Trellises for stabilizer codes: Definition and uses

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Trellises play an important theoretical and practical role for classical codes. Their main utility is to devise complexity-efficient error estimation algorithms. Here, we describe trellis representations for quantum stabilizer codes. We show that they share the same properties as their classical analogs. In particular, for any stabilizer code it is possible to find a minimal trellis representation. Our construction is illustrated by two fundamental error estimation algorithms.

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I. INTRODUCTION

Since the discovery of efficient quantum algorithms for solving hard classical problems, many efforts have been devoted to building quantum processing devices. While small-scale prototypes are readily available, scalability remains a practical issue because of the extreme sensibility of quantum devices to external noise. Fortunately, theoretical advances, such as the discovery of error-correction schemes and fault-tolerant implementations, have paved the way for quantum computing. One way of building quantum codes is through the stabilizer formalism [1,2]. An (n,k) stabilizer code protects k qubits by encoding them into an n-qubit register. The error recovery procedure involves the measurement of a syndrome (i.e., a vector in $\mathbb{F}_2^{n-k}$), which is used to partially discriminate the actual error E from all possible ones. The role of the error model, and in general of any *a priori* information about E, is to allow further discrimination in order to find the most likely guess $\hat{E}$. While one can imagine computing the likelihood of all possible errors compatible with the measured syndrome, such a method is impractical when codes are large. This is because there are 2^{n+k} such elements. This problem is well known in classical coding theory, but was rightfully ignored in the quantum case as block codes that can be physically implemented have extremely small length. However, with the development of quantum communication and the advent of other coding strategies, this shall no longer be the case.

In classical coding theory, one often relies on a graphical representation of the code, called a *trellis*, to perform error estimation. For instance, trellises yield many complexity-efficient error estimation schemes for memoryless channels as well as the means to estimate the noise parameters. In particular, it can be used to calculate with linear complexity the most likely error for a convolutional code of bounded memory over any memoryless channel. In this paper, we apply general results concerning group codes for classical communication [3] to show that a similar representation is available for quantum stabilizer codes. Two error estimation schemes that exploit trellises are introduced for memoryless channels. We show that the complexity of these algorithms is related to the number of trellis vertices, and provide a construction of a trellis that minimizes this quantity. One of these algorithms achieves the performance of an algorithm for convolutional codes that was proposed in [4], but with a significantly lower complexity.

II. STABILIZER CODES: ELEMENTARY FACTS

In the rest of this paper, some familiarity with quantum computation is assumed. This section provides a brief introduction to stabilizer codes; for a more detailed introduction, the reader is redirected to [1,2,5] and references therein.

In what follows, without loss of generality, the quantum register of interest has n physical qubits.

Preliminaries. Stabilizer codes rely heavily on properties of $\mathcal{G}_n$, the n-qubit Pauli group. This group is defined in terms of the Pauli matrices for a single qubit: $\mathcal{I}=\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, $\mathcal{X}=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, $\mathcal{Y}=\left(\begin{smallmatrix}0&-i\\i&0\end{smallmatrix}\right)$, $\mathcal{Z}=\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$. The group $\mathcal{G}_n$ is the multiplicative group generated by the n-fold tensor products of single-qubit Pauli matrices.

For our purpose here, phases are irrelevant and it will be more convenient to work with the effective Pauli group $G_n \triangleq \mathcal{G}_n/\{\pm\mathcal{I}^{\otimes n};\pm i\mathcal{I}^{\otimes n}\}$ (see [2]). The elements of G_1 will be denoted by $I\triangleq[\mathcal{I}]$, $X\triangleq[\mathcal{X}]$, $Y\triangleq[\mathcal{Y}]$, and $Z\triangleq[\mathcal{Z}]$. Here, $[\mathcal{P}]$ denotes the equivalence class of $\mathcal{P}\in\mathcal{G}_n$, that is, $\{\pm\mathcal{P},\pm i\mathcal{P}\}$. Note that G_n is Abelian, so that we will use the additive notation for its group operation. Since $G_n\cong G_1^n$, we often view $P\in G_n$ as an n-tuple $(P^i)_{i=1}^n$ with entries in G_1.

