Dynamical Casimir effect instabilities

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(Received 22 September 2005; revised manuscript received 27 March 2006; published 5 September 2006)

The dynamic Casimir effect, which concerns two photon radiation processes due to time dependent frequency modulations, is computed in the one photon loop approximation. An instability is signaled by the production of an unphysically large number of photons. We show how it is tamed and a saturation in the number of photons reached through higher order processes. Explicit results are obtained for a recently proposed experiment.

DOI: 10.1103/PhysRevA.74.032101 PACS number(s): 12.20.−m, 03.70.+k, 32.80.−t

I. INTRODUCTION

In this paper we discuss inherent instabilities plauging the dynamical Casimir effect process. We first isolate the different physical reasons causing instabilities and then present a theoretical formalism to deal with them thereby achieving a coherent description for these important phenomena.

Casimir [1], through considerations of the vacuum energy stored between two parallel capacitor plates, showed the existence of an attractive force between the plates. The seminal Casimir calculation was then generalized and interpreted as photon mode frequency shifts (Lamb shifts). Such mode frequency shifts induce changes in the free energy, which in the zero temperature limit [2], reduce to changes in the zero point energy [3,4] $\delta E_0=(\hbar/2)\Sigma_a\delta\Omega_a$. These Lamb frequency shifts are usually quite small and can thus be understood from a perturbation theory viewpoint. The conventional Casimir effect theory thereby considers Feynman diagram corrections to the free energy containing one photon loop [5,6].

An instability in the static Casimir process, which we shall not discuss in detail in this paper, may arise due to *giant* Lamb shifts which can occur if the coupling between a photon mode and the surrounding becomes too strong. Obviously then the one photon loop expansion fails. Various ramifications of this issue have been discussed at length in Refs. [7–11].

In the present paper, we exhibit such phenomenon through a circuit model. Such a photon mode Lagrangian is discussed in Sec. II along with a description of the Casimir effect induced shifts (i) in the real part of the frequency (Lamb shifts) and (ii) in the imaginary part of the frequency (the damping) of radiation modes due to their interaction with the electrical currents. The expressions for damping are developed in Sec. III.

We then turn our attention to the dynamical Casimir effect which may be *defined* as follows. If the damping functions and frequency shifts are also oscillating functions of time, then (over and above single photon absorption and emission processes) there is the absorption and emission of photon pairs [12–14]. The photon pair processes constitute a dynamical Casimir effect [15–17]. However, such frequency modulations tend to heat up the cavity. For their study, it is useful to consider a noise temperature description which is developed in Sec. IV. In Sec. V, the heating of a cavity mode by periodic frequency modulation is explored. In an unstable regime, the temperature of (say) a microwave cavity mode grows exponentially. The implied purely theoretical microwave oven would be much hotter than that which could be observed in experimental reality. A possible stabilizing decoherence mechanism has been discussed via leakage to an external reservoir [18], however, ultimately the stabilization must proceed via processes of higher order than one photon loop. Nonlinear higher loop photon processes producing dynamic microwave intensity stability are required in order to obtain saturation, i.e., limit the number of photons produced. These issues are discussed and resolved in Sec. VI where a finite expression (similar to that for a laser) for the final number of produced photons is derived. To make the theoretical expressions less abstract, in Sec. VII, we present numerical answers for the expected number of photons for a proposed dynamical Casimir effect experiment [19] which is based on some earlier theoretical work [20]. We close the paper with some concluding remarks in Sec. VIII.

II. LAGRANGIAN CIRCUIT MODE DESCRIPTION

Our purpose is to provide a Lagrangian description of a single microwave cavity mode which follows from the action principle formulation of electrodynamics [21–23]. For this purpose we employ the Coulomb gauge, for the vector potential $\mathrm{div}\,\mathbf{A}_{\mathrm{mode}}=0$. The vector potential representing the cavity mode may be written

$$\mathbf{A}_{\mathrm{mode}}(\mathbf{r},t)=\Phi(t)\mathbf{K}(\mathbf{r}). \tag{1}$$

