Dark periods and revivals of entanglement in a two-qubit system

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In a recent paper Yu and Eberly [Phys. Rev. Lett. **93**, 140404 (2004)] have shown that two initially entangled and afterward not interacting qubits can become completely disentangled in a finite time. We study transient entanglement between two qubits coupled collectively to a multimode vacuum field, assuming that the two-qubit system is initially prepared in an entangled state produced by the two-photon coherences, and find the unusual feature that the irreversible spontaneous decay can lead to a revival of the entanglement that has already been destroyed. The results show that this feature is independent of the coherent dipole-dipole interaction between the atoms but it depends critically on whether or not collective damping is present.

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The problem of controlling the evolution of entanglement between atoms (or qubits) that interact with the environment has received a great deal of attention in recent years [1–4]. The environment may be treated as a reservoir and it is well known that the interaction of an excited atom with the reservoir leads to spontaneous emission which is one of the major sources of decoherence. In light of the experimental investigations, the spontaneous emission leads to irreversible loss of information encoded in the internal states of the system and thus is regarded as the main obstacle in practical implementations of entanglement.

This justifies the interest in finding systems where spontaneous emission is insignificant. However, in many treatments of the entanglement creation and entanglement dynamics, the coupling of atoms to the environment is simply ignored or limited to the interaction of the atoms with a single-mode cavity [5–7].

It is well known that under certain circumstances a group of atoms can act collectively so that the radiation field emitted by an atom of the group may influence the dynamics of the other atoms [8–11]. The resulting dynamics and the spontaneous emission from the atoms may be considerably modified. It was recently suggested that two suitably prepared atoms can be entangled through the mutual coupling to the vacuum field [1,2,12–14]. Stationary two-atom entanglement is possible when the atoms are damped to the squeezed vacuum [15]. It has also been predicted that two initially entangled and afterward not interacting atoms can become completely disentangled in a time much shorter than the decoherence time of spontaneous emission. This feature has been studied by Yu and Eberly [16] and Jakóbczyk and Jamróz [17], who termed it the "sudden death" of entanglement, and elucidated many new characteristics of entanglement evolution in systems of two atoms. Their analysis, however, concentrated exclusively on systems of independent atoms.

In this paper, we consider a situation where the atoms are coupled to the multimode vacuum field and demonstrate the occurrence of multiple dark periods and revivals of entanglement induced by the irreversible spontaneous decay. We fully incorporate collective interaction between the atoms and study in detail the dependence of the revival time on the initial state of the system and on the separation between the atoms. We emphasize that the revival of entanglement in a pure spontaneous emission process contrasts with the situation of the coherent exchange of entanglement between atoms and a cavity mode [5,6].

We consider two identical two-level atoms (qubits) having lower levels $|g_i\rangle$ and upper levels $|e_i\rangle$ $(i=1,2)$ separated by energy $\hbar\omega_0$, where ω_0 is the transition frequency. The atoms are coupled to a multimode radiation field whose modes are initially in the vacuum state $|\{0\}\rangle$. The atoms radiate spontaneously and their radiation fields exert a strong dynamical influence on one another through the vacuum field modes. The time evolution of the system is studied using the Lehmberg–Agarwal [9–11] master equation, which reads

$$\frac{\partial\rho}{\partial t} = -i\omega_0\sum_{i=1}^{2}[S_i^z,\rho] - i\sum_{i\neq j}^{2}\Omega_{ij}[S_i^+S_j^-,\rho] - \frac{1}{2}\sum_{i,j=1}^{2}\gamma_{ij}([\rho S_i^+,S_j^-] + [S_i^+,S_j^-\rho]), \qquad (1)$$

where S_i^+ (S_i^-) are the dipole raising (lowering) operators and S_i^z is the energy operator of the ith atom; $\gamma_{ii}\equiv\gamma$ are the spontaneous decay rates of the atoms caused by their direct coupling to the vacuum field. The parameters γ_{ij} and Ω_{ij} $(i\neq j)$ depend on the distance between the atoms and describe the collective damping and the dipole-dipole interaction defined, respectively, by

$$\gamma_{ij} = \frac{3}{2}\gamma\left(\frac{\sin kr_{ij}}{kr_{ij}} + \frac{\cos kr_{ij}}{(kr_{ij})^2} - \frac{\sin kr_{ij}}{(kr_{ij})^3}\right) \qquad (2)$$

and

$$\Omega_{ij} = \frac{3}{4}\gamma\left(-\frac{\cos kr_{ij}}{kr_{ij}} + \frac{\sin kr_{ij}}{(kr_{ij})^2} + \frac{\cos kr_{ij}}{(kr_{ij})^3}\right), \qquad (3)$$

where $k=\omega_0/c$, and $r_{ij}=|\vec{r}_j-\vec{r}_i|$ is the distance between the atoms. Here, we assume, with no loss of generality, that the atomic dipole moments are parallel to each other and are polarized in the direction perpendicular to the interatomic axis. The effect of the collective parameters on the time evolution of the entanglement in the system is the main concern of this paper.

