b^r \frac{dr'}{v_0^2} \int_0^{r'} F_n dr'' + C_n v_0, \tag{41}$$

where C_n is a constant and

$$F_n = v_0 \rho v_{n-1}'' + v_0 \rho' v_{n-1}' - \frac{v_0 v_{n-1}}{r} \rho'. \tag{42}$$

We shall start by considering the first-order correction v_1. In the vicinity of the node at $r=0$ the unperturbed s-wave solution v_0 vanishes like r and we shall assume that $\rho(r) \sim r^p$ as $r \to 0$. Consequently, F_1 approaches the origin at least like r^p $(p>0)$ or r^2 $(p=0)$. Hence, in the vicinity of the origin the first-order correction behaves like

$$v_1 \sim r(r^p - b^p) + C_1 r, \quad p > 0$$

$$\sim r(r^2 - b^2) + C_1 r, \quad p = 0. \tag{43}$$

Clearly, for $p \geq 0$, v_1 is finite and approaches the origin at least as fast as r. This behavior is seen in the exact expression for v_1 in Eq. (39) where $\rho(r)=\beta$ is a constant. The same type of argument above can be used to prove that the second-order correction v_2 is finite and also vanishes at least as fast as r close to the origin. Repeating the process for v_3, v_4, v_5,… order by order up to v_n one can show that each v_n is finite and vanishes at least as fast as r in the vicinity of the origin provided that $\rho \sim r^p$, where $p \geq 0$, as $r \to 0$. Although we have not given the exact expression for v_2 in Sec. III A, the fact that it is finite and vanishes at $r=0$ can be seen in Fig. 1. It is worth noting that the behavior of v_1'' close to a given node, which is needed to establish the behavior of v_2, can be determined by dividing Eq. (12) by v_0 and then employing the behaviors of v_0, v_1, and ρ as $r \to 0$. For $-1 < p < 0$ the integrand concerning the integral over r' in Eq. (41) diverges like r^{p-1} for $n=1$. However, the integral is multiplied by v_0, which vanishes like r near the origin. Keeping this in mind, the divergence problem can be overcome by displacing the pole at $r=0$ into the lower-half r-plane by an amount $i\epsilon$, where ϵ is vanishingly small and positive, and then taking the limit $\epsilon \to 0$ at the end of the evaluation. The

result shows that v_1 vanishes at least as fast as r^{p+1} in the vicinity of the origin. Repeating the process order by order one concludes that $v_n \sim r^{np+1}$ as $r \to 0$. Consequently, for some value of n, the corresponding v_n will diverge as the node at $r=0$ is approached. Hence we must set the condition

$$\rho \sim r^p, \quad r \to 0, \tag{44}$$

where $p \geq 0$.

In order to investigate the behavior of v_n in the vicinity of the remaining nodes of v_0 we may write Eq. (41) in the form

$$v_n(r) = v_0(r) \int_b^\infty \frac{dr'}{v_0^2(r')} \Theta(r-r') \int_0^\infty F_n(r'') \Theta(r'-r'') dr'' + C_n v_0(r). \tag{45}$$

where $\Theta(x-y)$ is a step function. Consequently, the above equation may be recast in the form

$$v_n(r) = \int_0^r \left[\frac{d}{dr'}(\rho v'_{n-1}) - \frac{v_{n-1}}{r}\rho' \right] dr' K(r',r) + C_n v_0(r), \tag{46}$$

where the kernel $K(r',r)$ is defined by

$$K(r',r) = v_0(r) v_0(r') \int_{r'}^r \frac{dr''}{v_0^2(r'')}, \tag{47}$$

and we have interchanged r' and r''. In the vicinity of a node at point a, then v_0 vanishes like $(r-a)$. Consequently, the term $1/v_0^2$ is singular at $r=a$, but the kernel $K(r',r)$ is finite. To see this let us evaluate the kernel by displacing the zeros of v_0 into the lower-half r plane by $i\epsilon$ where ϵ is vanishingly small and positive. At the end of the evaluation the limit $\epsilon \to 0$ is taken. This results in

$$K(r',r) = \lim_{\epsilon \to 0} B(r-a+i\epsilon)(r'-a+i\epsilon) \int_{r'}^r \frac{dr''}{(r''-a+i\epsilon)^2} = B(r-r'), \tag{48}$$

where B is a constant. Clearly, $K(r',r)$ is finite as r or r' approaches a node of v_0. For $n>1$, the kernel contains all the dependence on the unperturbed solution v_0. However, for $n=1$, the first term in the square brackets of Eq. (46) depends on v'_0 while the second term depends on v_0. In the vicinity of a node at $r=a$, then v'_0 approaches a constant value while v_0 vanishes like $(r-a)$ as noted before. Consequently, keeping the result of Eq. (48) in mind, one concludes that the wave-function corrections are finite at all the zeros of the unperturbed wave function v_0 provided Eq. (44) is satisfied.

