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Genuine tripartite entangled states with a local hidden-variable model

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We present a family of three-qubit quantum states with a basic local hidden-variable model. Any von Neumann measurement can be described by a local model for these states. We show that some of these states are genuine three-partite entangled and also distillable. The generalization for larger dimensions or higher number of parties is also discussed. As a by-product, we present symmetric extensions of two-qubit Werner states.

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I. INTRODUCTION

One of the most striking characteristics of quantum mechanics is nonlocality. If quantum mechanics could be described by a local hidden-variable model (LHV) then the values measured for multiparticle correlations could be reproduced assuming that all measurable single-particle operators had already a value before the measurement. Bell showed that there are quantum states for which the many-body correlations cannot be explained based on this assumption [1]. However, such correlations arise only for *some* entangled quantum states while for separable states the correlations can always be mimicked by a LHV model [2].

Proving that the measurement results on a quantum state cannot be obtained from a LHV model is done by finding a Bell inequality which is violated by the state [1]. However, this is difficult since the determination of all Bell inequalities is a computationally hard problem [3]. To prove that any measurement on a given state can be described by a LHV model is perhaps even more challenging. This is because in order to do that one has to find a LHV model for any number of arbitrary operators measured at each party.

Due to the difficulty of the problem, LHV models have a quite limited literature. The first and most fundamental result of the subject was presented by Werner in Ref. [2]. He described a LHV model for arbitrary von Neumann measurements for some $U\otimes U$ symmetric bipartite states [2]. For the qubit case these are of the form

$$\rho_W = p|\psi^-\rangle\langle\psi^-| + (1-p)\frac{\mathbb{1}}{4}, \tag{1}$$

where $|\psi^-\rangle=(|01\rangle-|10\rangle)/\sqrt{2}$ is the singlet state. In the same paper Werner also gave the first modern definition of quantum entanglement, thus distinguishing it from nonlocality. Indeed, states ρ_W are entangled for $p>1/3$ and local for $p\leqslant 1/2$ [2]. It is hard to overestimate the importance of these results for the development of quantum information science. Later, Barrett obtained a model for general measurements, also called positive operator valued measures (POVMs), for a subset of Werner states [4]. LHV models were also constructed for finite number of settings for states with positive partial transpose exploiting symmetric extensions [5]. Apart from their fundamental interest, these results are also relevant from a quantum information theory viewpoint. Simulating entanglement by classical means (e.g., Ref. [6]) sheds light on the power of entanglement as information resource. In this context, those quantum states for which the correlations can be reproduced by a LHV model are useless for communication tasks, since they do not provide any advantage over shared classical randomness [7].

New and interesting open questions on the relationship between nonlocality and entanglement appear in the multipartite scenario. Recall that multipartite entanglement is known to be inequivalent to bipartite entanglement [8]. Moreover, genuine multipartite entanglement is the property most often detected in experiments (e.g., Ref. [9]). We know that the Bell inequality violation required for genuine multipartite entanglement (i.e., when all parties are entangled with each other [10]) increases exponentially with the number of parties [11]. Hence one could expect that entanglement of this type for large enough number of parties provides a sufficient condition for a state to be nonlocal. However, beyond the bipartite case, the connection between nonlocality and entanglement remains largely unexplored. Indeed, LHV models for multipartite entangled systems are still missing.

In this paper, we present a one-parameter family of three-qubit states whose correlations for von Neumann measurements can be reproduced by a LHV model. Thus these states do not violate any Bell inequality. Then, we prove that, remarkably, some of these states have genuine three-qubit entanglement [10] and we also show that they are distillable. To our knowledge, these are the first examples of genuine multipartite entangled states allowing for a local description. The generalization of the construction to other situations, more parties or higher dimensional systems, is also discussed.