The mode electromagnetic fields are then given by

$$\mathbf{E}_{\mathrm{mode}}(\mathbf{r},t)=-\frac{1}{c}\left[\frac{\partial\mathbf{A}_{\mathrm{mode}}(\mathbf{r},t)}{\partial t}\right]=-\frac{\dot{\Phi}(t)}{c}\mathbf{K}(\mathbf{r}),$$

$$\mathbf{B}_{\mathrm{mode}}(\mathbf{r},t)=\mathrm{curl}\,\mathbf{A}_{\mathrm{mode}}(\mathbf{r},t)=\Phi(t)\,\mathrm{curl}\,\mathbf{K}(\mathbf{r}). \tag{2}$$

The Lagrangian

$$L_F = \int_{\text{cavity}} \frac{d^3\mathbf{r}}{8\pi}[|\mathbf{E}_{\text{mode}}(\mathbf{r},t)|^2 - |\mathbf{B}_{\text{mode}}(\mathbf{r},t)|^2] \tag{3}$$

describes the mode in terms of a simple oscillator circuit. The capacitance C and inductance Λ of the circuit are defined, respectively, by

$$C = \frac{1}{4\pi}\int_{\text{cavity}} |\mathbf{K}(\mathbf{r})|^2 d^3\mathbf{r},$$

$$\frac{1}{\Lambda} = \frac{1}{4\pi}\int_{\text{cavity}} |\text{curl } \mathbf{K}(\mathbf{r})|^2 d^3\mathbf{r}. \tag{4}$$

The circuit electromagnetic field Lagrangian follows from Eqs. (2), (3), and (4). It is of the simple ΛC oscillator form

$$L_{\text{field}}(\dot{\Phi},\Phi) = \frac{C}{2c^2}\dot{\Phi}^2 - \frac{1}{2\Lambda}\Phi^2, \tag{5}$$

wherein the bare circuit frequency obeys

$$\Omega_\infty^2 = \frac{c^2}{\Lambda C}. \tag{6}$$

The interactions between cavity wall currents and an electromagnetic mode are conventionally described by

$$L_{\text{int}} = \frac{1}{c}\int \mathbf{J}\cdot\mathbf{A}_{\text{mode}} d^3\mathbf{r},$$

$$L_{\text{int}} = \frac{1}{c} I\Phi,$$

$$I(t) = \int \mathbf{J}(\mathbf{r},t)\cdot\mathbf{K}(\mathbf{r})d^3\mathbf{r}, \tag{7}$$

where the current I drives the oscillator circuit. In total, the circuit mode Lagrangian follows from Eqs. (5) and (7) as

$$L = \frac{C}{2c^2}\dot{\Phi}^2 - \frac{1}{2\Lambda}\Phi^2 + \frac{1}{c}I\Phi + L', \tag{8}$$

wherein L' describes all of the other degrees of freedom which couple into the mode coordinate. Maxwell's equations for a single microwave mode then take the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\Phi}}\right) = \left(\frac{\partial L}{\partial \Phi}\right),$$

$$C\left(\ddot{\Phi} + \Omega_\infty^2\Phi\right) = cI. \tag{9}$$

The damping of the oscillator will first be discussed from a classical electrical engineering viewpoint and only later from a fully quantum electrodynamic viewpoint.

III. OSCILLATOR CIRCUIT DAMPING

From an electrical engineering viewpoint, let us consider a small external current source δI_{ext} which drives the mode coordinate $\delta\Phi$. Equation (9) now reads

$$\frac{C}{c^2}\delta\ddot{\Phi} + \frac{1}{\Lambda}\delta\Phi = \frac{1}{c}\delta I = \frac{1}{c}(\delta I_{\text{ext}} + \delta I_{\text{ind}}), \tag{10}$$

were δI_{ind} is the current induced by the coordinate response $\delta\Phi$. In the complex frequency ζ domain we have in (the upper half Im $\zeta > 0$ plane)

$$\delta I_{\text{ext}}(t) = \text{Re}\{\delta I_{\text{ext},\zeta} e^{-i\zeta t}\},$$

$$\delta\Phi(t) = \text{Re}\{\delta I_{\text{ext},\zeta}\mathcal{D}(\zeta)e^{-i\zeta t}\}. \tag{11}$$