The phase-shift corrections contain all the wave-function dependence in the numerator. Since we have shown that the wave-function corrections are finite at the zeros of v_0 then the phase-shift corrections are also finite at those points. However, in the vicinity of the origin, special attention must be paid to the term

$$\int_0^\infty \frac{v_0 v_{n-1}}{r} dr', \tag{49}$$

which occurs in the expression corresponding to each δ_n. Clearly, provided that the condition in Eq. (44) is satisfied, then the integrand vanishes at least as fast as r in the neighborhood of the node at $r=0$. Consequently, the integral is finite at the origin.

A. Convergence of the phase-shift perturbation series

The convergence of the phase-shift perturbation series of Eq. (8) can be investigated by making use of Levinson's theorem which, for each partial wave, may be stated as

$$\delta(0) - \delta(\infty) = \pi N. \tag{50}$$

Clearly, this theorem establishes a connection between the difference in the scattering phases at zero and infinite energies and the number of bound states N. In our example of nucleon-nucleon scattering the unperturbed and exact-perturbed systems each have one bound state. So by Levinson's theorem δ_0 and δ, the exact-unperturbed and perturbed phase shifts respectively, span the same range; the perturbation series then converges rapidly [17]. For a finite-range potential, which is the case we considered in this work, $\delta(\infty)=0$ and hence the value of $\delta(0)$ is determined by the number of bound states. Consequently, the scattering phase shifts of the exact perturbed and unperturbed systems coincide at zero energies leading to a rapidly converging series at low energies. This is clearly seen in our results of Table I where the agreement between the perturbed $\delta^p$and exact phase-shift values δ is best at the low energy of $E_{\mathrm{Lab}}=20$ MeV. The same effect is also seen in the results of Au *et al.* [17], who also considered the case when the exact perturbed and unperturbed systems sustain different numbers of bound states. In such a case, their results showed that the series is extremely poor at low energies. This highlights the importance of choosing an unperturbed system that has the same number of bound states as the exact perturbed problem being considered. The fact that our perturbation approach requires only on-shell information for the unperturbed system simplifies this task to a great extent.

V. HIGHER ORBITAL ANGULAR MOMENTA

In this section we shall briefly outline the generalization of the developed formalism to higher orbital angular momenta. For $l>0$, Eq. (3) becomes

$$[1-\lambda\rho(r)]\tilde{v}''(r) - \left[\tilde{v}'(r) - \frac{\tilde{v}(r)}{r}\right]\lambda\rho'(r) = \frac{2m}{\hbar^2}\left(V(r) + \frac{[1-\lambda\rho(r)]l(l+1)}{r^2} - E\right)\tilde{v}(r). \tag{51}$$

Although $V(r)$ and $\rho(r)$ are assumed to vanish identically for $r \geq b$, there is still a scattering contribution arising from the centrifugal barrier term $l(l+1)/r^2$ on the right-hand side of the above equation. Consequently, the boundary condition in Eq. (4) must be replaced by

$$\tilde{v}(r) \to \frac{1}{k}\sin\left(kr + \tilde{\delta} - \frac{l\pi}{2}\right), \quad r \to \infty, \tag{52}$$

where $\tilde{\delta}$ is the exact perturbed scattering phase shift corresponding to a given $l>0$. When the velocity-dependent perturbing potential is switched off ($\lambda=0$) then Eq. (51) reduces to the unperturbed equation

$$\tilde{v}_0''(r) = \frac{2m}{\hbar^2}\left(V(r) + \frac{l(l+1)}{r^2} - E\right)\tilde{v}_0(r), \tag{53}$$

which, in the asymptotic region, has the solution

$$\tilde{v}_0(r) \to \frac{1}{k}\sin\left(kr + \tilde{\delta}_0 - \frac{l\pi}{2}\right), \quad r \to \infty, \tag{54}$$

