Quantum teleportation of composite systems via mixed entangled states

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We analyze quantum teleportation for composite systems, specifically for concatenated teleporation (decomposing a large composite state into smaller states of dimension commensurate with the channel) and partial teleportation (teleporting one component of a larger quantum state). We obtain an exact expression for teleportation fidelity that depends solely on the dimension and singlet fraction for the entanglement channel and entanglement (measures by I concurrence) for the state; in fact quantum teleportation for composite systems provides an operational interpretation for I concurrence. In addition we obtain tight bounds on teleportation fidelity and prove that the average fidelity approaches the lower bound of teleportation fidelity in the high-dimension limit.

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I. INTRODUCTION

Quantum teleportation is a method of indirectly transmitting quantum information via dual resources of classical communication and previously shared entanglement [1]. In the ideal and the simplest case, quantum teleportation consumes one ebit and two classical bits to transmit one qubit, and this can be generalized to qudit [1] and continuous-variable quantum teleportation [2]. Quantum teleportation has been tested experimentally for polarization qubits in light [3], continuous-variable encoding in light [4], and qubits in ions [5], but not without controversy: debates have raged over whether postselection legitimately establishes that teleportation has occurred [6], what level of fidelity must be achieved [7,8], and whether teleportation must hold for all decompositions of the density matrix of the state to be teleported [9].

In this paper we study quantum teleportation for composite systems via mixed entangled states whereby just one component of the state is teleported (called partial quantum teleportation) and where the entire state is teleported by sending down parallel quantum teleportation channels or, sequentially, down the same quantum teleportation channel (called concatenated quantum teleportation). This description complies with any quantum information protocol that requires teleportation of the state of a composite quantum system via teleportation of the subsystems. We point out that, by concatenated teleportation, one can transmit "large" quantum states of dimension D through "small" teleportation channels designed to transmit low-dimensional quantum states of dimension d where $d < D$. Previously, only very special cases of noisy partial and concatenated teleportation have been studied restricted to two-qubit composite input states [10].

The faithfulness of quantum teleportation is characterized by its fidelity. The fidelity for a given state is the overlap of the initial state with its teleported counterpart. Fidelity may vary for each input state, and an average fidelity can be used to characterize teleportation performance overall: the fidelity is averaged over all input states, for example, over the Haar measure. In some cases, for maximally entangled states, or for mixed entangled states like Werner states [11] as the shared entangled state, fidelity is independent of the input state. Fortunately any shared mixed entangled state can be converted to a Werner state without changing its average teleportation fidelity [12], so the case of fidelity being independent of the input state is quite general. This independence of fidelity does *not* apply, however, for input states that are correlated or entangled with degrees of freedom that are not entangled, namely for partial quantum teleportation. Even if the entire state is ultimately teleported by concatenated teleportation, the principle that the fidelity is independent of the input state does not hold, in general. Thus previous notions of fidelity are challenged by generalizing quantum teleportation to the cases of partial quantum teleportation and concatenated quantum teleportation.

We introduce the "global" teleportation fidelity, which accounts for the overall performance of a finite set of teleportation channels used independently to teleport general composite input pure quantum states regardless of whether a part or the whole system is being teleported. The fidelity also indicates how well entanglement is transferred or preserved when teleported across noisy channels.

Furthermore we obtain quite general fidelity results for partial and concatenated quantum teleportation via noisy entangled states (Werner states), where each subsystem is teleported independently. For teleportation of a given input state, we show that (a) when the composite input state is a product, the global teleportation fidelity simply reduces to the product of teleportation fidelities of the individual teleportation channels and input states independent, (b) the fidelity is decreasing with increasing entanglement of the input state, (c) the upper bound is a product of the teleportation fidelities of the individual Werner states for concatenated telportation (for partial teleportation it is simply the teleportation fidelity of the only Werner state), and corresponds to a completely product input state, and (d) the lower bound corresponds to the case when the input state is maximally entangled for partial teleportation and maximally entangled across *all* bipartitions for concatenated teleportation. Whether this lower bound for concatenated teleportation can always be achieved

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remains an open problem. For partial teleportation this lower bound is just the *fully entangled fraction* [13] of the Werner state.

We also compute the *average* global teleportation fidelity where the average has been obtained for a uniform distribution of pure input states. We show that the average fidelity for partial teleportation approaches the lower bound as the dimension of the nonteleported component increases. This behavior is also noted in concatenated teleportation. We also obtain the bound on the initial entanglement of the input pure state such that the final state is also entangled for a given set of channel parameters.

The plan of the paper is as follows. In Sec. II we discuss teleportation fidelity and how the existing definition needs to be modified for teleportation of composite systems. Sections III and IV deal with partial and concatenated quantum teleportation. In Sec. V we discuss the problem of sending large quantum states through small quantum channels and we conclude this paper in Sec. VI with a summary of the main results and discussion of open problems.

II. TELEPORTATION FIDELITY

A. Teleportation fidelity for $d\otimes d$ mixed states

For a mixed entangled state described by density operator ρ over the $d\otimes d$ Hilbert space, one may characterize the degree of entanglement by the fully entangled fraction, or singlet fraction $F=\max_{\Psi}\langle\Psi|\rho|\Psi\rangle$, where the maximum is taken over all maximally entangled states $|\Psi\rangle$ [13]. For teleporting unknown d-dimensional pure states with a mixed entangled state ρ, Alice and Bob may achieve teleportation fidelity [12]

$$f=\frac{Fd+1}{d+1} \tag{1}$$

with trace-preserving local operations. For a perfect entangled channel, $F=1$ so $f=1$, indicating perfect teleportation capability, as expected.

Although the shared state ρ can be of a general form, every two-qudit state can be transformed to an isotropic mixed state (Werner state) [11]

$$\varrho(x)=x|\Phi^+\rangle\langle\Phi^+|+(1-x)\frac{\mathbb{1}}{d^2} \tag{2}$$

with $\mathbb{1}$ the identity operator over $d\otimes d$, and

$$|\Phi^+\rangle=\frac{1}{\sqrt{d}}\sum_{k=0}^{d-1}|k\rangle|k\rangle \tag{3}$$

a two-qudit maximally entangled state by local operations and classical communication (LOCC) with F remaining invariant [12]. We use the notation ϱ to designate the shared noisy entangled state and ρ for other density operators. The teleportation fidelity f_{w} and singlet fraction F_{w} for the state $\rho(x)$ are given by

$$f_{\text{w}}=\frac{1}{d}+x\left(\frac{d-1}{d}\right), \tag{4}$$

$$\mathcal{F}_{\text{w}}=\frac{1}{d^2}[1+x(d^2-1)], \tag{5}$$

respectively. This fidelity is independent of the input state if and only if the input state is pure which does not hold, in general. If the input state is mixed, then fidelity does not remain independent of the input state. This leads us to define global teleportation fidelity.