The crucial fact about $\mathcal{G}_n$ is that any pair of elements $\mathcal{P},\mathcal{Q}$ either commutes or anticommutes. This leads to the definition of an inner product "$\star$" for elements of G_n such that $(P^i)_i\star(Q^i)_i=\Sigma_i P^i\star Q^i \bmod 2$. Here, $P^i\star Q^i=1$ if $P^i\neq Q^i$, $P^i\neq I$, and $Q^i\neq I$; and $P^i\star Q^i=0$ otherwise. One can then check easily that $\mathcal{P},\mathcal{Q}\in\mathcal{G}_n$ commute if and only if $[\mathcal{P}]\star[\mathcal{Q}]=0$.

Error model. Stabilizer codes can accommodate for a broad class of channels. For simplicity, only memoryless Pauli channels will be considered, although the tools presented in this paper extend to other memoryless channels. Memoryless Pauli channels act on the whole n-qubit register as $\sigma\rightarrow\Psi(\sigma)=(\Psi^1\otimes\Psi^2\otimes\cdots\otimes\Psi^n)(\sigma)$. Above Ψ^i is a one-qubit channel whose action on ρ, a single-qubit density operator, is given by $\Psi^i(\rho)=\Sigma_{\mathcal{E}\in\{\mathcal{I},\mathcal{X},\mathcal{Y},\mathcal{Z}\}}\Pr_i([\mathcal{E}])\mathcal{E}\rho\mathcal{E}$, with $\Pr_i(\cdot)$ a probability distribution over G_1.

Definition of the code subspace. The code subspace C of an (n,k) stabilizer code is the largest subspace stabilized by the action of $\mathcal{S}$, an Abelian subgroup of $\mathcal{G}_n$. For the code to

protect k qubits using n, i.e., to be of rate k/n, $\mathcal{S}$ must be generated by $n-k$ independent operators $\mathcal{S}_j$, and be such that $-\mathcal{I}^{\otimes n} \notin \mathcal{S}$. The code subspace is equivalently defined by $n-k$ eigenvalue equations: $|\psi\rangle \in C$ if and only if $\forall\, j \in \{1,\ldots,n-k\}, \mathcal{S}_j|\psi\rangle = |\psi\rangle$.

To study the main properties of these codes, phases are again irrelevant. More precisely, it is sufficient to represent the set of generators of the stabilizer group $\{\mathcal{S}_j\}_j$ by the set of equivalence classes $\{S_j\}_j$ where $S_j=[\mathcal{S}_j]$, which generate a subgroup S of G_n. Using a slight abuse in terminology, we also call S the stabilizer group of the code and $\{S_j\}_j$ the stabilizer set of the code.

Error estimation. The goal of error estimation is to infer channel errors from their action on the state of the quantum register. In the context of stabilizer codes, the necessary information is provided by the measurement of the Hermitian operators $\mathcal{S}_j$.

Let $\mathcal{E} \in \mathcal{G}_n$ be the actual, yet unknown, quantum error that affected the state $|\psi\rangle \in C$. The measurement of the operators $\mathcal{S}_j$ on $|\psi'\rangle \triangleq \mathcal{E}|\psi\rangle$ defines a binary vector of length $n-k$ called the syndrome of $\mathcal{E}$: $s(\mathcal{E}) \triangleq (s^j(\mathcal{E}))_{j=1}^{n-k} \triangleq \frac{1}{2}(1-\langle\psi'|\mathcal{S}_j|\psi'\rangle) = (S_j \bigstar [\mathcal{E}])_{j=1}^{n-k}$. Among all possible error operators $\mathcal{F} \in \mathcal{G}_n$, only some of them are compatible with $s(\mathcal{E})$; i.e., they satisfy $s(\mathcal{F})=s(\mathcal{E})$. It is readily checked that they belong to a coset of $N(\mathcal{S})$, the normalizer of $\mathcal{S}$ in $\mathcal{G}_n$. Equivalently, these compatible errors $\mathcal{F}$ are such that $[\mathcal{F}]$ belongs to a coset of $S^\perp \triangleq \{P \in G_n : P \bigstar Q = 0, \forall\, Q \in S\}$. While the knowledge of the syndrome restricts the class of errors that could have happened during the transmission, the error model further discriminates between these elements by assigning them probabilities. Error recovery then uses these probabilities to find a best guess $\hat{\mathcal{E}}$ for the actual error $\mathcal{E}$. For instance, maximum likelihood error estimation consists in finding a most likely $\hat{\mathcal{E}}$ compatible with the measured syndrome $s(\mathcal{E})$. Trellises are both aimed at computing these probabilities and at choosing a best guess efficiently.