It will prove convenient to work in the basis of four collective states, so-called Dicke states, defined as [8]

$$|e\rangle = |e_1\rangle \otimes |e_2\rangle,$$

$$|g\rangle = |g_1\rangle \otimes |g_2\rangle,$$

$$|s\rangle = (|g_1\rangle \otimes |e_2\rangle + |e_1\rangle \otimes |g_2\rangle)/\sqrt{2},$$

$$|a\rangle = (|g_1\rangle \otimes |e_2\rangle - |e_1\rangle \otimes |g_2\rangle)/\sqrt{2}. \quad (4)$$

In this basis, the two-atom system behaves as a single four-level system with the ground state $|g\rangle$, two intermediate states $|s\rangle$ and $|a\rangle$, and the upper state $|e\rangle$.

In order to determine the amount of entanglement between the atoms and the entanglement dynamics, we use the concurrence, which is a widely accepted measure of entanglement. The concurrence introduced by Wootters [18] is defined as

$$\mathcal{C}(t) = \max(0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}), \quad (5)$$

where $\{\lambda_i\}$ are the eigenvalues of the matrix

$$R = \rho\tilde{\rho} \quad \text{with } \tilde{\rho} = \sigma_y \otimes \sigma_y \rho^* \sigma_y \otimes \sigma_y, \quad (6)$$

and σ_y is the Pauli matrix. The range of concurrence is from 0 to 1. For unentangled atoms $\mathcal{C}(t)=0$ whereas $\mathcal{C}(t)=1$ for maximally entangled atoms.

The density matrix, which is needed to calculate $\mathcal{C}(t)$, is readily evaluated from the master equation (1). Following Yu and Eberly, we choose the atoms to be at the initial time ($t=0$) prepared in an entangled state of the form

$$|\Psi_0\rangle = \sqrt{p}|e\rangle + \sqrt{1-p}|g\rangle, \quad (7)$$

where $0 \leq p \leq 1$. The state $|\Psi_0\rangle$ is a linear superposition of only those states of the system in which both or neither of the atoms is excited. As discussed in Refs. [16,17], in the absence of the coupling between the qubits, the initial entangled state of the form (7) disentangles in a finite time. They termed this feature the sudden death of entanglement.

In what follows, we examine the time evolution of the entanglement of two atoms coupled to the multimode vacuum field. If the atoms are initially prepared in the state (7), it is not difficult to verify that the initial one-photon coherences are zero, i.e., $\rho_{es}(0)=\rho_{ea}(0)=\rho_{sg}(0)=\rho_{ag}(0)=\rho_{as}(0)=0$. This implies that, for all times, the density matrix of the system represented in the collective basis (4), is given in the block-diagonal form

$$\rho(t) = \begin{pmatrix} \rho_{ee}(t) & \rho_{eg}(t) & 0 & 0 \\ \rho_{eg}^*(t) & \rho_{gg}(t) & 0 & 0 \\ 0 & 0 & \rho_{ss}(t) & 0 \\ 0 & 0 & 0 & \rho_{aa}(t) \end{pmatrix}, \quad (8)$$

with the density matrix elements evolving as

$$\rho_{ee}(t) = pe^{-2\gamma t},$$

$$\rho_{eg}(t) = \sqrt{p(1-p)}e^{-\gamma t},$$

$$\rho_{ss}(t) = pe^{-\gamma t}\frac{\gamma+\gamma_{12}}{\gamma-\gamma_{12}}(e^{-\gamma_{12}t} - e^{-\gamma t}),$$

$$\rho_{aa}(t) = pe^{-\gamma t}\frac{\gamma-\gamma_{12}}{\gamma+\gamma_{12}}(e^{\gamma_{12}t} - e^{-\gamma t}), \quad (9)$$

subject to conservation of probability $\rho_{gg}(t)=1-\rho_{ss}(t)-\rho_{aa}(t)-\rho_{ee}(t)$.