B. Global teleportation fidelity

As we are concerned with teleportation of composite quantum systems (subsystems are correlated, in general) by teleporting the individual subsystems, we measure the faithfulness of quantum teleportation in the following way: for a pure composite input state $|\Psi\rangle$, suppose the output state described by a density matrix is Ω, irrespective of whether a part or the complete state is teleported; then the global teleportation fidelity is defined as

$$\mathcal{F}=\langle\Psi|\Omega|\Psi\rangle. \tag{6}$$

The fidelity function defined here is a special case of Uhlmann's fidelity function for two mixed states [14]. From the definition, fidelity is a function of the input state and the channel parameters. Average fidelity is defined as

$$\langle\mathcal{F}\rangle=\int d\Psi\langle\Psi|\Omega|\Psi\rangle \tag{7}$$

with the integral carried out over a uniform distribution $d\Psi$ of all pure input states obeying the Haar measure.

If the state $|\Psi\rangle$ is a composite system of two or more subsystems and we are teleporting only one subsystem (partial teleportation), then the global teleportation fidelity is identical to Schumacher's entanglement fidelity [18]. In this paper we will continue to refer to global teleportation fidelity (for concatenated teleportation) or entanglement fidelity (partial teleportation) as simply the fidelity of teleportation.

III. PARTIAL QUANTUM TELEPORTATION

We now analyze quantum teleportation where a component of a state is teleported via a Werner state and the remaining part of the state is retained, perhaps for later verification of teleportation fidelity. As far as Alice and Bob are concerned they have been supplied with a mixed input state.

A. Fidelity

We begin with Alice and Bob (henceforth AB) sharing a Werner state in $d\otimes d$ as shown in Fig. 1. Victor supplies Alice with a qudit, which can be in a mixed state or entangled with a larger state held by Victor. In fact these two cases are equivalent: every mixed qudit state can be realized from a larger entangled pure state by performing some measurement on the larger space [20]. Hence we assume, without loss of generality, that Victor supplies part of a pure entangled qudit state $|\phi\rangle_{\text{VA}}$ in $d'\otimes d$ to Alice and retains the d'-dimensional component. Alice and Bob are also denied any knowledge of Victor's state. The teleportation protocol

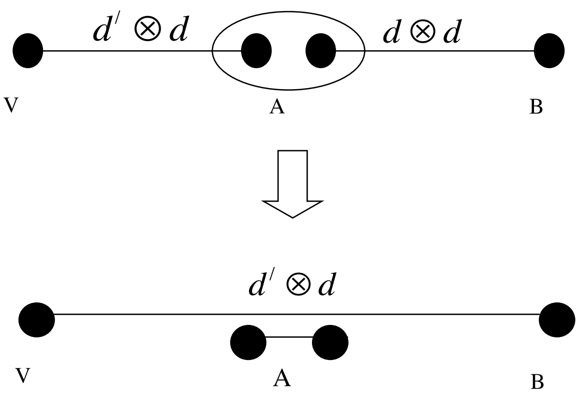

FIG. 1. Partial quantum teleportation. Alice (A) and Bob (B) share a Werner state ϱ_W in $d\otimes d$ space, and Victor (V) possesses a state in $d'\otimes d$. The d-dimensional subset of Victor's state is sent to Alice, who performs a Bell measurement (depicted by the ellipse). The resultant state is shown below. Specifically, Victor and Bob share a state in $d'\otimes d$, and Alice holds a two-qudit Bell state in $d\otimes d$.

follows the standard one introduced in Ref. [1]. The canonical generalized Bell basis states corresponding to the generalized Bell measurement are given by

$$|\psi\rangle_{nm} = \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1} e^{2\pi i jn/d}|j\rangle|j\oplus m\rangle, \tag{8}$$

where, $n,m=0,\ldots,d-1$ and $\oplus$ defines addition modulo d.

After Alice informs Bob of her measurement outcome nm, Bob performs a unitary operation on his qudit to recover the original state. The corresponding unitary operator is

$$U_{nm} = \frac{1}{\sqrt{d}}\sum_{k} e^{2\pi i kn/d}|k\rangle\langle k\oplus m|. \tag{9}$$

Subsequent to teleportation, which includes generalized Bell basis measurements [1] and transmitting classical information to Bob about the measurement outcomes, Victor and Bob share

$$\Omega_{\rm VB} = x|\phi\rangle_{\rm VB}\langle\phi| + \frac{1-x}{d}(\omega_{\rm V}\otimes \mathbb{1}_{\rm B}). \tag{10}$$

The faithfulness of this partial teleportation is quantified by the *fidelity*

$$\mathcal{F} = {}_{\rm VA}\langle\phi|\Omega_{VB}|\phi\rangle_{\rm VA} \tag{11}$$

$$= x + \frac{1-x}{d}\,{}_{\rm VA}\langle\phi|(\omega_{\rm V}\otimes \mathbb{1}_{\rm B})|\phi\rangle_{\rm VA}. \tag{12}$$

We note that the above fidelity is identical to entanglement fidelity introduced by Schumacher [18]. Observe that the second term on the right-hand side of Eq. (12) does not depend on representations of the states involved. Therefore, to evaluate it we employ the Schmidt decomposition for the state $|\phi\rangle_{\rm VA}$,

$$|\phi\rangle_{\rm VA} = \sum_{n=0}^{r-1}\lambda_n|n\rangle|n\rangle_{\rm VA} \tag{13}$$

with $r\leqslant \min(d,d')$ being the Schmidt rank.

The Schmidt coefficients $\{\lambda_n\}$ are real and non-negative, and normalization requires $\Sigma_{n=0}^{r-1}\lambda_n^2=1$. Thus, the reduced density matrix is given by

$$\omega = \sum_{n=0}^{r-1}\lambda_n^2|n\rangle\langle n| \tag{14}$$

and, using $\mathbb{1}_B=\Sigma_{k=0}^{d-1}|k\rangle_{\rm B}\langle k|$, we have,

$$\omega\otimes \mathbb{1}_{\rm B} = \sum_{n,k}\lambda_n^2|nk\rangle_{\rm VB}\langle nk|, \tag{15}$$

which leads to

$${}_{\rm VA}\langle\phi|(\omega\otimes \mathbb{1}_{\rm B})|\phi\rangle_{\rm VA} = \sum_{n=0}^{r-1}\lambda_n^4. \tag{16}$$

Hence,

$$\mathcal{F} = x + \frac{1-x}{d}\sum_{n=0}^{r-1}\lambda_n^4. \tag{17}$$

As $\Sigma_n\lambda_n^4$ is simply the purity $\mathcal{P}=\mathrm{Tr}(\omega^2)$ for ω the reduced density matrix and the I concurrence [17], $C_\phi=\sqrt{2(1-\mathcal{P})}$, we have,

$$\mathcal{F} = x + \frac{1-x}{d}\mathcal{P} \tag{18}$$

$$= f_{\rm w} - \frac{1-f_{\rm w}}{2(d-1)}C_\phi^2. \tag{19}$$

Consequences of this result include the following key points: (a) if $|\phi\rangle_{\rm VA}$ is a product state, then $C(\phi)=0$, and teleportation fidelity reduces to $f_{\rm w}$, and (b) whenever the input state is entangled, i.e., $C(\phi)>0$, the fidelity seen by Victor is strictly less than $f_{\rm w}$. Note that the fidelity decreases with increasing entanglement of the input state.