III. TRELLISES FOR STABILIZER CODES

Considering the previous remarks about error estimation, it is natural to seek a representation of cosets of $S^\perp$ in which it is easy to search for their most likely element. This is the main motivation for the definition of trellises for quantum as well as for classical codes. In a broader context, this motivation extends to the calculation of quantities on which error estimation is based, e.g., the likelihood function, the *a posteriori* qubit error probability, etc.

Definition. In the rest of this section, S denotes the stabilizer group of a quantum code with parameters (n,k), and $\{S_j\}_j$ is its stabilizer set.

An *n-section trellis* relative to the stabilizer set $\{S_j\}_j$ and syndrome $s \in \mathbb{F}_2^{n-k}$ is a directed multigraph with the following properties:

(1) Its vertices can be grouped into $n+1$ sets V_i, with $|\mathsf{V}_0| = |\mathsf{V}_n| = 1$. The set V_i is called the ith *state space* of the trellis.

(2) Its edges are directed and can be grouped into n sets E_i. An edge $\mathsf{e} \triangleq (\mathsf{vw})$ is said to be issued at vertex v and ending at vertex w. Edges in E_i are issued from a vertex of V_{i-1} and end at a vertex of V_i. The set E_i is called the ith *section* of the trellis.

(3) An edge $\mathsf{e} \in \mathsf{E}_i$ bears a label $l(\mathsf{e})$ such that $l(\mathsf{e}) \in G_1$. We say that $l(\mathsf{e})$ is the *Pauli label* of e.

(4) Each element $P \in G_n$ with syndrome s is associated to a unique directed path $(\mathsf{e}^1,\ldots,\mathsf{e}^n)$ such that $l(\mathsf{e}^i)=P^i$.

An example of a trellis is considered in Fig. 1 for the four-qubit code with stabilizer group generated by $\{XXXX, ZZZZ\}$ and relative to syndrome $s=(0,0)$.

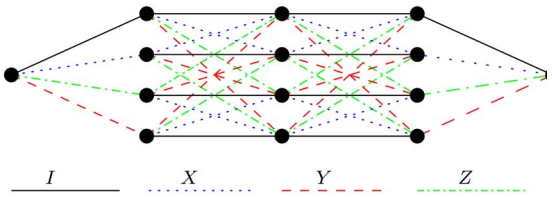

FIG. 1. (Color online) Trellis representation for the four-qubit code and syndrome (0,0). Here, $|\mathsf{V}_0| = |\mathsf{V}_4| = 1$ and $|\mathsf{V}_1| = |\mathsf{V}_2| = |\mathsf{V}_3| = 4$.

Construction. Given a code stabilized by S, there are many possible trellises representing the coset of $S^\perp$ of syndrome s. We provide here a simple construction for which the number of vertices in each V_i is bounded by 2^{n-k}. In analogy with classical codes, this trellis will be called the *Wolff trellis* of the code relative to the stabilizer set $\{S_j\}_j$ and to the syndrome s.