Before proceeding, let us introduce the notation. We denote von Neumann measurements on n parties $A,B,C,\ldots$, as $M_A,M_B,M_C,\ldots$. The spectral decomposition of M_A is given as $M_A=\Sigma_{k=1}^{d}\alpha_k P_k$. In the case of qubits, which we mostly consider in this work, $\alpha_1=+1$ and $\alpha_2=-1$, while $M_A=\hat{n}_A\cdot\vec{\sigma}$, where $\hat{n}_A$ is the normalized vector defining the direction of the von Neumann measurement and $\vec{\sigma}$ is the vector of Pauli matrices $\vec{\sigma}=(\sigma_x,\sigma_y,\sigma_z)$.

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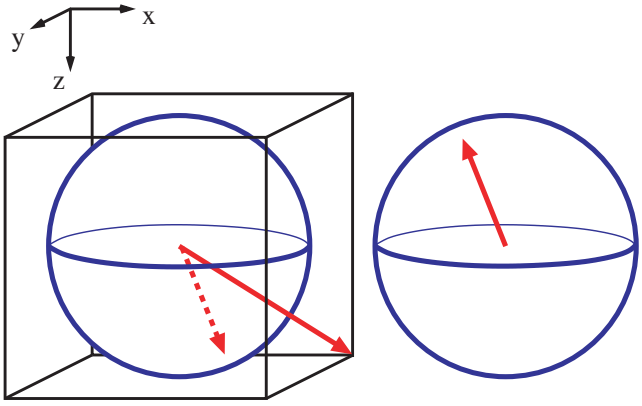

FIG. 1. (Color online) Schematic representation of our two-qubit hidden variable model. Party B receives a standard Bloch vector. The dashed arrow on the left-hand side points opposite to this vector. Party A receives a vector pointing to one of the eight vertices of a cube tangent to the Bloch sphere. The vertex is chosen such that the overlap with the dashed vector is maximal.

II. TWO-QUBIT CASE

The key point for the construction of our LHV model for three qubits is an alternative derivation of Werner's original result for two qubits. This new derivation has the advantage of being easily generalizable to the case of three qubits. Consider the two-qubit operators

$$\rho^{(2,c)} := \int_{\omega\in\mathbb{C}^2,|\omega|=1} M(d\omega)\varrho_\omega \otimes \rho_\omega, \tag{2}$$

where

$$\varrho_\omega := \frac{1}{2}\left[\mathbb{1} - c\sum_{k=x,y,z} \mathrm{sgn}(\langle\sigma_k\rangle_\omega)\sigma_k\right],$$

$$\rho_\omega := |\omega\rangle\langle\omega|. \tag{3}$$

Here $|\omega\rangle$ is a two-element state vector and M is the unique probability measure invariant under all unitary rotations. Direct calculation shows that $\rho^{(2,c)}$ are Werner states (1) with $p=c/2$. Based on this construction, the following statement can be made.

Theorem 1 [2]. There exists a LHV model for von Neumann measurements on states $\rho^{(2,c)}$ for $c\leqslant 1$.

Before starting the proof, let us explain the intuition behind it. First of all, note that one can restrict the analysis to $c=1$. In this case, the decomposition (2) can be understood as a sort of local model for which party B receives a standard Bloch vector $\hat{n}_\omega$, while A receives one of the vectors pointing to the vertices of a cube. This cube is tangent to the Bloch sphere for the Bloch vectors, which point to the directions $\pm x$, $\pm y$, and $\pm z$ (see Fig. 1). Actually, A receives the vector with maximum overlap with $-\hat{n}_\omega$. Using the standard trace rule $\langle M_A\otimes M_B\rangle = \int M(d\omega)\mathrm{Tr}(M_A\varrho_\omega)\mathrm{Tr}(M_B\rho_\omega)$, we have a LHV model for $M_A=\pm\sigma_{x/y/z}$ and arbitrary M_B. The choice of M_A is restricted since if party A chooses other operators to measure, for some ω she would obtain $|\langle M_A\rangle_{\varrho_\omega}|>1$. We then say that party B has a physical qubit, while A is receiving a nonphysical Bloch vector. As we have already said, state $\rho^{(2,1)}$ is a Werner state, i.e., it is invariant under transformations of the form $U\otimes U$ where U is an arbitrary unitary matrix. Using this symmetry, we can construct a LHV model for all measurements. The detailed proof goes as follows.