As we have considered the most general input state $|\phi\rangle_{\rm VA}$ (any mixed state can necessarily be prepared from a pure state by measurement) part of which being teleported, we have the following proposition: $\mathcal{F}=f_{\rm w}$ if and only if $|\phi\rangle_{VA}$ is a product state of the form $|\phi_1\rangle_V|\phi_2\rangle_A$.

We have thus far calculated the fidelity for partial teleportation for a particular input state. Given the functional dependence of the fidelity on the entanglement of the input state it is natural to ask for the bounds on fidelity. In the next subsection we obtain these bounds.

B. Bounds on fidelity

The upper bound is obviously $f_{\rm w}$ as the right-hand term in Eq. (19) vanishes when $C_\phi=0$ corresponding to a product input state of the form $|\phi\rangle=|\phi_1\rangle_{\rm V}|\phi_2\rangle_{\rm A}$. The lower bound can be obtained when, for a given input state $|\phi\rangle$, C_ϕ is a maximum. This corresponds to $|\phi\rangle_{\rm VA}$ being a maximally entangled state of Schmidt rank d where $d=\min\{d,d'\}$ and subsequently $C_\phi=\sqrt{2(d-1/d)}$. Thus the lower bound is

$$\mathcal{F}\geqslant x+\frac{1-x}{d^2}=F_\mathrm{w}, \tag{20}$$

which is the fully entangled fraction of the Werner state. The bounds on $\mathcal{F}$ can now be completed:

$$F_\mathrm{w}\leqslant \mathcal{F}\leqslant f_\mathrm{w}. \tag{21}$$

The lower bound is saturated if and only if the state is maximally entangled of Schmidt rank d. Next we answer the question of what fidelity is expected on an average for randomly picked input states.

C. Average fidelity

Equation (19) shows that the deviation of $\mathcal{F}$ from f_w is only a function of "entanglement" of the input state $|\phi\rangle_{\mathrm{VA}}$ and it is quite possible that the entanglement of the input state may not be known beforehand. Thus the average fidelity will provide a good indication of what is expected for randomly picked input states. From Eq. (18), it follows that the average fidelity is

$$\langle\mathcal{F}\rangle=x+\frac{1-x}{d}\langle\mathcal{P}\rangle \tag{22}$$

taken over the Haar measure. Thus we only need to evaluate the average purity of the input state. From [19],

$$\langle\mathcal{P}\rangle=\frac{d+d'}{dd'+1}, \tag{23}$$

where the average is over all pure states in $d'\otimes d$ distributed according to the unitarily invariant or uniform distribution. Substituting, we obtain

$$\langle\mathcal{F}\rangle=x+\frac{1-x}{d}\left(\frac{d+d'}{dd'+1}\right)=f_\mathrm{w}-\frac{d'-1}{dd'+1}(1-f_\mathrm{w}). \tag{24}$$

The average fidelity is not only a function of channel parameter f_w and the dimension d of the state teleported (also the channel is a $d\otimes d$ Werner state), it is also a function of the dimension of the subsystem not teleported. In fact, this shows that for partial teleportation the average fidelity can be affected by changing the dimension of the idle subsystem.

To understand how changing d' (the dimension of the idle subsystem), affects the average fidelity while keeping d a constant, we note that $(d'-1)/(dd'+1)$ is a monotonically increasing function of d'. Thus $\langle\mathcal{F}\rangle$ decreases monotonically with increasing d'. In the limit of large $d'\gg d$, and in particular when $d'\to\infty$,

$$\langle\mathcal{F}\rangle\to x+\frac{1-x}{d^2}=F_\mathrm{w}. \tag{25}$$

Thus the average fidelity approaches the lower bound when the dimension of the idle subsystem is very large compared to that of the particle teleported ($d'\gg d$). Physically it means, as we increase the dimension of Victor's subsystem, and randomly pick entangled states from the composite Hilbert space, we will find states that are highly entangled (close to being maximally entangled) with a high probability. Let us note that the phenomenon of the purity of an average bipartite state tending towards maximally mixed is also a consequence of Levy's lemma [15] (for an application see [16]). Next we discuss how fidelity can be connected to entanglement of the teleported state and how well entanglement is conserved in the process of partial teleportation.

D. Entanglement of the output state

A strong indication of entanglement preservation may be obtained by evaluating the fully entangled fraction of the output state Ω (denoted by F_Ω). To make matters simple, we assume that the dimension of Victor's system is also d.

Suppose $|\phi\rangle$ is the composite pure input state in $d\otimes d$ whose one share is teleported. Let $|\Phi\rangle$ be the maximally entangled state in the Schmidt basis of $|\phi\rangle$. Therefore, $F_\phi=|\langle\Phi|\phi\rangle|^2$. One can easily show that $\langle\Phi|\omega\otimes\mathbb{1}|\Phi\rangle=1/d$, where ω is the reduced density matrix of the subsystems corresponding to the state $|\phi\rangle$. Let Ω be the output state after teleportation as given by Eq. (10). Then the fully entangled fraction is

$$F_\Omega=\langle\Phi|\Omega|\Phi\rangle=xF_\phi+\frac{(1-x)}{d^2}. \tag{26}$$

It immediately follows that $F_\Omega\leqslant F_\mathrm{w}=x+(1-x)/d^2$ where the equality holds if $|\phi\rangle$ is a maximally entangled state. Let us note that this is consistent with the fundamental notion that entanglement across a bipartition can never increase on an average by LOCC. To find out if Ω is entangled or not, we use a sufficient criterion of entanglement [12] namely: if the fully entangled fraction of a mixed is state in $d\otimes d$ is greater than $1/d$ then it is entangled. Therefore, if the condition

$$F_\Omega>\frac{1}{d} \tag{27}$$

is satisfied, then Ω is certainly entangled. It is important to note that, if the above condition is violated, it does not imply that Ω is separable. Rewriting Eq. (27),

$$F_\phi>1-\frac{d^2-1}{F_\mathrm{w}d^2-1}\left(F_\mathrm{w}-\frac{1}{d}\right), \tag{28}$$

puts a lower bound on the entanglement of all input states, which are guaranteed to remain entangled after partial teleportation, for a given Werner state. Note that if the Werner state is poorly entangled, i.e., $F_\mathrm{w}\approx 1/d$, then the above inequality shows that, to remain entangled after teleportation, the input state must be highly entangled.