For every $i \in \{0,\ldots,n\}$, let π_i be a mapping from G_n to itself defined by $\pi_i(P^1,P^2,\ldots,P^n) = (P^1,\ldots,P^i,I,\ldots,I)$ [with the convention that $\pi_0(P^1,P^2,\ldots,P^n) = (I,\ldots,I)$]. Let P_s be an arbitrary, but fixed, element of G_n with syndrome s. For every $i \in \{0,\ldots,n\}$, V_i is a subset of $\mathbb{F}_2^{n-k}$ defined by $\{(S_j \bigstar \pi_i(P))_{j=1}^{n-k} : P \in P_s + S^\perp\}$. A vertex $\mathsf{v} \in \mathsf{V}_i$ is connected to vertex $\mathsf{w} \in \mathsf{V}_{i+1}$ with an edge labeled by E if there exists $P \in P_s + S^\perp$ such that $\mathsf{v} = (S_j \bigstar \pi_i(P))_{j=1}^{n-k}$, $\mathsf{w} = (S_j \bigstar \pi_{i+1}(P))_{j=1}^{n-k}$, and $P^{i+1} = E$.

An example of a trellis obtained in this way is given in Fig. 2 for the five-qubit code relative to the stabilizer set $\{ZXIII, XZXII, IXZXI, IIXZX\}$ and to syndrome $s=(0,0,1,1)$.

Since the V_i's are obviously binary affine subspaces, it follows that their cardinality is a power of 2. Let $\xi_i \triangleq \log_2|\mathsf{V}_i|$. The ξ_i's are easily upper-bounded by bringing in the following quantities. Let $S_{\text{start}\leqslant i}$ be the subset of operators of $\{\mathcal{S}_j\}_j$ that have at least one of their i-first components different from I. Let $S_{\text{end}\leqslant i}$ be the subset of operators of $\{\mathcal{S}_j\}_j$ that have all their last $n-i$ components equal to I. Observe that $S_j \bigstar \pi_i(P)$, the jth coordinate of the elements of V_i satisfies the following properties: (i) equal to zero when $\mathcal{S}_j$ does not belong to $S_{\text{start}\leqslant i}$, and (ii) equal to the jth coordinate s_j of the syndrome for $\mathcal{S}_j \in S_{\text{end}\leqslant i}$. This implies that the dimension of the affine subspace V_i is at most $|S_{\text{start}\leqslant i}| - |S_{\text{end}\leqslant i}|$. In other words, we have the following.

Lemma 1. $\xi_i \leqslant |S_{\text{start}\leqslant i}| - |S_{\text{end}\leqslant i}|$.

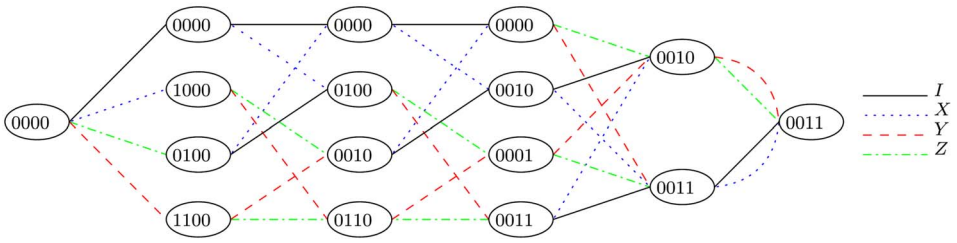

FIG. 2. (Color online) Trellis for the five-qubit code defined by the stabilizer set $\{ZXIII, XZXII, IXZXI, IIXZX\}$ and for the syndrome $s=(0011)$ obtained through the second construction. Here $|\mathsf{V}_0|=|\mathsf{V}_5|=1$, $|\mathsf{V}_1|=|\mathsf{V}_2|=|\mathsf{V}_3|=4$, and $|\mathsf{V}_4|=2$.

In particular, this lemma can be applied to quantum convolutional codes to show that they have trellises with bounded state-space size. Following the definition of [4,6], an (n,k) stabilizer code is convolutional with parameters (η,κ) if there exists a set of generators $\{S_j\}_j$ of its stabilizer group S with an η-qubit shift invariance property. More precisely, the values S_j^i must be equal to the entries $H_j^i \in G_1$ of an infinite matrix H that satisfies $H_{j+\eta-\kappa}^{i+\eta}=H_j^i$ for every i,j. In such a case, all $|S_{\text{start}\leqslant i}|-|S_{\text{end}\leqslant i}|$'s are obviously upper bounded by a common constant.