Proof of Theorem 1. The goal is to find a LHV model for the state (2) with $c=1$, that is to write its correlations as

$$\mathrm{Tr}(M_A\otimes M_B\rho^{(2,1)}) = \int_\omega M(d\omega)\langle M_A\rangle_\omega\langle M_B\rangle_\omega, \tag{4}$$

where $\langle M_{A/B}\rangle_\omega$ are the expectation values of $M_{A/B}$ if the value of the hidden variable is ω, and we require $|\langle M_A\rangle_\omega|, |\langle M_B\rangle_\omega|\leqslant 1$. Identifying the subensemble index ω in Eq. (2) with the hidden variable in Eq. (4), one has a LHV model with

$$\langle M_A\rangle_\omega = \mathrm{Tr}(M_A\varrho_\omega) = -\frac{1}{2}\mathrm{Tr}\left[M_A\sum_{k=x,y,z}\mathrm{sgn}(\langle\sigma_k\rangle_\omega)\sigma_k\right],$$

$$\langle M_B\rangle_\omega = \mathrm{Tr}(M_B\rho_\omega) = \mathrm{Tr}(M_B|\omega\rangle\langle\omega|). \tag{5}$$

It is clear that this model works only if $M_A=\pm\sigma_{x/y/z}$.

Now we modify our LHV model in order to allow arbitrary operators M_A of the type $\hat{n}_A\vec{\sigma}$. Such an operator can be written in the form

$$M_A = U_A^\dagger\sigma_z U_A. \tag{6}$$

We can take advantage of the invariance of Werner states under transformations of the form $U\otimes U$, so $\langle M_A\otimes M_B\rangle = \langle\sigma_z^{(A)}\otimes M_B'\rangle$, where $M_B'=U_AM_BU_A^\dagger$. Hence

$$\langle M_A\rangle_\omega = \mathrm{Tr}(\sigma_z\varrho_\omega) = -\mathrm{sgn}[\mathrm{Tr}(\sigma_z|\omega\rangle\langle\omega|)],$$

$$\langle M_B\rangle_\omega = \mathrm{Tr}(M_B'\rho_\omega) = \mathrm{Tr}(U_AM_BU_A^\dagger|\omega\rangle\langle\omega|). \tag{7}$$

Indeed, substituting Eq. (7) into Eq. (4) reproduces all two-qubit correlations $\langle M_A\otimes M_B\rangle$. However, Eq. (7) is not a LHV model for arbitrary M_A and M_B yet. This is because $\langle M_B\rangle_\omega$ depends on U_A, i.e, it depends on what operators are measured on party A. This dependence can be removed by defining $|\omega'\rangle = U_A^\dagger|\omega\rangle$. We obtain the desired LHV model with ω' as the hidden variable given as

$$\langle M_A\rangle_{\omega'} = -\mathrm{sgn}[\mathrm{Tr}(M_A|\omega'\rangle\langle\omega'|)],$$

$$\langle M_B\rangle_{\omega'} = \mathrm{Tr}(M_B|\omega'\rangle\langle\omega'|). \tag{8}$$

This can trivially be extended to arbitrary M_A [12]. One can recognize now Werner's model [2], where

$$\langle P_k\rangle_\omega = \begin{cases} 1 & \text{if } \langle\omega|P_k|\omega\rangle < \langle\omega|P_l|\omega\rangle \text{ for all } l\neq k, \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

for party A while

$$\langle M_B\rangle_\omega = \mathrm{Tr}(M_B|\omega\rangle\langle\omega|) \tag{10}$$

for B. This finishes the proof of theorem 1.