IV. CONCATENATED QUANTUM TELEPORTATION

Partial quantum teleportation is the simplest nontrivial case of teleportation of composite quantum systems where only one component of a composite system is teleported. More generally, for a composite system of a finite number of subsystems, there is the case for which every component or a subset are teleported. This is concatenated quantum teleportation. We first discuss representation and entanglement of multipartite pure states and how the structure of an input state plays a role in determining the final fidelity.

A. Representation of multipartite quantum states

Quantum information encoded in a d-dimensional pure quantum state $|\Psi\rangle \in \mathcal{H}_d$, where $\mathcal{H}_d$ is a d-dimensional Hilbert space, may be represented by

$$|\Psi\rangle = \sum_{i=0}^{d-1} a_i |i\rangle, \quad (29)$$

where the set of states $\{|i\rangle, i=0,\ldots,d-1\}$ is an orthonormal basis in $\mathcal{H}_d$. The expansion coefficients a_i's are, in general, complex and satisfy the normalization condition $\sum_{i=0}^{d-1}|a_i|^2 = 1$. Any N-qudit pure state $|\Psi\rangle \in \mathcal{H} = \otimes_{i=1}^{N}\mathcal{H}_i$ can be represented by

$$|\Psi\rangle = \sum_{k_1k_2,\ldots,k_N=0}^{d-1} a_{k_1k_2\ldots k_N}|k_1k_2,\ldots,k_N\rangle \quad (30)$$

where (a) the set of states $\{|k_1k_2,\ldots,k_N\rangle, k_i=0,\ldots,d-1\}$ form the standard basis in the Hilbert space $\mathcal{H}$ of N qudits and (b) the coefficients $a_{k_1k_2,\ldots,k_N}$ are, in general, complex and satisfy the normalization condition $\sum_{k_1k_2\ldots k_N}|a_{k_1k_2\ldots k_N}|^2 = 1$. A convenient way to express a general state of N qudits that takes into account all possible configurations including entanglement between only a subset of qudits, may be represented by a tensor product of say, $M(M \leqslant N)$ pure states $|\Phi_l\rangle$

$$|\Psi\rangle = \bigotimes_{l=1}^{M}|\Phi_l\rangle, \quad |\Phi_l\rangle \in \mathcal{H}_d^{\otimes\alpha_l}, \quad (31)$$

which is a composite nonseparable state of α_l qudits. In particular, the Hilbert space is decomposed as $\mathcal{H}_d^{\otimes N} = \otimes_{l=1}^{M}\mathcal{H}_d^{\otimes\alpha_l}$ and we also have $\sum_{l=1}^{M}\alpha_l = N$. Each state $|\Phi_l\rangle$ is defined such that no further product decomposition is possible. In other words, the state $|\Phi_l\rangle$ is entangled if and only if $|\alpha_l| > 1$. Thus a completely entangled state corresponds to the case when the decomposition has only one term. On the other hand, if the state is completely a product, the decomposition consists of N terms.

B. Entanglement of multipartite pure states

Entanglement of a bipartite pure quantum state can be represented by I concurrence defined by $C = \sqrt{1(1-\mathcal{P})}$, where $\mathcal{P} = \mathrm{Tr}(\omega^2)$ is the purity of the associated reduced density matrix [17]. For a pure state of more than two subsystems one can accordingly define I concurrence across every bipartition. For example, for a system of three qudits, one can have a set of three I concurrences, C_1, C_2, C_3, where C_1 is the entanglement across the bipartition 1:23 and so on. In the same way $\mathcal{P}_i = \mathrm{Tr}(\omega_i^2)$ is the purity of the ith reduced density matrix.

Generalizing this concept, entanglement of an N-qudit pure state may be characterized by a set I concurrences,

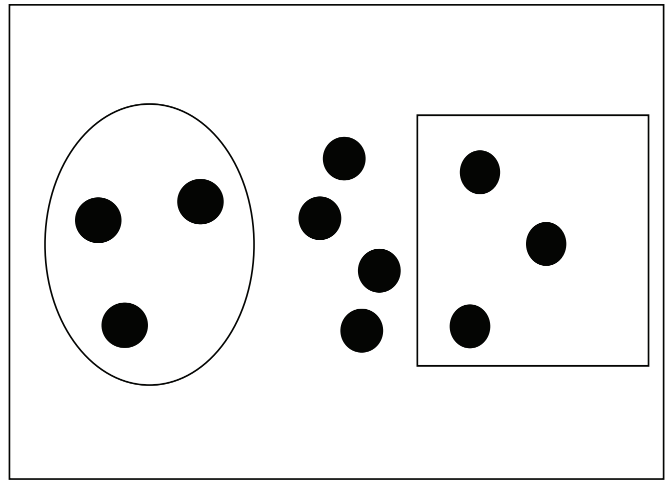

FIG. 2. Venn diagram representing two distinct bipartitions of a ten-qudit state such that three qudits are in one partition and seven in the other, i.e., the 3:7 bipartition. Two configurations m are shown, one three qudits on the left in one partition (depicted by a circle) and the remaining seven in the other partition, and the other configuration corresponds to the three qudits in the partition on the right (depicted by a square) vs the other seven qudits.

where each I concurrence quantifies entanglement across a bipartition. As $|\Psi\rangle$ is an N-qudit pure state, it is also a pure state across every bipartition and therefore admits a Schmidt decomposition. Consider a bipartition with k qudits on one side and $N-k$ on the other, where $1 \leqslant k \leqslant \lfloor N/2 \rfloor$. Associated with each k there can be $\binom{N}{k}$ distinct configurations. Let $C_{(k)m}(\Psi)$ be the I concurrence of the state across the bipartition k: $N-k$ in configuration m. The subscript m is a string denoting k qudits that are on one side of the partition.

For example, consider a pure state of ten qudits and the bipartition 3:7 (i.e., three qudits on one side and seven on the other side), which is depicted in Fig. 2: $\mathcal{H}_d^{\otimes 10} = \mathcal{H}_d^{\otimes 3} \otimes \mathcal{H}_d^{\otimes 7}$. Then $C_{(3)1,5,6}$ represents the I concurrence of the state in the partition 3:7 and in a configuration 1,5,6:2,3,4,7,8,9,10 (i.e., the qudits 1st, 5th, and 6th are on one side). Furthermore in a given configuration m which a string of length $k: i,j,\ldots,m$ (where each index refers to a qudit, and is assigned a value between 1 and N) we have the oredering $i < j < \cdots < m$.

The last constraint is necessary because, for a chosen set of k parties on one side of the bipartition, permutation among the same set of parties will be a different string but identical configuration as far as entanglement is concerned. In the above example, the last constraint implies the strings 1,5,6 and 5,6,1 although different but entanglement-wise correspond to identical configuration. Similarly the density matrix $\omega_{i,j,\ldots,m}^{(k)}$ represents the reduced density matrix of the pure state in the bipartite partition $k:N-k$ and configuration $i,j,\ldots,m$. Thus for any N-qudit pure state we can associate a set of functions $\{C_{(k)m}(\Psi)\}$ that can characterize entanglement of the state.