The induced current is determined by the "surface admittance" $Y(\zeta)$ of the cavity walls; In detail

$$\delta I_{\text{ind}}(t) = -\frac{1}{c}\int_0^\infty \mathcal{G}(t')\delta\dot{\Phi}(t-t')dt',$$

$$Y(\zeta) = \int_0^\infty e^{i\zeta t}\mathcal{G}(t)dt, \tag{12}$$

so that

$$\left\{-\frac{C}{c^2}\zeta^2 + \frac{1}{\Lambda} - \frac{i\zeta}{c^2}Y(\zeta)\right\}\mathcal{D}(\zeta) = \frac{1}{c},$$

$$-i\zeta\varepsilon(\zeta)C + i\zeta C = Y(\zeta), \tag{13}$$

wherein the effective frequency dependent capacitance $\varepsilon(\zeta)C$ determines the mode dielectric response function $\varepsilon(\zeta)$. The retarded propagator for the mode in the frequency domain obeys [9,21]

$$\mathcal{D}(\zeta) = \frac{\Lambda}{c}\left[\frac{\Omega_\infty^2}{\Omega_\infty^2 - \zeta^2 - \Pi(\zeta)}\right], \tag{14}$$

wherein the "self-energy" $\Pi(\zeta)$, or equivalently the "damping function" $\Gamma(\zeta)$, is determined by the induced current admittance via

$$\Pi(\zeta) = \frac{i\zeta Y(\zeta)}{C} = \Pi(0) + i\zeta\Gamma(\zeta). \tag{15}$$

The self-energy describes both frequency shift and damping properties of the mode.

Causality dictates that all engineering response functions obey analytic dispersion relations (Im $\zeta > 0$) of the form

$$\mathcal{D}(\zeta) = \frac{2}{\pi}\int_0^\infty \frac{\omega \,\text{Im}\, \mathcal{D}(\omega + i0^+)d\omega}{\omega^2 - \zeta^2},$$

$$\Pi(\zeta) = \frac{2}{\pi}\int_0^\infty \frac{\omega \,\text{Im}\, \Pi(\omega + i0^+)d\omega}{\omega^2 - \zeta^2}. \tag{16}$$

The damping rate for the oscillation is determined by

$$\text{Im}\, \Pi(\omega + i0^+) = \omega\, \text{Re}\, \Gamma(\omega + i0^+) = \frac{\omega\, \text{Re}\, Y(\omega + i0^+)}{C}. \tag{17}$$

The shifted frequency

$$\Omega_0^2 = \Omega_\infty^2 - \Pi(0), \tag{18}$$

is related to the damping rate via the dispersion relation sum rule [9]

$$\Omega_\infty^2 = \Omega_0^2 + \frac{2}{\pi}\int_0^\infty \mathrm{Re}\,\Gamma(\omega + i0^+)d\omega, \tag{19}$$

which follows from Eqs. (15)–(18). Finally, the quality factor Q for the mode frequency Ω_0 is well defined as

$$\frac{\Omega_0}{Q} = \mathrm{Re}\,\Gamma(\Omega_0 + i0^+) \tag{20}$$

if and only if the mode is under damped by a large margin; e.g. $Q \gg 1$. From Eq. (18), we see that if the damping is sufficiently strong $[\Pi(0) > \Omega_\infty^2]$, then the mode can go unstable as alluded to in the introduction. Let us apply these considerations to the dynamical Casimir effect.

IV. DYNAMICAL CASIMIR EFFECTS

For a discussion of the dynamical Casimir effect let us introduce the notion of a *noise* temperature. In the linear regime considered in this section, the dynamical Casimir effect is shown to be directly related to the production of a pair of photons.