Minimality. As will be seen below, the complexity of many useful algorithms using trellises is linear in their number of vertices. This raises the issue of finding a trellis that minimizes this quantity. If one is willing to change the stabilizer set for the code in order to put it in a *trellis-oriented form*, then the Wolff trellis is minimal. Indeed, the trellis obtained in this way does not only minimize the number of vertices but also the *state-space profile*. As for classical trellises, we define the state-space profile of the trellis of an (n,k) code by the $(n+1)$-tuple $(\xi_0,\xi_1,\ldots,\xi_n)$. In other words, we are going to prove that the Wolff trellis applied to a stabilizer set in trellis-oriented form minimizes each ξ_i individually. Without loss of generality, we now assume that the trellis is associated with the syndrome $s=(0,\ldots,0)$.

First, we define the trellis-oriented form of a stabilizer set. For $1\leqslant j\leqslant n-k$, let $c(j)$ and $d(j)$ be, respectively, the position of the first (last) component of S_j, which is different from I. We say that the stabilizer set $\{S_j\}_j$ is in trellis-oriented form if and only if for all j, (i) $c(j)$ is a nondecreasing function, (ii) $S_{j'}^{c(j)}=I$ for $j'>j+1$ and $S_{j+1}^{c(j)}\neq S_j^{c(j)}$, and (iii) there is at most one $j'\neq j$ such that $d(j)=d(j')$ and in such case $S_j^{d(j)}\neq S_{j'}^{d(j')}$. Note that any stabilizer set $\{S_j\}_j$ can be put in trellis-oriented form by permuting and adding the generators.

Second, we show a lower bound on ξ_i. For an (n,k) stabilizer code given by the stabilizer set $\{S_j\}_j$, and for $i\in\{0,\ldots,n\}$, let C_i^f (the *future* subgroup) be the subgroup of $S^\perp$ whose elements have their first i components equal to I, and let C_i^p (the *past* subgroup) be the subgroup of $S^\perp$ whose elements have their last $n-i$ components equal to I. We then have the following lemma.

Lemma 2. $\xi_i\geqslant n+k-\log_2|C_i^p|-\log_2|C_i^f|$.

Proof. Let $\mathsf{v}\in\mathsf{V}_i$, and C_v be the set of elements of $S^\perp$ that correspond to a path in the trellis associated with the all-zero syndrome and passing through v. Let P and F be the set of all paths that go from v_0 to v (and from v to v_n, respectively). Let Q be a fixed element of C_v and $\mathsf{q_P q_F}$ its corresponding path in the trellis, where $\mathsf{q_P}\in\mathsf{P}$ and $\mathsf{q_F}\in\mathsf{F}$. By construction of the trellis, we have $|C_\mathsf{v}|=|\mathsf{P}||\mathsf{F}|$. Note that any path $\mathsf{p}\in\mathsf{P}$ can be extended into a path $\mathsf{pq_F}$ from v_0 to v_n. Finally, observe that such a path $\mathsf{pq_F}$ is associated with an element of $S^\perp$ that belongs to the coset $Q+C_i^p$ and therefore $|\mathsf{P}|\leqslant|Q+C_i^p|=|C_i^p|$. Similarly $|\mathsf{F}|\leqslant|C_i^f|$, which yields $2^{n+k}=|S^\perp|=\Sigma_{\mathsf{v}\in\mathsf{V}_i}|C_\mathsf{v}|\leqslant|\mathsf{V}_i||C_i^p||C_i^f|$. This gives the desired bound on ξ_i by taking the logarithm. ■

Finally we conclude with the following theorem.

Theorem 1. The Wolff trellis achieves the previous bound on ξ_i for each i when the stabilizer set is in trellis-oriented form.