C. Representation of N-qudit input states and fidelity

The expansion in Eq. (31) is completely general. The key idea is teleportation fidelity of every term in the decomposition is independent of the other terms. Consider any one term in the separable expansion, say the state $|\Phi_j\rangle$ of α_j qudits. Clearly α_j entangled states are required to teleport the entire state. Let us denote the final density matrix after teleportation by Ω_j. Once every state in the decomposition is teleported, the resulting density matrix can be represented by

$$\Omega = \overset{M}{\underset{j=1}{\otimes}} \Omega_j. \tag{32}$$

The concatenated teleportation fidelity is

$$\mathcal{F} = \langle \Psi | \Omega | \Psi \rangle = \prod_{j=1}^{M} \langle \Phi_j | \Omega_j | \Phi_j \rangle = \prod_{j=1}^{M} \mathcal{F}_j, \tag{33}$$

where $\mathcal{F}_j$ is the concatenated teleportation fidelity for the state $|\Phi_j\rangle$. From the above expression it is clear that what we need to evaluate is the teleportation fidelity of an indecomposable pure state of j qudits for $j \geqslant 2$. We now consider the simplest case where a pure two-qudit state is teleported via two noisy Werner states.

D. Concatenated teleportation of a pure two-qudit state

1. Teleported density matrix

Any two-qudit pure input state in $d \otimes d$ can be written in the form

$$|\Psi\rangle = \sum_{i,j=0}^{d-1} a_{ij} |ij\rangle. \tag{34}$$

Let $\sigma = |\Psi\rangle\langle\Psi|$ and $C = \sqrt{2(1-\mathrm{Tr}(\omega^2)}$ be the I concurrence [17] with ω the reduced density matrix for each qudit. We represent each teleportation channel by a Werner state in $d \otimes d$. Let us denote them by $\varrho_{A_1B_1}$ and $\varrho_{A_2B_2}$ and the corresponding teleportation fidelities by $f_1(x_1), f_2(x_2)$ with x_1, x_2 the mixing parameters of the respective Werner states. Figure 3 depicts two-qudit concatenated teleportation, which is readily generalized to N-qudit concatenated teleportation.

The concatenated teleportation protocol is performed in sequence where each qudit is independently teleported according to the standard teleportation of a qudit [1] (also see the section of partial quantum teleportation for details). The final density matrix can be computed in a straightforward manner. Explicitly

$$\Omega_{B_1B_2} = x_1 x_2 \sigma_{B_1B_2} + \frac{(1-x_1)x_2}{d} \mathbb{1}_{B_2} \otimes \omega_{B_1} + \frac{(1-x_2)x_1}{d} \omega_{B_2} \otimes \mathbb{1}_{B_1} + \frac{(1-x_1)(1-x_2)}{d^2} \mathbb{1}_{B_1} \otimes \mathbb{1}_{B_2}. \tag{35}$$

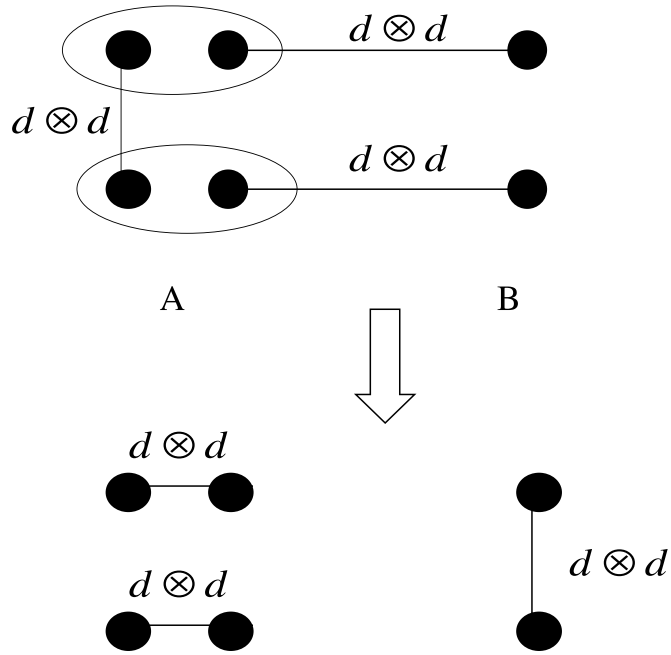

FIG. 3. Two-qudit concatenated quantum teleportation. Initially (seen in the upper part of the figure), Alice (A) and Bob (B) share two Werner states across two $d \otimes d$ channels, and Alice combines each of her qudits in the Werner states with the two qudits supplied by Victor to perform Bell measurements (depicted by the ellipses). Subsequent to this (shown in the lower part of the figure), Alice possesses two two-qudit Bell states, and Bob holds a two-qudit state that could be pure or mixed.

As it is understood that the above two-qudit density matrix belongs to Bob, we will not be using the subscript.

2. Fidelity

The teleportation fidelity is given by

$$\mathcal{F} = \langle \Psi | \Omega | \Psi \rangle = x_1 x_2 + \frac{(1-x_1)x_2}{d} \langle \Psi | \mathbb{1} \otimes \omega | \Psi \rangle + \frac{(1-x_2)x_1}{d} \langle \Psi | \omega \otimes \mathbb{1} | \Psi \rangle + \frac{(1-x_1)(1-x_2)}{d^2}. \tag{36}$$

To evaluate the terms $\langle \Psi | \mathbb{1} \otimes \omega | \Psi \rangle$ and $\langle \Psi | \omega \otimes \mathbb{1} | \Psi \rangle$ on the right-hand side of the above equation, we proceed as shown in the previous section, to obtain

$$\langle \Psi | \mathbb{1} \otimes \omega | \Psi \rangle = \langle \Psi | \omega \otimes \mathbb{1} | \Psi \rangle = \mathrm{Tr}(\omega^2) = \mathcal{P}; \tag{37}$$

and noting that $f_i = x_i + (1-x_i)/d$, $i=1,2$, the fidelity can be rewritten as

$$\mathcal{F} = f_1 f_2 - (1-\mathcal{P})G = f_1 f_2 - \tfrac{1}{2} G C^2, \tag{38}$$

where

$$G = \frac{1}{(d-1)^2}([1-f_1][f_2 d - 1] + [1-f_2][f_1 d - 1]) \tag{39}$$

and $C = \sqrt{2(1-\mathcal{P})}$. We also note that $\mathcal{F}$ reduces to a product of the individual fidelities of the Werner states if and only if

$C=0$, i.e., if the input state is a product state. On the other hand, $\mathcal{F}=1$ if and only if both $f_1=f_2=1$. The expression for fidelity reduces to that of partial quantum teleportation when one of the two fidelities becomes equal to unity as one would expect. Perhaps the most compelling resemblance with partial teleportation is that the fidelity is again a decreasing function of the entanglement of the input state. The more entangled the input state is, the lower the fidelity and vice versa.