Suppose that the dielectric response function $\varepsilon(\zeta)$ of the mode in Eq. (13) is made to vary with time; i.e.,

$$\varepsilon(\zeta) \Rightarrow \varepsilon(\zeta, t) \quad \text{equivalently } \Pi(\zeta) \Rightarrow \Pi(\zeta, t). \tag{21}$$

If the resulting differential equation for the $\Phi = \mathrm{Re}\{\phi\}$ signal is linear (to a sufficient degree of accuracy)

$$\ddot{\phi}(t) + \Omega^2(t)\phi(t) = 0,$$

$$\Omega(t \to \pm\infty) = \Omega_0, \tag{22}$$

then there exists a solution of the form

$$\phi(t \to \infty) = e^{i\Omega_0 t} + \rho e^{-i\Omega_0 t},$$

$$\phi(t \to -\infty) = \sigma e^{i\Omega_0 t},$$

$$|\rho|^2 + |\sigma|^2 = 1. \tag{23}$$

From a quantum mechanical viewpoint, the time variation $e^{i\Omega_0 t}$ may represent a photon moving backward in time and $e^{-i\Omega_0 t}$ may represent photon moving forward in time. In Eq. (23), the reflection amplitude for a photon moving backward in time to bounce forward in time is given by ρ. A backward in time moving photon reflected forward in time appears in the laboratory to be a pair of photons being created [14–20,24–28].

The probability of such a photon pair creation event defines a photon pair creation noise temperature T^* induced by the time varying frequency via

$$R = |\rho|^2 = e^{-\hbar\Omega_0/k_B T^*}. \tag{24}$$

The mean number $\bar{N}$ of photons which would be radiated from the vacuum by a time varying frequency modulation $\Omega(t)$ obeys a formal Planck law

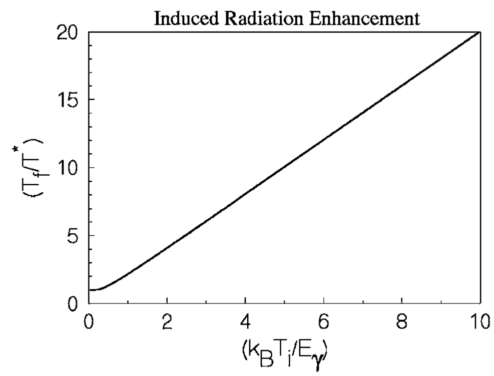

FIG. 1. If T_i represents the initial cavity mode temperature and T^* represents the noise temperature of the pair radiated photons, then the final temperature T_f of the cavity mode is enhanced (over and above T^*) via the initial photon population. The resulting radiation enhancement is plotted for photons with energy $E_\gamma = \hbar\Omega_0$.

$$\bar{N} = \frac{R}{1-R} = \frac{1}{e^{\hbar\Omega_0/k_B T^*} - 1}. \tag{25}$$

Suppose (for example) that a microwave cavity is initially in thermal equilibrium at temperature T_i. The mean number of initial microwave photons in a given normal mode is then given by

$$N_i = \frac{1}{e^{\hbar\Omega_0/k_B T_i} - 1}. \tag{26}$$

After a sequence of frequency modulation pulses the mean number of final photons in the cavity mode is

$$N_f = (2\bar{N} + 1)N_i + \bar{N} = N_i \coth\left(\frac{\hbar\Omega_0}{2k_B T^*}\right) + \frac{1}{e^{\hbar\Omega_0/k_B T^*} - 1}. \tag{27}$$

Note that the existence of an *initial* number of photons N_i in the cavity mode makes larger the final number of photons

$$N_f = \frac{1}{e^{\hbar\Omega_0/k_B T_f} - 1} \tag{28}$$

via the induced radiation of additional photon pairs. If the microwave frequency large margin inequality

$$\hbar\Omega_0 \ll k_B T^* \tag{29}$$

holds true, then Eqs. (25)–(29) imply the following approximate law for the *final* cavity mode noise temperature:

$$T_f \approx T^* \coth\left(\frac{\hbar\Omega_0}{2k_B T_i}\right). \tag{30}$$

The resulting enhancement (T_f/T^*) is plotted in Fig. 1. The dynamical Casimir effect for frequency modulation pulses is thereby described in terms of the amount of heat that raises the temperature $T_i \to T_f$ of the microwave cavity. This result has been previously derived in Ref. [24,25]. In the next section, we consider periodic frequency modulations in detail.