Proof. From Lemma 1 we know that $\xi_i\leqslant|S_{\text{start}\leqslant i}|-|S_{\text{end}\leqslant i}|$. Note that C_i^p is the subgroup of G_n with I on their $n-i$ last components and orthogonal to all elements of $S_{\text{start}\leqslant i}$. This implies $\log_2|C_i^p|=2i-|S_{\text{start}\leqslant i}|$. Using a similar argument, we get that $\log_2|C_i^f|=2(n-i)-(n-k-|S_{\text{end}\leqslant i}|)$. Adding these (in)equalities, we conclude that $\xi_i+\log_2|C_i^p|+\log_2|C_i^f|\leqslant n+k$. Since Lemma 2 gives the reverse inequality, we actually have $\xi_i+\log_2|C_i^p|+\log_2|C_i^f|=n+k$. ■

IV. USING TRELLISES OF STABILIZER CODES

Min-Sum (Viterbi) algorithm. The Min-Sum algorithm is certainly one of the most widely employed algorithms that benefits from the trellis representation of classical codes. Here, we present a Min-Sum algorithm for stabilizer codes that computes the most likely error for memoryless Pauli channels given a measured syndrome s by using a trellis associated with the code.

Consider an (n,k) stabilizer code with stabilizer set $\{S_j\}_j$. Define the likelihood of $P\in G_n$ as $\Sigma_{i=1}^n\log\Pr_i(P^i)$. Consider the n-section trellis for this quantum code associated with the syndrome s. The naming conventions for the vertices and edges are set as in previous sections. For each edge e^i of E_i, define its weight $\text{wt}(\mathsf{e}^i)=-\log\Pr_i(l(\mathsf{e}^i))$. By construction, the sum of weights along the path in the trellis that represents P is equal to the opposite of the likelihood. The task that consists in finding a most likely error $\hat{E}\in G_n$ with syndrome s is thus equivalent to finding a lowest weight path $(\mathsf{e}^1,\ldots,\mathsf{e}^n)$ in the trellis associated with s.

This can be done by constructing recursively some sets C_i of lowest weight error candidates. More precisely, C_i contains couples (c,w) where c is a path issued from v_0 that ends on a vertex of V_i and where $w=\text{wt}(\mathsf{c})$.

Min-Sum algorithm

Initialization: $\mathsf{C}_0 := \{(\mathsf{v}_0, 0)\}$ and $\mathsf{C}_i := \emptyset$ for $i \leqslant 1$

Main step:

for i from 1 to n **do**

for all $\mathsf{v} \in \mathsf{V}_i$ **do**

Put in C_i the pair $(\mathsf{c}', \mathrm{wt}(\mathsf{c}'))$, where c' is a path of minimum weight (ties are broken at random) among all paths that (1) end in v; (2) have their $i-1$ first vertices given by a path c of C_{i-1}.

Note that the time complexity of this algorithm is linear in the number of vertices in the trellis.

Sum-Product algorithm. While the Min-Sum finds a most likely error compatible with the observed syndrome s, the Sum-Product aims at calculating marginal error probabilities for physical qubits. That is, $p_i(P) \triangleq \Pr(\text{error at qubit } i = P \mid s)$, where $P \in G_1$. By definition of the trellis, this probability is equal to the probability that a path $(\mathsf{e}^i)_{i=1}^n$ from v_0 to v_n is such that $l(\mathsf{e}^i) = P$.

The Sum-Product algorithm computes for each vertex v of the trellis associated with s a "forward" probability $f(\mathsf{v})$ and a "backward" probability $b(\mathsf{v})$. Both are then used to calculate the marginal probabilities $p_i(P)$.

Sum-Product algorithm

Initialization: $f(\mathsf{v}_0) := 1$ and $b(\mathsf{v}_n) := 1$; and $f(\mathsf{v}) := 0$ and $b(\mathsf{v}) := 0$ for all other vertices.