IV. THREE-QUBIT CASE

The previous construction is, in principle, easy to generalize to more parties as follows:

$$\rho^{(n,c)} := \int_{\omega\in\mathbb{C}^2,|\omega|=1} M(d\omega)\varrho_\omega \otimes \rho_\omega^{\otimes(n-1)}. \tag{11}$$

Here, party A receives again a nonphysical Bloch vector, while $B,C,\ldots$, get a standard qubit. We are now in the position to prove the main result of this work.

Theorem 2. There exists a LHV model for von Neumann measurements on states $\rho^{(3,c)}$ for $c\leqslant 1$. These states contain genuine three-qubit entanglement if $c>(\sqrt{13}-1)/3\approx 0.869$.

*Proof of Theorem 2.*Based on our discussion on the two-qubit case, it is clear that there is a LHV model for von Neumann measurements on $\rho^{(3,c)}$ if, and this is an important condition, this state is $U^{\otimes 3}$ invariant. After long but straightforward calculation, one obtains

$$\rho^{(3,c)} = \frac{1}{8}\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1} + \sum_{k=x,y,z}\frac{1}{24}\mathbb{1}\otimes\sigma_k\otimes\sigma_k$$

$$-\frac{c}{16}(\sigma_k\otimes\mathbb{1}\otimes\sigma_k + \sigma_k\otimes\sigma_k\otimes\mathbb{1}). \tag{12}$$

This state is invariant $U\otimes U\otimes U$ by inspection if we know that $\Sigma_k\sigma_k\otimes\sigma_k$ is $U\otimes U$ invariant. Thus correlation measurements on this state fit the LHV model given in Eqs. (9) and (10) for parties A/B, while we have $\langle M_C\rangle_\omega=\mathrm{Tr}(M_C|\omega\rangle\langle\omega|)$ for C.

We now show that $\rho^{(3,c)}$ is genuine three-partite entangled when $c>(\sqrt{13}-1)/3$, i.e., it cannot be constructed by mixing pure states with two-qubit entanglement. In particular, it cannot be constructed by mixing different bipartite Werner states of the form $\rho^{(2,c)}\otimes\rho_C$, $\rho_A\otimes\rho^{(2,c)}$, etc. In what follows, we adopt the definitions of Ref. [13]:

$$R_1 = \frac{1}{3}(2V_{BC} - V_{CA} - V_{AB}),$$

$$R_2 = \frac{1}{\sqrt{3}}(V_{AB} - V_{CA}), \tag{13}$$

where V_{kl} exchanges two qubits, and we use the notation $r_k=\langle R_k\rangle$. In order to examine the entanglement properties of the three-qubit states $\rho^{(3,c)}$, we consider the projection of the set of biseparable states on the r_1/r_2 plane [13], as shown in Fig. 2. The union of the three solid disks corresponds to the union of biseparable pure states of the three possible bipartitioning. One of these disks has the equation [13]

$$\left(\frac{\sqrt{3}}{2}r_1 + \frac{1}{2\sqrt{3}}\right)^2 + r_2^2 \leqslant \frac{1}{3}. \tag{14}$$

The other two can be obtained through ±120° rotations around the origin. The straight solid lines indicate the boundary of the convex hull of these sets. Any biseparable mixed state corresponds to a point within this set. Based on Eq. (14) and its rotated versions one can see that for biseparable states