V. PERIODIC FREQUENCY MODULATIONS

For periodic modulations in the frequency, one must examine the differential equation

$$\ddot{\phi}(t) + \Omega^2(t)\phi(t) = 0,$$

$$\Omega_0^2 + \nu^2(t) = \Omega^2(t),$$

$$\nu(t+\tau) = \nu(t). \tag{31}$$

From a mathematical viewpoint, Eq. (31) has been well studied [29]. If $\nu(t)$ can be represented as a nonoverlapping pulse sequence of the form

$$\nu(t) = \sum_{n=-\infty}^{\infty} \varpi(t - n\tau), \tag{32}$$

then the transmission problem for a single pulse

$$\ddot{\phi}_1(t) + \{\Omega_0^2 + \varpi^2(t)\}\phi_1(t) = 0 \tag{33}$$

yields a complete solution to the general problem. In particular, we examine the two photon creation problem as in Eq. (23); i.e.,

$$\phi_1(t \to \infty) = e^{i\Omega_0 t} + \rho_1 e^{-i\Omega_0 t},$$

$$\phi_1(t \to -\infty) = \sigma_1 e^{i\Omega_0 t},$$

$$|\rho_1|^2 + |\sigma_1|^2 = R_1 + P_1 = 1,$$

$$\sigma_1 = \sqrt{P_1}e^{-i\Theta_1}. \tag{34}$$

Employing the characteristic function

$$\mu(\Omega_0) = \frac{\cos[\Omega_0\tau + \Theta_1(\Omega_0)]}{\sqrt{P_1(\Omega_0)}}, \tag{35}$$

one may study the stability problem for the dynamic Casimir effect. For *periodic* frequency modulations there are two cases of interest [14].

Case I. Stable Motions $:-1 < \mu(\Omega_0) < +1$

$$\mu(\Omega_0) = \cos(\Omega\tau),$$

$$\phi_\pm(t+\tau) = e^{\pm i\Omega t}\phi_\pm(t). \tag{36}$$

Case II. Unstable Motions: $\mu(\Omega_0) > +1$ or $\mu(\Omega_0) < -1$.

$$\mu(\Omega_0) = \cosh(\gamma\tau) \quad \text{or} \quad \mu(\Omega_0) = -\cosh(\gamma\tau)$$

$$\phi_\pm(t+\tau) = e^{\pm\gamma t}\phi_\pm(t). \tag{37}$$

In the unstable regime, 2γ represents the number of cavity photons being produced per unit time. If the cavity mode has a high quality factor $Q \gg 1$, then photons are also absorbed at a rate (Ω_0/Q). The net photon production rate in this approximation would then be

$$\Gamma_1 \simeq \left(2\gamma - \frac{\Omega_0}{Q}\right), \tag{38}$$

and the theoretical noise temperature after n_p pulses would be

$$k_B T_1^* \approx \hbar\, \Omega_0 \exp(n_p \tau \Gamma_1). \tag{39}$$

As an example, let us suppose a sequence of rectangular pulse sequences of the form

$$\Omega(t) = \Omega_0 \quad \text{if } t_0 + n\tau < t < t_0 + (n+1/2)\tau,$$

$$\Omega(t) = (1+\alpha)\Omega_0 \quad \text{if } t_0 + (n+1/2)\tau < t < t_0 + (n+1)\tau, \tag{40}$$

wherein $n=1,2,\ldots,n_p$. The estimate

$$\exp(n_p\tau\Gamma_1) \sim \exp(n_p\alpha/2) \quad \text{for } 1 \gg \alpha \gg (\Omega_0\tau)/Q \tag{41}$$

is not unreasonable.

Equation (41) shows the inherent instability present in the dynamical Casimir effect since for case II, the number of produced photons increases exponentially. In fact, were it not controlled, the exponential temperature *instability* for high quality cavity modes, i.e., $\Gamma_1 > 0$ in Eqs. (38)–(41), would be sufficient for large n_p to *melt* the cavity. No microwave oven works that efficiently even if the dynamic Casimir effect were employed for exactly that purpose. The one loop photon approximation is evidently at fault and higher loops (nonlinear processes) must be invoked for the noise temperature of the mode to be theoretically stable as would be laboratory microwave cavities.