Note that the evolution of the system depends crucially on the initial conditions, and for the present initial conditions the evolution of the density matrix elements is independent of the dipole-dipole interaction between the atoms, but is profoundly affected by the collective damping γ_{12}. If the initial conditions are chosen differently, such that there is a nonzero coherence ρ_{as} which oscillates with the frequency $2\Omega_{12}$, the dipole-dipole interaction manifests its presence in oscillatory behavior of the concurrence [14].

Given the density matrix Eq. (8), we can now calculate the concurrence $\mathcal{C}(t)$ and examine the transient dynamics of the entanglement. First, we find that the square roots of the eigenvalues of the matrix R are

$$\sqrt{\lambda_{1,2}(t)} = |\rho_{ge}(t)| \pm [\rho_{ss}(t) + \rho_{aa}(t)],$$

$$\sqrt{\lambda_{3,4}(t)} = [\rho_{ss}(t) - \rho_{aa}(t)] \pm \sqrt{\rho_{gg}(t)\rho_{ee}(t)}, \quad (10)$$

from which it is easily verified that for a particular value of the matrix elements there are two possibilities for the largest eigenvalue, either $\sqrt{\lambda_1(t)}$ or $\sqrt{\lambda_3(t)}$. The concurrence is thus given by

$$\mathcal{C}(t) = \max\{0, \mathcal{C}_1(t), \mathcal{C}_2(t)\}, \quad (11)$$

with

$$\mathcal{C}_1(t) = 2|\rho_{ge}(t)| - [\rho_{ss}(t) + \rho_{aa}(t)],$$

$$\mathcal{C}_2(t) = |\rho_{ss}(t) - \rho_{aa}(t)| - 2\sqrt{\rho_{gg}(t)\rho_{ee}(t)}. \quad (12)$$

From this it is clear that the concurrence $\mathcal{C}(t)$ can always be regarded as being made up of the sum of nonnegative contributions of the weights $\mathcal{C}_1(t)$ and $\mathcal{C}_2(t)$ associated with two different classes of entangled states that can be generated in a two-qubit system. From the form of the entanglement weights it is obvious that $\mathcal{C}_1(t)$ provides a measure of an entanglement produced by linear superpositions involving the ground $|g\rangle$ and the upper $|e\rangle$ states of the system, whereas $\mathcal{C}_2(t)$ provides a measure of an entanglement produced by a distribution of the population between the symmetric and antisymmetric states. Inspection of Eq. (12) shows that for $\mathcal{C}_1(t)$ to be positive it is necessary that the two-photon coherence ρ_{eg} is different from zero, whereas the necessary condition for $\mathcal{C}_2(t)$ to be positive is that the symmetric and antisymmetric states are not equally populated.

We consider first the effect of the collective damping on the sudden death of an initial entanglement determined by the state (7). The entanglement weights $\mathcal{C}_1(t)$ and $\mathcal{C}_2(t)$, which are needed to construct $\mathcal{C}(t)$, are readily calculated from Eqs. (7) and (12). We see that the system initially prepared in the state (7) can be entangled according to the criterion $\mathcal{C}_1$, and the degree to which the system is initially entangled is $\mathcal{C}_1(0)=2\sqrt{p(1-p)}$.

If the atoms radiate independently, $\gamma_{12}=0$, and then we

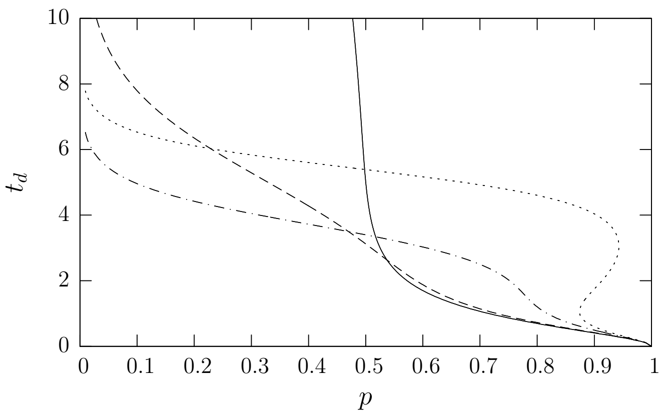

FIG. 1. The death time of the entanglement prepared according to the criterion $\mathcal{C}_1$ and plotted as a function of p for different separations between the atoms: $r_{12}=\lambda$ (solid line), $\lambda/3$ (dashed line), $\lambda/6$ (dash-dotted line), and $r_{12}=\lambda/20$ (dotted line).