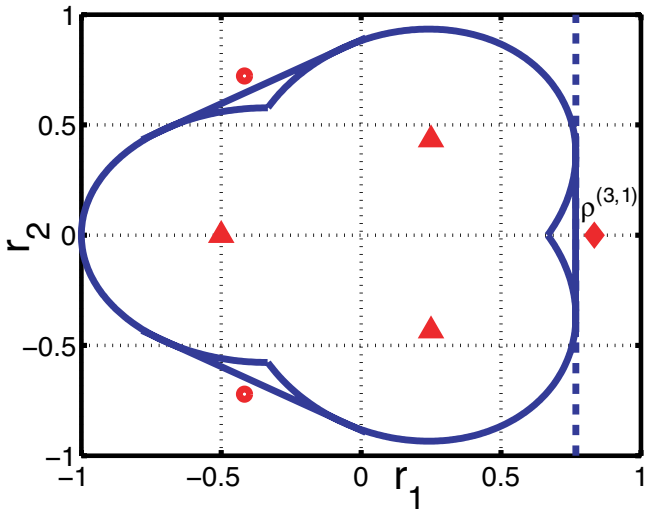

FIG. 2. (Color online) Union of three solid ellipses: Projection of the points corresponding to pure biseparable states on the r_1/r_2 plane. Together with the solid straight lines: Set of mixed biseparable states. (Diamond) State $\rho^{(3,1)}$. (Circles) States obtained from $\rho^{(3,1)}$ by permuting its qubits. (Triangles) Two-qubit Werner state of the form $\rho^{(2,1)}\otimes\mathbb{1}/2$ and states obtained from it by permuting the qubits. For more details see text.

$$\langle R_1\rangle \leqslant \frac{\sqrt{13}+1}{6} \approx 0.77. \tag{15}$$

In Fig. 2 points fulfilling Eq. (15) are on the left-hand side of the dashed vertical line. For $\rho^{(3,c)}$ we have $r_1=(2+3c)/6$ thus the state is genuine three-party entangled if $c>(\sqrt{13}-1)/3$. This finishes the proof of Theorem 2.

For the state $\rho^{(3,c)}$ the reduced two-qubit states ρ_{AB} and ρ_{AC} are entangled if $c>2/3$. This means that for $c>2/3$ a singlet between A and B, and also between A and C can be distilled. Hence from many copies of our state any three-qubit state can be obtained with local operations and classical communication.

V. STATES $\rho^{(n,c)}$ AND $U^{\otimes n}$-INVARIANT SYMMETRIC EXTENSIONS OF TWO-QUBIT WERNER STATES

At first sight, there is no reason to limit our discussion to $n=3$. Direct calculation, however, shows that $\rho^{(4,1)}$ is not $U^{\otimes 4}$ invariant, so the previous local model cannot be applied. Nevertheless, the states of Eq. (2) can be used to construct symmetric extensions [14,15]. Recall that a $(1,n-1)$ symmetric extension of a two-party state ρ is an n-party operator H_ρ such that $\rho=\mathrm{Tr}_{3,4,\ldots,n}H_\rho$ and H_ρ is symmetric under the permutation of parties $2,3,\ldots,n$. For being an extension, we also need that H_ρ is positive semidefinite [14] and for a quasiextension that it is positive on product states [5].

It can be seen by inspection that if $\rho^{(n,c)}$ is positive semidefinite then it is an extension of $\rho^{(2,c)}$. Symmetric extensions can be constructed for a larger range of c if the matrix $\rho^{(n,c)}$ is twirled:

$$\rho_T^{(n,c)} := \int_{U\in U(2)} dU U^{\dagger\otimes n}\rho^{(n,c)}U^{\otimes n}$$

$$= \int_{\omega\in\mathbb{C}^2,|\omega|=1} M(d\omega)\tau_\omega\otimes\rho_\omega^{\otimes(n-1)}, \qquad (16)$$

where dU denotes the Haar measure and ρ_ω is defined in Eq. (3) and

$$\tau_\omega := \frac{1}{2}\left[\mathbb{1}-\frac{3}{2}c\sum_{k=x,y,z}\langle\sigma_k\rangle_\omega\sigma_k\right]. \qquad (17)$$

Direct calculation shows that for Werner states with $p=2/3$, $5/9$, and $1/2$, $(1,m)$ symmetric extensions can be obtained from Eq. (16) for $m=2$, 3, and 4, respectively.