Once again an analogy might be helpful (this time with lasers). The master equation for an *ideal* laser with a production rate γ_0 satisfies the linear differential equation

$$\frac{dn(t)}{dt} = \gamma_0 n(t), \tag{42}$$

whose solution leads to an exponentially large number of photons n for large t. This unphysical growth is curtailed through considerations of nonlinear processes which leads to saturation in the mean number of produced photons. The more correct equation reads

$$\frac{d\hat{n}(t)}{dt} = \gamma_0\hat{n}(t) - \gamma_1\hat{n}^2(t), \tag{43}$$

and leads to saturation with a mean number $\bar{n} = \frac{\gamma_0}{\gamma_1}$. Theoretically, one would calculate γ_0 and γ_1 in different orders of loop perturbation theory. In the next section, we shall develop a similar strategy to obtain saturation and an explicit, finite expression for the mean number of produced photons.

VI. MICROWAVE INTENSITY STABILITY

The issue of stability for a microwave cavity is intimately related to the fact that the modulation is induced by a *pump* which supplies the energy for the induced cavity radiation. One may define a *pump coordinate* η which, in general, is a quantum mechanical operator. In principle, one might mechanically vibrate a wall in the cavity in which case η would be proportional to a mechanical displacement. In practice, changing the frequency by electronic means may well be more efficient [19]. Be that as it may, let us define the coordinate so that

$$\langle \eta(t) \rangle = \frac{\nu^2(t)}{\Omega_0^2}, \tag{44}$$

wherein the quantities on the right-hand side of Eq. (44) are given in Eq. (31).

If the quantum pump coordinate exhibits stationary fluctuations

$$\Delta \eta = \eta - \langle \eta \rangle \tag{45}$$

with quantum noise

$$\frac{1}{2}\langle \Delta \eta(t) \Delta \eta(t') + \Delta \eta(t') \Delta \eta(t) \rangle = \int_{-\infty}^{\infty} \bar{S}_\eta(\omega) e^{-i\omega(t-t')} d\omega, \tag{46}$$

then two photon absorption and two photon emission processes are described by the additional noise Hamiltonian

$$\Delta H = \frac{1}{4} \hbar\, \Omega_0 (a^\dagger a^\dagger + aa) \Delta \eta. \tag{47}$$

The usual mode photon creation and destruction operators are $a^\dagger$ and a, respectively. When the Hamiltonian in Eq. (47) is taken to second order in perturbation theory, the resulting energies involve four boson processes which thereby generates multiphoton loop processes.

With the pump coordinate positive and negative frequency spectral functions

$$\langle \Delta \eta(t) \Delta \eta(t') \rangle = \int_{-\infty}^{\infty} S_\eta^{+}(\omega) e^{-i\omega(t-t')} d\omega,$$

$$\langle \Delta \eta(t') \Delta \eta(t) \rangle = \int_{-\infty}^{\infty} S_\eta^{-}(\omega) e^{-i\omega(t-t')} d\omega, \tag{48}$$

the two photon Fermi golden rule transition rates which follow from Eqs. (47) and (48) read

$$\Gamma^{+}(n \to n-2) = \frac{\pi \Omega_0^2}{8} S_\eta^{+}(\omega = 2\Omega_0) n(n-1),$$

$$\Gamma^{-}(n-2 \to n) = \frac{\pi \Omega_0^2}{8} S_\eta^{-}(\omega = 2\Omega_0) n(n-1). \tag{49}$$

The pump coordinate also has a noise temperature T_η, which is defined via

$$S_\eta^{-}(2\Omega_0) = e^{-2\hbar\Omega_0/k_B T_\eta} S_\eta^{+}(2\Omega_0). \tag{50}$$

If there are many photons in the mode, then the net rate of photon absorption is given by

$$\Gamma_{\text{absorption}} \simeq \left(\frac{\pi \Omega_0^2 \tau_\eta}{4} \right) \tanh\left(\frac{\hbar \Omega_0}{k_B T_\eta} \right) n^2,$$

where

$$n \gg 1, \quad \tau_\eta \equiv \bar{S}_\eta(\omega = 2\Omega_0). \tag{51}$$

On the other hand, the frequency modulation produces photons at a rate

$$\Gamma_{\text{emission}} \simeq 2\gamma n, \quad \text{where } n \gg 1, \tag{52}$$

and γ is defined in Eq. (37). We may now state the central result of this section.