where $\tilde{v}_0$ and $\tilde{\delta}_0$ are the exact unperturbed wave-function and scattering phase shift, respectively, corresponding to some $l>0$, and both are assumed to be known. Expanding $\tilde{v}(r)$ and $\tilde{\delta}$ as in Eqs. (7) and (8) and proceeding in the same manner as in Sec. II then, for $n \geq 1$, Eq. (11) is modified to

$$\tilde{v}_n''\tilde{v}_0 - \tilde{v}_0''\tilde{v}_n = \tilde{v}_0\frac{d}{dr}(\rho\tilde{v}_{n-1}') - \frac{\tilde{v}_0\tilde{v}_{n-1}}{r}\rho' - \rho\frac{l(l+1)}{r^2}\tilde{v}_0\tilde{v}_{n-1}. \tag{55}$$

Further, the asymptotic forms for the first- and second-order wave-function corrections are now given by

$$\tilde{v}_1 \to \frac{\tilde{\delta}_1}{k}\cos\left(kr + \tilde{\delta}_0 - \frac{l\pi}{2}\right), \quad r \to \infty \tag{56}$$

and

$$\tilde{v}_2 \to \frac{1}{k}\left\{\tilde{\delta}_2\cos\left(kr + \tilde{\delta}_0 - \frac{l\pi}{2}\right) - \frac{1}{2}\tilde{\delta}_1^2\sin\left(kr + \tilde{\delta}_0 - \frac{l\pi}{2}\right)\right\}, \quad r \to \infty. \tag{57}$$

Taking the centrifugal barrier term in Eq. (55) into account and following similar steps as in Secs. II A and II B one may show that the first-order phase-shift and wave-function corrections, respectively, are

$$\tilde{\delta}_1 = k\int_0^\infty \frac{\tilde{v}_0^2}{r}\rho' dr + k\int_0^\infty \tilde{v}_0'\rho\tilde{v}_0' dr + k\int_0^\infty \rho\frac{l(l+1)}{r^2}\tilde{v}_0^2 dr, \tag{58}$$

$$\tilde{v}_1 = \tilde{v}_0\int_b^r \frac{dr'}{\tilde{v}_0^2}\int_0^{r'}\left[\tilde{v}_0\frac{d}{dr''}(\rho\tilde{v}_0') - \frac{\tilde{v}_0^2}{r''}\rho' - \rho\frac{l(l+1)}{r''^2}\tilde{v}_0^2\right]dr'' + \tilde{\delta}_1\cot\left(kb + \tilde{\delta}_0 - \frac{l\pi}{2}\right)\tilde{v}_0. \tag{59}$$

Formulas for the higher-order phase-shift and wave-function corrections may be easily obtained by following the same procedure. Provided $\rho \sim r^p$ where $p>0$ as r approaches the origin, then using an analysis similar to that presented after Eq. (41) one can show that the wave-function corrections are finite at the zeros of the unperturbed solution v_0. In particular, v_n vanish at least as fast r^{l+1} as r approaches the origin. The condition on ρ stated above is also sufficient for the phase-shift corrections to be finite at the nodes of v_0.

VI. CONCLUSIONS

In this work we considered the time-independent Schrödinger equation with a small perturbing velocity-dependent potential. Exact analytical formulas for the scattering phase-shift and wave-function corrections were derived. No prior knowledge of the scattering-states continuum is required. Only the zeroth-order solution and the perturbing potential are needed to determine the changes in the phase shifts and wave functions. The derived exact formulas show that the corrections depend not only on the form of $\rho(r)$, but also on its derivative with respect to the radial variable r. This seems to be a specific feature for perturbative velocity-dependent potentials. Although, for simplicity, the s-wave case was considered first, we extended the perturbation formalism for an arbitrary orbital angular momentum $l>0$. In addition, formulas for the phase-shift and wave-function corrections may be obtained to any given order in the perturbing potential. Although the formulas for the changes in the wave functions contain the square of the unperturbed solution in the denominator, we have shown that the wave-function corrections are finite as a given node of the unperturbed solution is approached. This is true provided the boundary condition in Eq. (44) is satisfied with $p \geq 0$ for the s-wave case and $p>0$ for $l>0$. In such cases the phase-shift corrections are also finite at the zeros of v_0 and $\tilde{v}_0$.