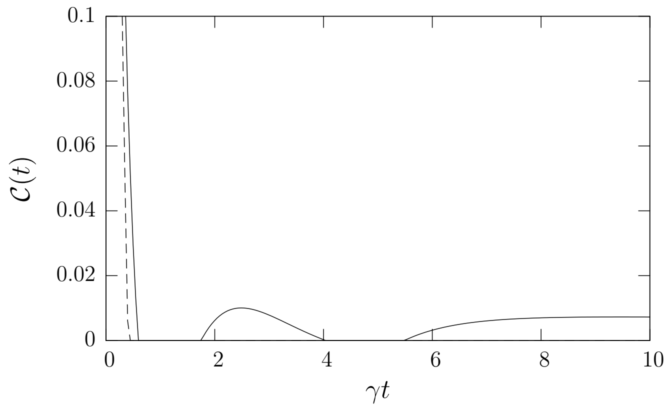

FIG. 2. Transient evolution of the concurrence $\mathcal{C}(t)$ for the initial state $|\Psi_0\rangle$ with $p=0.9$. The solid line represents $\mathcal{C}(t)$ for the collective system with the interatomic separation $r_{12}=\lambda/20$. The dashed line shows $\mathcal{C}(t)$ for independent atoms, $\gamma_{12}=0$.

find from Eq. (9) that $\rho_{ss}(t)=\rho_{aa}(t)$. It is clear by inspection of Eq. (12) that in this case $\mathcal{C}_2(t)$ is always negative, so we immediately conclude that no entanglement is possible according to the criterion $\mathcal{C}_2$, and the atoms can be entangled only according to the criterion $\mathcal{C}_1$. The initial entanglement decreases in time because of the spontaneous emission and disappears at the time

$$t_d=\frac{1}{\gamma}\ln\left(\frac{p+\sqrt{p(1-p)}}{2p-1}\right), \tag{13}$$

from which we see that the time it takes for the system to disentangle is a sensitive function of the initial atomic conditions. We note from Eq. (13) that the sudden death of the entanglement of independent atoms is possible only for $p>1/2$. Since $\rho_{ee}(0)=p$, we must conclude that entanglement sudden death is ruled out for a system that is initially not inverted.

For a collective system, when the atoms are close to each other, $\gamma_{12}\neq 0$, and then the sudden death appears in less restricted ranges of the parameter p. This is shown in Fig. 1, where we plot the death time as a function of p for several separations between the atoms. We see that the range of p over which the sudden death occurs increases with decreasing r_{12}, and for small separations the sudden death occurs over the entire range of p.

The most interesting consequence of the collective damping is the possibility of entanglement revival. We now use Eqs. (9) and (12) to discuss the ability of the system to revive entanglement in the simple process of spontaneous emission. Figure 2 shows the deviation of the time evolution of the concurrence for two interacting atoms from that of independent atoms. In both cases, the initial entanglement falls as the transient evolution is damped by the spontaneous emission. For independent atoms we observe the collapse of the entanglement without any revivals. However, for interacting atoms, the system collapses over a short time and remains disentangled until a time $t_r\approx 1.7/\gamma$ at which, somewhat counterintuitively, the entanglement revives. This revival then decays to zero, but after some period of time a new revival begins. Thus, we see two time intervals (dark periods) at which the entanglement vanishes and two time intervals at which the entanglement revives. To estimate the death and revival times, we use Eqs. (12) and (9), and find that for $\gamma_{12}\approx\gamma$, the entanglement weight $\mathcal{C}_1(t)$ vanishes at times satisfying the relation

$$\gamma t\exp(-\gamma t)=\sqrt{\frac{1-p}{p}}, \tag{14}$$

which for $p>0.88$ has two nondegenerate solutions t_d and $t_r>t_d$. The time t_d gives the collapse time of the entanglement beyond which the entanglement disappears. The death zone of the entanglement continues until the time t_r at which the entanglement revives. Thus, for the parameters of Fig. 2, the entanglement collapses at $t_d=0.6/\gamma$ and revives at the time $t_r=1.7/\gamma$.

The origin of the dark periods and the revivals of the entanglement can be understood in terms of the populations of the collective states and the rates with which the populations and the two-photon coherence decay. One can note from Eq. (9) that for short times $\rho_{aa}(t)\approx 0$, but $\rho_{ss}(t)$ is large. Thus, the entanglement behavior can be analyzed almost entirely in terms of the population of the symmetric state and the coherence $\rho_{eg}(t)$.