Theorem 1. If the pump coordinate pushes the cavity mode into a modulation dynamic Casimir instability, then the quantum noise will stabilize the cavity mode according to the equation

$$\frac{dn}{dt} = 2(\gamma n - \tilde{\gamma} n^2),$$

$$\tilde{\gamma} = \frac{\pi \Omega_0^2 \tau_\eta}{8} \tanh\left(\frac{\hbar \Omega_0}{k_B T_\eta} \right). \tag{53}$$

The cavity photon occupation number will then saturate according to

$$\bar{n}_{\text{sat}} = \frac{8\gamma}{\pi \Omega_0^2 \tau_\eta} \coth\left(\frac{\hbar \Omega_0}{k_B T_\eta} \right) \equiv \frac{k_B T_{\text{sat}}}{\hbar \Omega_0} \gg 1. \tag{54}$$

An alternative, simpler expression may be derived through the fluctuation dissipation theorem and the notion of a relaxation time [30] for the pump coordinate.

With the response function

$$\chi(\zeta) = \frac{i}{\hbar} \int_0^{\infty} \langle [\eta(t), \eta(0)] \rangle e^{i\zeta t} dt, \tag{55}$$

and the fluctuation dissipation theorem

$$\bar{S}_\eta(\omega) = \frac{\hbar}{2\pi} \coth\left(\frac{\hbar \omega}{2 k_b T_\eta} \right) \text{Im}\, \chi(\omega + i0^{+}), \tag{56}$$

Eq. (54) reads

$$\bar{n}_{\text{sat}} = \frac{16\gamma}{\Omega_0^2 [\hbar\, \text{Im}\, \chi(2\Omega_0 + i0^{+})]}. \tag{57}$$

The relaxation time $\tau^\dagger$ for the parameter η may be conveniently defined by [30]

$$\chi(0) \tau^\dagger = \lim_{\omega \to 0} \frac{\text{Im}\, \chi(\omega + i0^{+})}{\omega}, \tag{58}$$

so as to obtain

$$\bar{n}_{\text{sat}} \approx \left[\frac{8\gamma}{\Omega_0^3 \tau^\dagger \hbar\, \chi(0)} \right]. \tag{59}$$

Equation (59) is our answer for the final number of produced photons.

VII. A NUMERICAL EXAMPLE

In order to render our final answer less abstract, consider a proposed experiment [19] and let us provide a concrete numerical estimate for the expected number of photons for their setup. In that proposal, the parameter η describes the metallic conductivity in a semiconductor plate due to a laser beam inducing particle hole pairs. If we let τ_R represent the recombination time taken to annihilate a particle hole pair in the semiconductor and let ω_L represent the laser frequency, then we estimate that

$$\frac{1}{\tau^{\dagger}} \approx \frac{\hbar \omega_L \chi(0)}{\tau_R}, \tag{60}$$

which implies

$$\bar{n}_{\text{saturate}} \approx \left(\frac{8\gamma}{\Omega_0}\right)\left(\frac{1}{\Omega_0 \tau_R}\right)\left(\frac{\omega_L}{\Omega_0}\right). \tag{61}$$

The following estimates are reasonable for the proposal [19]:

$$\left(\frac{\gamma}{\Omega_0}\right) \approx 0.05,$$

$$\left(\frac{1}{\Omega_0 \tau_R}\right) \approx 10,$$

$$\left(\frac{\omega_L}{\Omega_0}\right) \approx 2 \times 10^5, \tag{62}$$

so that our estimate is $\bar{n}_{\text{saturate}} \approx 10^6$ microwave photons.

VIII. CONCLUSION

We have explored the concept of induced instabilities in dynamical Casimir effect. Even if the frequency shifts are small, perfect periodicity in modulation pulses can build up to exponentially large proportions leading to an instability. Dynamic quartic terms are invoked to stabilize the cavity modes. The basic principle involved is that the shifted frequencies themselves must undergo fluctuations. We prove in theorem 1 how the quantum noise in the *pump* coordinate which supplies energy to the cavity mode will stabilize the system. Given the noise fluctuations in the pump coordinate, the final saturation temperature of the microwave cavity can be computed from Eq. (59). We have illustrated our method by estimating the expected number of produced photons through a concrete application to a recently proposed experiment [19] to measure the dynamical Casimir effect.