The upper bound on the fidelity is f_1f_2 for a product input state ($C=0$) and the lower bound is obtained when, for a given $|\Psi\rangle$, C is maximum. This occurs when $|\Psi\rangle$ is a maximally entangled state of Schmidt rank d for which $C=\sqrt{2(d-1)/d}$. Therefore,

$$f_1f_2-\frac{d-1}{d}G\leqslant\mathcal{F}\leqslant f_1f_2. \tag{40}$$

From Eq. (38) the average fidelity is computed,

$$\langle\mathcal{F}\rangle=f_1f_2-(1-\langle\mathcal{P}\rangle)G=f_1f_2-\frac{(d-1)^2}{d^2+1}G,$$

where we have used $\langle\mathcal{P}\rangle=2d/(d^2+1)$ [19]. When d is large, i.e., $d\gg1$, neglecting terms of the order of $1/d^2$ and higher yields $\langle\mathcal{F}\rangle\approx f_1f_2-((d-1)/d)G$ which is the lower bound on $\mathcal{F}$. Once again we see that, for large quantum states, the average fidelity is approximately close to the lower bound.

So far, we have studied teleportation of two body composite systems, be it partial where one subsystem is teleported or concatenated, where both subsystems are teleported. Not surprisingly essentially the same features in both the cases have been observed. In the next section we discuss the simplest case of few body system teleportation: concatenated teleportation of three qudits.

E. Concatenated teleportation of three-qudit pure states

This involves teleportation of a three-qudit pure state via three Werner states, each capable of teleporting one qudit. The fidelity of the ith, $i=1,2,3$, Werner state is f_i, and the mixing parameter is x_i. Let ω_i be the reduced density matrix for the ith qudit. Proceeding along the lines of the previous sections, the teleportation fidelity is

$$\mathcal{F}=\prod_{i=1}^{3}x_i+\frac{1}{d^3}\prod_{i=1}^{3}(1-x_i)+\sum_{k=1}^{3}\mathrm{Tr}(\omega_k^2)\left(\frac{1-x_k}{d}\prod_{\substack{i=1\\ i\neq k}}^{3}x_i+\frac{x_k}{d^2}\prod_{\substack{i=1\\ i\neq k}}^{3}(1-x_i)\right). \tag{41}$$

Denoting

$$G_k=\left(\frac{1-x_k}{d}\prod_{\substack{i=1\\ i\neq k}}^{3}x_i+\frac{x_k}{d^2}\prod_{\substack{i=1\\ i\neq k}}^{3}(1-x_i)\right) \tag{42}$$

and using the relations $f_i=x_i+1-x_i/d$ and $\mathcal{P}_k=\mathrm{Tr}(\omega_k^2)$, the fidelity

$$\mathcal{F}=\prod_{i=1}^{3}f_i-\sum_{k=1}^{3}G_k(1-\mathcal{P}_k)=\prod_{i=1}^{3}f_i-\frac{1}{2}\sum_{k=1}^{3}G_kC_k^2 \tag{43}$$

is a function of the channel parameters f_i and dimension d and also depends on the complete set of I concurrences $\{C_i\},i=1,2,3$. If the input state is completely entangled, i.e., $C_i\neq0$ for all i, or partially entangled, i.e., $C_i\neq0$ for some i, then $\mathcal{F}<\Pi_{i=1}^3f_i$. The equality $\mathcal{F}=\Pi_{i=1}^3f_i$ is achieved if and only if the state is a product state of the form $|\phi\rangle_1\otimes|\phi\rangle_2\otimes|\phi\rangle_3$. This means that the presence of entanglement between any two subsystems will necessarily reduce the fidelity.

We now obtain the bounds on $\mathcal{F}$. The upper bound is given by $\Pi_{i=1}^3f_i$. The lower bound corresponds to the case when $\Sigma_{k=1}^3G_kC_k^2$ takes a maximum value. As G_k for all k is only a function of dimension and channel parameters and does not depend on the state teleported, the maximum value corresponds to a three-qudit pure state for which the set of I concurrences, $\{C_i\}$ will maximize $\Sigma_{k=1}^3G_kC_k^2$. Now every C_k is associated with a bipartition in $d\otimes d^2$ Hilbert space where, for a maximally entangled state, the maximum Schmidt rank is d and therefore corresponds to the maximum I concurrence for all pure states in the Hilbert space $d\otimes d^2$. In principle, the lower bound, therefore, corresponds to a given input state, such that $C_i=\sqrt{2(d-1)/d}$ for all i. The primary concern is whether such a state actually exists. In a system of three qudits, it turns out that such states do exist, and below is an example Greenberger-Horne–Zeilinger (GHZ) state:

$$|\Psi\rangle_{123}=\frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|iii\rangle_{123}. \tag{44}$$

Thus the bounds on $\mathcal{F}$ are given by

$$\prod_{i=1}^{3}f_i-\frac{d-1}{d}\sum_{k=1}^{3}G_k\leqslant\mathcal{F}\leqslant\prod_{i=1}^{3}f_i. \tag{45}$$

The average fidelity is given by

$$\langle\mathcal{F}\rangle=\prod_{i=1}^{3}f_i-\sum_{k=1}^{3}G_k(1-\langle\mathcal{P}_k\rangle). \tag{46}$$

To evaluate $\langle\mathcal{P}_k\rangle$ we need to compute the average over all pure states in $d\otimes d^2$. From [19] we obtain $\langle\mathcal{P}_k\rangle=d+d^2/(d^3+1)$. Substituting

$$\langle\mathcal{F}\rangle=\prod_{i=1}^{3}f_i-\frac{d^2-2d+1}{d^2-d+1}\sum_{k=1}^{3}G_k. \tag{47}$$

For large $d\gg1$, neglecting terms of the order $1/d^2$ and higher, one can show that

$$\langle\mathcal{F}\rangle\approx\prod_{i=1}^{3}f_i-\frac{d-1}{d}\sum_{k=1}^{3}G_k, \tag{48}$$

which is the lower bound on $\mathcal{F}$. This again shows that for large dimensional quantum states, the average teleportation fidelity can be well approximated by the lower bound.