Forward pass:

for $i = 1$ to n **do**

for all $\mathsf{v} \in \mathsf{V}_i$ **do** $f(\mathsf{v}) := \Sigma_{\mathsf{w} \in \mathsf{V}_i^-(\mathsf{v})} f(\mathsf{w}) \Pr_i(l(\mathsf{wv}))$

$F_i := \Sigma_{\mathsf{v} \in \mathsf{V}_i} f(\mathsf{v})$

for all $\mathsf{v} \in \mathsf{V}_i$ **do** $f(\mathsf{v}) := f(\mathsf{v}) / F_i$

Backward pass:

for $i = n-1$ down to 0 **do**

for all $\mathsf{v} \in \mathsf{V}_i$ **do** $b(\mathsf{v}) = \Sigma_{\mathsf{w} \in \mathsf{V}_i^+(\mathsf{v})} b(\mathsf{w}) \Pr_{i+1}(l(\mathsf{vw}))$

$B_i := \Sigma_{\mathsf{v} \in \mathsf{V}_i} b(\mathsf{v})$

for all $\mathsf{v} \in \mathsf{V}_i$ **do** $b(\mathsf{v}) := b(\mathsf{v}) / B_i$

Final pass:

for $i = 1$ to n **do**

for all $P \in G_1$ **do**

$p_i(P) := \Sigma_{\mathsf{vw} \in \mathsf{E}_i(P)} f(\mathsf{v}) b(\mathsf{w}) \Pr_i(P)$

Above, (1) $\mathsf{V}_i^-(\mathsf{v})$ is the set of vertices w in V_{i-1} that are adjacent to v; (2) $\mathsf{V}_i^+(\mathsf{v})$ is the set of vertices w in V_{i+1} which are adjacent to v; and (3) $\mathsf{E}_i(P)$ is the set of edges between V_{i-1} and V_i that bear the Pauli-label P.

Once again, the practical relevance of this algorithm is due to the fact that its complexity is linear in the number of vertices in the trellis.

Computing the weight enumerator polynomial. The *weight enumerator polynomial* is a trivariate polynomial given by $A(x,y,z) \triangleq \Sigma_{0 \leqslant u,v,w \leqslant n} a_{u,v,w} x^u y^v z^w$, where $a_{u,v,w} \triangleq |\{P \in N(S) : |P|_X = u, |P|_Y = v, |P|_Z = w\}|$ and where $|P|_E$ denotes the number of coordinates of P that are equal to E. It is possible to extract from $A(x,y,z)$ a lot of useful information, e.g., bounds on the fidelity of the recovered state after decoding [7]. As for classical codes, the weight enumerator can be computed with linear complexity in the number of vertices of the trellis. For this purpose, intermediate polynomials $A_{\mathsf{v}}(x,y,z)$ are calculated for each vertex of the trellis associated with $s = (0, \ldots, 0)$. Also, define the polynomials $Q_I(x,y,z) \triangleq 1$, $Q_X(x,y,z) \triangleq x$, $Q_Y(x,y,z) \triangleq y$, and $Q_Z(x,y,z) \triangleq z$.

Computation of $A(x,y,z)$

Initialization: $A_{\mathsf{v}_0}(x,y,z) = 1$

Main step:

for i from 1 to n **do**

for all $\mathsf{v} \in \mathsf{V}_i$ **do**

$A_{\mathsf{v}}(x,y,z) := \Sigma_{\mathsf{w} \in \mathsf{V}_i^+(\mathsf{v})} A_{\mathsf{w}}(x,y,z) Q_{l(\mathsf{wv})}(x,y,z)$

$A(x,y,z) := A_{\mathsf{v}_n}(x,y,z)$

Above, $\mathsf{V}_i^-(\mathsf{v})$ is the set of vertices w in V_{i-1} that are adjacent to v.

The proof of correctness for these three algorithms follows from the correctness of the more general Min-Sum and Sum-Product algorithms presented in [8].

V. CONCLUSION

From a practical point of view, we have proposed a definition for the trellis of a stabilizer code together with two constructions and three algorithms that take advantage of this representation. Following the same path, several algorithms for classical codes running on trellises can be generalized to the quantum case. Among them, estimation of noise parameters (or equalization) seems a promising avenue for enhancing the performance of quantum optics fiber communications using near-future quantum technology.

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