Figure 3 shows the time evolution of $\mathcal{C}(t)$, the population $\rho_{ss}(t)$, and the coherence $\rho_{eg}(t)$. As can be seen from the graphs, the entanglement vanishes at the time at which the population of the symmetric state is maximal and remains zero until the time t_r at which $\rho_{ss}(t)$ becomes smaller than $\rho_{eg}(t)$. We may conclude that the first dark period arises due to the significant accumulation of the population in the symmetric state. The impurity of the state of the two-atom system is rapidly growing and entanglement disappears.

The reason for the occurrence of the first revival seen in Fig. 2 is that the two-photon coherence $\rho_{eg}(t)$ decays more slowly than the population of the symmetric state. Once $\rho_{ss}(t)$ falls below $2|\rho_{eg}(t)|$, entanglement emerges again. Thus, the coherence can become dominant again and entanglement regenerated over some period of time during the

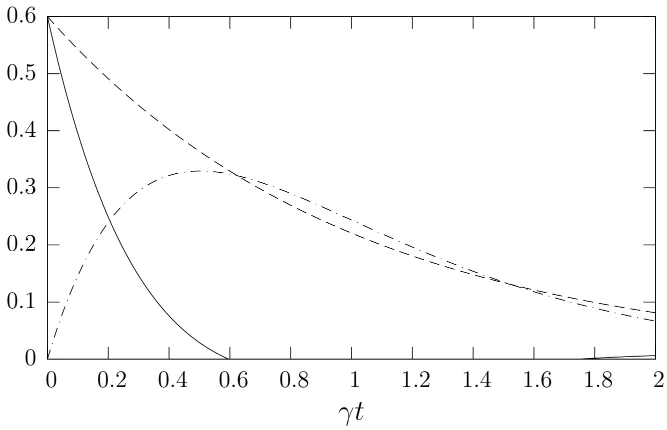

FIG. 3. Origin of the first dark period and revival of the entanglement of a collective system. The time evolution of the coherence $2|\rho_{eg}(t)|$ (dashed line) is compared with the evolution of the population $\rho_{ss}(t)$ (dash-dotted line) for the same parameters as in Fig. 2. The solid line is the time evolution of the concurrence $\mathcal{C}(t)=\mathcal{C}_1(t)$.

decay process. This is the same coherence that produced the initial entanglement. Therefore, we may call the first revival an "echo" of the initial entanglement that has been unmarked by destroying the population of the symmetric state. It is interesting to note that the entanglement revival appears only for large values of p, and is most pronounced for $p>0.88$. This is not surprising because for $p>1/2$ the system is initially inverted, which increases the probability of spontaneous emission.

We have seen that the short-time behavior of the entanglement is determined by the population of the symmetric state of the system. A different situation occurs at long times. As is seen from Fig. 2, the entanglement revives again at longer times and decays asymptotically to zero as $t\to\infty$. The second revival has completely different origin from the first one. At long times both $\rho_{ss}(t)$ and $\rho_{eg}(t)$ are almost zero. However, the population $\rho_{aa}(t)$ is sufficiently large as it accumulates on the time scale $t=1/(\gamma-\gamma_{12})$ which is very long when $\gamma_{12}\approx\gamma$. A careful examination of Eq. (9) shows that $\mathcal{C}_1(t)<0$ at long times, so that the long-time entanglement is determined solely by the weight $\mathcal{C}_2$, which is negative for short times, and it becomes positive after a finite time t_{r_2} (second revival time) given approximately by the formula

$$t_{r_2}\approx\frac{1}{\gamma_{12}}\ln\left(\frac{1}{\sqrt{p}}\frac{4\gamma}{\gamma-\gamma_{12}}\right). \quad (15)$$

It follows from the above analysis and Fig. 2 that the entanglement prepared according to the criterion $\mathcal{C}_1$ is a rather short-lived affair compared with the long-lived entanglement prepared with the criterion $\mathcal{C}_2$. Asymptotically, the concurrence is equal to the population $\rho_{aa}(t)$.

We conclude with an example of possible experimental observation of the features predicted in this paper. In principle this system may be realized in a scheme similar to that used by Osnaghi *et al.* [6] to observe entanglement between two atoms. The scheme involves two Rydberg atoms traversing a semiconductor microwave cavity of the resonant wavelength ~0.6 cm. At such long wavelengths, the interatomic separations smaller than the resonant wavelength that we have assumed here could be realized without much difficulty.

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