VI. GENERALIZATION TO MORE PARTIES OR HIGHER DIMENSION

Now rather than fixing the quantum state from the very beginning, we will look for the four-qubit quantum state for which the correlations fit the local model of Eqs. (9) and (10), when party C and D also get physical qubits. Note, however, that there is no *a priori* reason why a LHV model should give correlations compatible with a quantum state. The desired state must be $U^{\otimes 4}$ invariant and thus it must be a linear combination of the $4!=24$ permutation operators [13]. It must fit all three-qubit correlations of our LHV model and must be invariant under the permutation of qubits B, C, and D. It can be proved that this state must have the form $\rho' := \rho_{N4}-K\{3\Sigma_{k=x,y,z}\sigma_k\otimes\sigma_k\otimes\sigma_k\otimes\sigma_k+\Sigma_{l<k}\Pi[\sigma_k\otimes\sigma_k\otimes\sigma_l\otimes\sigma_l]\}$, where $\Pi[A]$ denotes the sum of all distinct permutations of A and K is a constant. Here ρ_{N4} contains the terms that do not affect four-qubit correlations of the form $\langle\sigma_a\otimes\sigma_b\otimes\sigma_c\otimes\sigma_d\rangle$. Setting $K=1/128$ our state gives the same four-qubit correlations as the LHV model for $M_{A/B}=\sigma_x$ and $M_{C/D}=\sigma_y$. Due to the finite number of free parameters there is only one such Hermitian matrix with unit trace. However, for $M_{A/B/C/D}=\sigma_x$ this matrix fails to reproduce correlations of the LHV model (i.e., $-1/4$). Thus, there is not a matrix corresponding to the n-qubit version of the LHV model given in Eqs. (9) and (10) for $n\geq 4$ qubits.

Finally, one can explore whether the local model Eqs. (9) and (10), with $\langle M_C\rangle_\omega=\mathrm{Tr}(M_C|\omega\rangle\langle\omega|)$ can be associated to a three-qudit state. Surprisingly, it turns out that the model obtained this way is not a valid LHV model for a quantum state when $d>2$. In order to see this, let us consider the case $d=3$. Take $M_A=|1\rangle\langle 1|$, defined by $\{\alpha_k\}=\{1,0,0\}$ and $\{P_k\}=\{|1\rangle\langle 1|,|2\rangle\langle 2|,|3\rangle\langle 3|\}$, and $M_A'=|1\rangle\langle 1|$, which is actually equal to M_A but defined by $\{\alpha_k'\}=\{1,0,0\}$ and $\{P_k'\}=\{|1\rangle\langle 1|,|2'\rangle\langle 2'|,|3'\rangle\langle 3'|\}$, where $|2'\rangle=\alpha|2\rangle+\beta|3\rangle$ and $|3'\rangle=\beta^*|2\rangle-\alpha^*|3\rangle$, with $|\alpha|^2+|\beta|^2=1$. Moreover, on the other two qudits we measure $M_B=M_C=|2\rangle\langle 2|$. Using the methods of Ref. [2], we obtain $\langle M_A\otimes M_B\otimes M_C\rangle=13/162$, while $\langle M_A'\otimes M_B\otimes M_C\rangle=15/162$ for $\alpha=\beta=1/\sqrt{2}$. Thus $\langle M_A\otimes M_B\otimes M_C\rangle\neq\langle M_A'\otimes M_B\otimes M_C\rangle$. A similar lack of self-consistency can be found for $d>3$.

VII. CONCLUSIONS

We presented a family of three-qubit states for which correlations for all von Neumann measurements can be described by a LHV model. We proved that some of these states are genuine three-qubit entangled and distillable, so three-qubit entanglement is not sufficient for a state to be nonlocal. We also showed that there is not a quantum state corresponding to our model with more parties or higher dimension. In the future, it would be interesting to extend our model to general measurements. For details of our calculation, see [16].

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