In order to establish the validity of our perturbation approach we applied it to a simple, exactly solvable model for a nucleon-nucleon interaction system proposed in Ref. [1]. The derived formulas produced exact analytical expressions for the phase-shift and wave-function corrections as can be seen, for example, in Eqs. (37) and (39). The calculated phase-shift changes are presented in columns 4–6 of Table I, while the perturbed phase-shift values δ^p determined up to and including the third-order correction are shown in the last column. The perturbed values compare quite well with the exactly obtained ones δ, which are reported in Ref. [1] and are shown in the third column of Table I. Further, the unperturbed wave function $v_0(r)$ as well as the first- and second-order corrections $v_1(r)$ and $v_2(r)$ are shown in Fig. 1.

The convergence properties of the phase-shift perturbation series given in Eq. (8) has been investigated using Levinson's theorem. Provided the unperturbed and exact-perturbed systems sustain the same number of bound states the series converges rapidly. For a finite-range potential the agreement is best at low energies. When the unperturbed and exact-perturbed systems have different numbers of bound states the perturbation series is extremely poor at low energies [17]. This illustrates the importance of choosing an unperturbed system that has the same number of bound states as the exact perturbed problem.

The developed formalism may be applied in studies considering nucleon-nucleon [1], pion-nucleon [2], and electron-atom scattering [3] as velocity-dependent potentials have been employed to model such interactions. Furthermore, the

Schrödinger equation with a nonconstant term by the second-order derivative (which is considered in this work) has been proposed to describe particles with a PDEM. The Schrödinger equation for a PDEM is extensively used in describing the dynamics of electrons in semiconductor heterostructures such as, compositionally graded crystals [4], quantum dots [5], and liquid crystals [6]. For example, electron scattering on disordered double-barrier heterostructures has been considered in Ref. [7]. It is hoped that the developed formalism may be applied in such fields, which are important in designing semiconductor devices. In addition, the Schrödinger equation for a PDEM is used in the energy-density-functional approach to the nuclear many-body problem [11] and its applications in the framework of nonlocal terms of the accompanying potential [18,19]. The developed formalism may also be of use in such studies.

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[1] M. Razavy, G. Field, and J. S. Levinger, Phys. Rev. **125**, 269 (1962).
[2] L. S. Kisslinger, Phys. Rev. **98**, 761 (1955).
[3] A. E. S. Green, D. E. Rio, and T. Ueda, Phys. Rev. A **24**, 3010 (1981).
[4] M. R. Geller and W. Kohn, Phys. Rev. Lett. **70**, 3103 (1993).
[5] L. I. Serra and E. Lipparini, Europhys. Lett. **40**, 667 (1997).
[6] M. Barranco, M. Pi, S. M. Gatica, E. S. Hernández, and J. Navarro, Phys. Rev. B **56**, 8997 (1997).
[7] I. Gómez, E. Diez, F. Domínguez-Adame, and P. Orellana, Physica E (Amsterdam) **18**, 372 (2003).
[8] A. R. Plastino, A. Rigo, M. Casas, F. Gracias, and A. Plastino, Phys. Rev. A **60**, 4318 (1999).
[9] A. de souza Durta, M. Hott, and C. A. S. Almeida, Europhys. Lett. **62**, 8 (2003).
[10] B. Bagchi, A. Banerjee, C. Quesne, and V. M. Tkachuk, J. Phys. A **38**, 2929 (2005).
[11] P. Ring and P. Schuck, *The Nuclear Many Body Problem* (Springer-Verlag, New York, 1980), p. 211.
[12] M. I. Jaghoub, Eur. Phys. J. A **27**, 99 (2006).
[13] M. I. Jaghoub, Eur. Phys. J. A **28**, 253 (2006).
[14] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill, New York, 1978), p. 330.
[15] E. Merzbacher, *Quantum Mechanics*, 2nd ed. (Wiley, New York and London, 1970), p. 244.
[16] G. C. Milward and C. Wilkin, J. Phys. A **34**, 5101 (2001).
[17] C. K. Au, C. K. Chow, C. S. Chu, P. T. Leung, and K. Young, Phys. Lett. A **164**, 23 (1992).
[18] F. Arias de Saavedra, J. Boronat, A. Polls, and A. Fabrocini, Phys. Rev. B **50**, 4248 (1994).
[19] A. Puente, L. I. Serra, and M. Casas, Z. Phys. D: At., Mol. Clusters **31**, 283 (1994).