ACKNOWLEDGMENTS

Y.S. would like to thank INFN, Italy for supporting his research and members of the INFN experimental group [19] for inviting him to their meetings and for useful discussions about the instability problem as well as for providing us with the details of their experimental setup.

[1] H. B. G. Casimir, Proc. K. Ned. Akad. Wet. **51**, 793 (1948).
[2] H. B. G. Casimir and D. Polder, Phys. Rev. **73**, 360 (1948).
[3] M. Bordag, U. Mohideen, and V. M. Mostepanenko, Phys. Rep. **205**, 353 (2001).
[4] K. A. Milton, *The Casimir Effect* (World Scientific, River Edge, NJ, 2001).
[5] I. E. Dzyaloshinski, E. M. Lifshitz, and L. P. Pitayevski, Sov. Phys. JETP **10**, 161 (1960).
[6] I. E. Dzyaloshinski, E. M. Lifshitz, and L. P. Pitayevski, Adv. Phys. **10**, 165 (1961).
[7] E. Sassaroli, Y. Srivastava, and A. Widom, Comments Mod. Phys. **D1**, 369 (1998).
[8] S. Sivasubramanian, Y. Srivastava, and A. Widom, Int. J. Mod. Phys. B **15**, 537 (2001).
[9] S. Sivasubramanian, Y. Srivastava, and A. Widom, Mod. Phys. Lett. B **16**, 1201 (2002).
[10] S. Sivasubramanian, Y. Srivastava, and A. Widom, J. Phys.: Condens. Matter **15**, 1109 (2003).
[11] S. Sivasubramanian, Y. Srivastava, and A. Widom, Physica A **345**, 356 (2005).
[12] J. Schwinger, Lett. Math. Phys. **24**, 59 (1992); **24**, 227 (1992).
[13] J. Schwinger, Proc. Natl. Acad. Sci. U.S.A. **89**, 4091 (1992); **89**, 11118 (1992); **90**, 958 (1993); **90**, 2015 (1993); **90**, 7285 (1993).
[14] E. Sassaroli, Y. N. Srivastava, and A. Widom, Phys. Rev. A **50**, 1027 (1994).
[15] V. V. Dodonov, J. Phys. A **31**, 9835 (1998).
[16] D. A. R. Dalvit and F. D. Mazzitelli, Phys. Rev. A **59**, 3049 (1999).
[17] V. V. Dodonov and M. A. Andreata, J. Phys. A **32**, 6711 (1999).
[18] R. Schützhold and M. Tiersch, J. Opt. B: Quantum Semiclassical Opt. **7**, S120 (2005).
[19] C. Braggio, G. Bressi, G. Carugno, C. Del Noce, G. Galeazzi, A. Lombardi, A. Palmieri, G. Ruoso, and D. Zanello, Europhys. Lett. **70**, 754 (2005).
[20] E. Yablonovitch, Phys. Rev. Lett. **62**, 1742 (1989).
[21] Y. Srivastava and A. Widom, Phys. Rep. **148**, 1 (1987).
[22] A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics* (Prentice Hall, Englewood Cliffs, NJ, 1963), Chap. 6.
[23] S. Sivasubramanian, A. Widom, and Y. N. Srivastava, cond-mat/0403592 (unpublished).
[24] G. Plunien, R. Schutzhold, and G. Soff, Phys. Rev. Lett. **84**, 1882 (2000).
[25] M. Crocce, D. A. R. Dalvit, and F. D. Mazzitelli, Phys. Rev. A **64**, 013808 (2001).
[26] M. Crocce, D. A. R. Dalvit, F. C. Lombardo, and F. D. Mazzitelli, Phys. Rev. A **70**, 033811 (2004).
[27] A. V. Dodonov and V. V. Dodonov, J. Opt. B: Quantum Semiclassical Opt. **7**, S47 (2005).
[28] E. Sassaroli, Y. Srivastava, and A. Widom, Nucl. Phys. B (Proc. Suppl.) **33**, 209 (1993).
[29] W. Magnus and S. Winkler, *Hill's Equation* (Dover Publications, New York, 2003).
[30] P. C. Martin, *Measurements and Correlation Functions* (Gordon and Breach, New York, 1968).