F. Concatenated teleportation of an entangled state of N qudits

In this section we study the most general case where a pure state of N qudits are being teleported via N noisy Werner states. The teleportation fidelity derived in the Appendix is given by $\langle\Psi|\Omega|\Psi\rangle$

$$\mathcal{F}=\prod_{i=1}^{N} f_i-\frac{1}{2}\sum_{k=1}^{N} C_k^2 G_k^{(1)}-\frac{1}{2}\sum_{\substack{k,l=1\\k<l}}^{N} C_{kl}^2 G_{kl}^{(2)}-\frac{1}{2}\sum_{\substack{k,l,m=1\\k<l<m}}^{N} C_{klm}^2 G_{klm}^{(3)}-\cdots . \quad (49)$$

Interestingly the fidelity shows that entanglement across every bipartition matters in determining the total fidelity. Depending on the structure of the state, some of the concurrence functions may be zero. The upper bound is $\Pi_{i=1}^{N} f_i$ and that corresponds to the case of a pure product input state. If there is entanglement between any subset of qudits, which means there must be at least one concurrence function not equal to zero, then the fidelity is always strictly less than the upper bound. The lower bound, in principle, can be attained by a state that is maximally entangled across all bipartitions. We do not know if such a state actually exists, in general.

It is also possible to compute the average fidelity, which is

$$\langle\mathcal{F}\rangle=\prod_{i=1}^{N} f_i-\frac{1}{2}\sum_{k=1}^{N} \langle C_k^2\rangle G_k^{(1)}-\frac{1}{2}\sum_{\substack{k,l=1\\k<l}}^{N} \langle C_{kl}^2\rangle G_{kl}^{(2)}-\frac{1}{2}\sum_{\substack{k,l,m=1\\k<l<m}}^{N} \langle C_{klm}^2\rangle G_{klm}^{(3)}-\cdots . \quad (50)$$

Across any bipartition, $C^2=2(1-\mathcal{P})$. Thus $\langle C^2\rangle=2(1-\langle\mathcal{P}\rangle)$. Using the result in [19] we get

$$\langle C_k^2\rangle=2\frac{(d^{N-1}-1)(d-1)}{d^N+1}=2h_1(d)\ \ \forall\, k,$$

$$\langle C_{kl}^2\rangle=2\frac{(d^{N-2}-1)(d^2-1)}{d^N+1}=2h_2(d)\ \ \forall\, k,l;k<l,$$

$$\langle C_{klm}^2\rangle=2\frac{(d^{N-3}-1)(d^3-1)}{d^N+1}=2h_3(d)\ \ \forall\, k,l,m;k<l<m. \quad (51)$$

Substituting, we obtain

$$\langle\mathcal{F}\rangle=\prod_{i=1}^{N} f_i-h_1\sum_{k=1}^{N} G_k^{(1)}-h_2\sum_{\substack{k,l=1\\k<l}}^{N} G_{kl}^{(2)}-h_3\sum_{\substack{k,l,m=1\\k<l<m}}^{N} G_{klm}^{(3)}-\cdots . \quad (52)$$

We can further compress the above expression by noting that all Gs are just functions of the channel parameters and dimension. Thus, denoting $\Sigma_{k=1}^{N} G_k^{(1)}=G_1, \Sigma_{\substack{k,l=1\\k<l}}^{N} G_{kl}^{(2)}=G_2$ and so on, we can rewrite the average fidelity as

$$\langle\mathcal{F}\rangle=\prod_{i=1}^{N} f_i-\sum_{i=1}^{M} h_i G_i \quad (53)$$

where $M=\lfloor N/2\rfloor$. Note that the i^{th} term pertains to the bipartition $i:N-i$.

V. APPLICATION: TELEPORTATION OF LARGE QUANTUM STATES VIA SMALL QUANTUM CHANNELS

As an application of our results of concatenated teleportation we consider the problem of sending large quantum states (say of dimension D) using small quantum channels designed to teleport states of lower dimensions (say $d<D$). A simple instance of the problem applies for the case that Charlie wants to teleport a three-dimensional pure state

$$|\Psi\rangle=a_0|0\rangle+a_1|1\rangle+a_2|2\rangle \quad (54)$$

with fidelity, say, f_0. However, Alice and Bob can only teleport qubits, i.e., they have only qubit-teleportation channels; entangled states that are in $2\otimes 2$. Surely Alice and Bob cannot directly teleport the state of Charlie. A solution, however, lies in Charlie being able to map, in principle, his qutrit onto a state of two qubits. For example,

$$|\Psi_1\rangle=a_0|00\rangle+a_1|01\rangle+a_2|10\rangle, \quad (55)$$

following which Alice and Bob can teleport each qubit via their qubit-teleportation channels. It is clear that in this case two-qubit-teleportation channels suffice to teleport the complete quantum state. If we assign fidelities f_1,f_2 for the respective qubit-teleportation channels, then certain conditions must be satisfied by the channel fidelities to attain an overall teleportation fidelity f_0.

Our results show that if Charlie wants to maximize the fidelity the optimal mapping is the one with the least entanglement. Clearly in this case one cannot have a two-qubit product state representing the three-qubit state of Charlie. Thus the two-qubit state must be entangled but minimally so. It is evident that there are many possible mappings of the state (54) onto a pure entangled state of two qubits [one of the mappings is (55)]. An example of an alternative mapping is the state

$$|\Psi_2\rangle=a_0|01\rangle+a_1|10\rangle+a_2|11\rangle. \quad (56)$$

With respect to quantum information, the states (55) and (56) carry the same amount of information. However, one can easily check that the entanglement of the two mappings is different which implies $\mathcal{E}(\Psi_1)\neq\mathcal{E}(\Psi_2)$, where $\mathcal{E}$ is an entanglement measure. Therefore, one has to choose the one mapping which provides the least entanglement, i.e., $\min\{\mathcal{E}(\Psi_1),\mathcal{E}(\Psi_2)\}$. Of course, these two mappings are not the only possibilities.

In a general instance of the problem, the input state to be teleported is a D-dimensonal quantum state and the teleportation channels can teleport quantum states of dimension d

(qudit), $d<D$. The first step is to map the D-dimensional state onto a state of N qudits where $N=\lceil\frac{\log_{10}D}{\log_{10}d}\rceil$. Note that after such a mapping the N-qudit state may be entangled. The teleportation of this N-qudit state requires N teleportation channels. This is essentially an optimizing problem where fidelity must be optimized over all possible mappings. This is because different mappings will lead to different entanglement in the mapped states without changing the information content. However, one should choose a mapping that minimizes the entanglement of the mapped state.

VI. CONCLUSION

We have presented an in-depth analysis of quantum teleportation of composite systems via mixed entangled states. In the case of partial teleportation, a subset of the state is teleported, and fidelity is identified with how well the entire state (the teleported component and the unteleported comment) is preserved by teleportation. In the case of concatenated teleportation, the entire state is teleported albeit after segmenting it in states in smaller Hilbert spaces and teleported "piece by piece."

Our analysis produces exact expressions for the fidelity of teleportation in both cases. Furthermore we obtain strict upper and lower bounds on exact fidelity and prove that, for a large-dimensional system, $\langle\mathcal{F}\rangle\to\mathcal{F}_{\text{lower bound}}$. An intuitive picture of the expression of the multichannel fidelity can be given by expanding the initial set of Werner states, thus fidelity is simply a convex sum of the fidelities given by various possibilities.

The fidelity is a function of the concurrences for all bipartitions of the state; thus concatenated teleportation, provides an operational context for employing I concurrence. Our analysis presents a couple of interesting challenges that will be considered in a future work. One issue concerns hybridization of qudits so that a multiqudit state, with each qudit of d dimensions, can be converted to a multiqudit state, with each qudit of $d'<d$ qudits. Although quantum gates on hybrid qudits have been studied [21], actual conversion between qudits of different dimensions, which would be needed for concatenated quantum teleportation, has not yet been studied.

Another issue concerns optimal performance of concatenated teleportation. In concatenated teleportation, Alice and Bob employ parallel channels for teleporting a large state through small channels. We assumed that states in each of the channels are uncorrelated and unentangled with each other, and the generalized Bell state measurements on each channel are assumed to be independent of each other. At the other extreme, the entanglement resoures for all channels could be entangled and collective Bell measurements are made over all channels; this approach could take us back to the case of full teleportation of the state with unit fidelity. It would be interesting to explore the relaxation of our assumptions of independent channels and distinct Bell state measurements and observe the improvement of fidelity, as well as ascertain the separate powers of collective measurements vs entangled channels.

In summary we have significantly extended studies of quantum teleportation to partial teleportation and concatenated teleportation with exact results and bounds for fidelity. Moreover this analysis opens up promising new areas of investigation and may ultimately lead to improved teleportation for 'large' states through 'small' channels.

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APPENDIX

In this Appendix we explicitly derive the expression of fidelity for the general case of concatenated teleportation where a N-qudit pure state is teleported via N teleportation channels. The fidelity of the ith Werner state is f_i, and the mixing parameter is x_i. Let the density matrix corresponding to the N-qudit input state be $|\Psi\rangle\langle\Psi|=\sigma$. Let the teleported density matrix be Ω. It can be expressed as

$$
\begin{aligned}
\Omega = {} & \prod_{i=1}^{N} x_i\sigma + \frac{1}{d^N}\prod_{i=1}^{N}(1-x_i)\bigotimes_{i-1}^{N}\mathbb{1}_i + \sum_{k=1}^{N}\left((\mathbb{1}_k\otimes\omega_{\neq k}^{(N-1)})\frac{1-x_k}{d}\prod_{\substack{i=1\\ i\neq k}}^{N}x_i + (\omega_k^{(1)}\otimes\mathbb{1}_{\{\neq k\}}^{\otimes N-1})\frac{x_k}{d^{N-1}}\prod_{\substack{i=1\\ i\neq k}}^{N}(1-x_i)\right)\\
& + \sum_{\substack{k,l=1\\ k<l}}^{N}\left((\mathbb{1}_l\otimes\omega_{\neq kl}^{(N-2)})\frac{(1-x_k)(1-x_l)}{d^2}\prod_{\substack{i=1\\ i\neq k,l}}^{N}x_i + (\omega_{kl}^{(2)}\otimes\mathbb{1}_{\{\neq kl\}}^{\otimes N-2})\frac{x_kx_l}{d^{N-2}}\prod_{\substack{i=1\\ i\neq k,l}}^{N}(1-x_i)\right)\\
& + \sum_{\substack{k,l,m=1\\ k<l<m}}^{N}\left(\frac{1}{d^3}(\mathbb{1}_{klm}\otimes\omega_{\neq klm}^{(N-3)})(1-x_k)(1-x_l)\times(1-x_m)\prod_{\substack{i=1\\ i\neq k,l,m}}^{N}x_i + (\omega_{klm}^{(3)}\otimes\mathbb{1}_{\{\neq klm\}}^{\otimes N-3})\frac{x_kx_lx_m}{d^{N-3}}\times\prod_{\substack{i=1\\ i\neq k,l,m}}^{N}(1-x_i)\right) + \cdots\\
& + \sum_{\{g\}}\left((\mathbb{1}_{g=\{x_j\}}^{\otimes g}\otimes\omega_{\neq\{g\}}^{(N-g)})\frac{\prod_{j=\{g\}}(1-x_j)}{d^g}\times\prod_{\substack{i=1\\ i\neq\{g\}}}^{N}x_i + (\omega_{=\{g\}}^{(g)}\otimes\mathbb{1}_{\{\neq g\}}^{\otimes N-g})\frac{\prod_{k=\{g\}}x_k}{d^{N-g}}\times\prod_{\substack{i=1\\ i\neq\{g\}}}^{N}(1-x_i)\right),
\end{aligned}
\tag{A1}
$$

where (a) $\{g\}$ stands for a set of k qudits $i,j,\ldots,p$ with $i<j<,\ldots,<p$ where $k=N/2$ (N even), $N-1/2$ (N odd) (b) for

example, $\omega^{(2)}_{\neq kl}$ is the reduced density operator of $N-2$ qudits excluding qudits k and l. Note that $\omega^{(2)}_{\neq kl}=\omega^{(2)}_{kl}$ and $\omega^{(3)}_{\neq klm}=\omega^{(3)}_{klm}$ and so on due to Schmidt decomposition across every bipartite partition.

The fidelity is computed using the formula

$$\mathcal{F}=\langle\Psi|\Omega|\Psi\rangle \tag{A2}$$

and is given by

$$\mathcal{F}=\prod_{i=1}^{N}x_i+\frac{1}{d^N}\prod_{i=1}^{N}(1-x_i)+\sum_{k=1}^{N}\mathrm{Tr}(\omega^{(1)}_k)^2G^{(1)}_k+\sum_{k,l=1;k<l}^{N}\mathrm{Tr}(\omega^{(2)}_{kl})^2G^{(2)}_{kl}+\sum_{\substack{k,l,m=1\\k<l<m}}^{N}\mathrm{Tr}(\omega^{(3)}_{klm})^2G^{(3)}_{klm}+\cdots, \tag{A3}$$

where

$$G^{(1)}_k=\frac{1-x_k}{d}\prod_{i=1,i\neq k}^{N}x_i+\frac{x_k}{d^{N-1}}\prod_{\substack{i=1\\i\neq k}}^{N}(1-x_i), \tag{A4}$$

$$G^{(2)}_{kl}=\frac{(1-x_k)(1-x_l)}{d^2}\prod_{\substack{i=1\\i\neq k,l}}^{N}x_i+\frac{x_kx_l}{d^{N-2}}\prod_{\substack{i=1\\i\neq k,l}}^{N}(1-x_i), \tag{A5}$$

$$G^{(3)}_{klm}=\frac{(1-x_k)(1-x_l)(1-x_m)}{d^3}\prod_{\substack{i=1\\i\neq k,l,m}}^{N}x_i+\frac{x_kx_lx_m}{d^{N-3}}\prod_{\substack{i=1\\i\neq k,l,m}}^{N}(1-x_i), \tag{A6}$$

etc. The total number of indices is constrained by the possible bipartitions.

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