Fractional revivals and quantum entanglement via coupled angular momenta

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We examine the entanglement dynamics generated by two coupled angular momenta. For initially spin-coherent states, we show that fractional revivals can lead to a well-defined sequence of fractional disentanglements, at which the purity of each subsystem is almost a fractional number $1/n$ (where n is an integer). This interesting effect is interpreted as the formation of entangled macroscopic superposition states.

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When two or more quantum systems interact with each other, the evolution of quantum entanglement provides a fundamental picture describing how nonclassical correlations develop in time. By analyzing the quantum entanglement, one can also identify correlated observables relevant to the entangled states involved. Apart from the most extensively studied two-qubit problems, the dynamics of quantum entanglement has been discussed in various basic physical processes. These include, for examples in atomic physics, the ionization processes [1,2], two-body scattering processes [3], and spontaneous emission [4].

In this paper, we address the interesting question of how two large spin systems become entangled via angular momentum coupling. Specifically, we consider two spin-s systems A and B. The Hamiltonian is a simple form of angular momentum coupling:

$$H = g\mathbf{J}_A \cdot \mathbf{J}_B \tag{1}$$

where $\mathbf{J}_A$ and $\mathbf{J}_B$ are the angular momentum operators of A and B, and g is the coupling strength. Such an interaction appears in many physical situations, and high-spin systems could be realized by wave-mixing processes in Bose-Einstein condensates [5]. Classically, $\mathbf{J}_A$ and $\mathbf{J}_B$ execute precessions about a fixed axis, with the precession speed determined by the total angular momentum. In quantum mechanics, if the initial states of A and B are wave packets described by spin-coherent states [6], then the wave packets will follow the classical motion at short times. However, since different parts of wave packets precess at different speeds, the wave packets spread in a correlated manner. Consequently, the two systems become entangled as time increases.

For pure systems, quantum entanglement can be characterized by the loss of purity of subsystems [1]. Interestingly, for the interaction described by (1), we find that the purity can exhibit a kind of fractional revival phenomenon. The partial recovery of purity implies that the system is partially disentangled. We note that fractional revival is a general consequence of wave-packet dynamics in systems with discrete and quadratic energy spacings [7,8], and previous investigations have considered mainly the dynamics in single systems. In this paper we investigate a composite system in which a close relation between fractional revivals and quantum entanglement can be found. In contrast to single systems where autocorrelation functions are sufficient to reveal the revival behavior [7], here the correlations between subsystems require a different indicator. As we shall see below, the purity function provides a useful way to detect both quantum entanglement and fractional revivals.

To begin with, we employ the addition of angular momentum $\mathbf{J}=\mathbf{J}_A+\mathbf{J}_B$, so that the Hamiltonian takes the form $H=g(\mathbf{J}^2-\mathbf{J}_A^2-\mathbf{J}_B^2)/2$. Since J_A^2 and J_B^2 are constants of motion, the general solution of the time-dependent system state vector is given by

$$|\Psi(\tau)\rangle = e^{is(s+1)\tau}\sum_{J=0}^{2s}\sum_{M=-J}^{J} C_{JM}e^{-iJ(J+1)\tau/2}|J,M\rangle \tag{2}$$

where $\tau=gt$ is the dimensionless time, and $|J,M\rangle$ are eigenvectors of J^2 and J_z. The amplitude C_{JM} is determined by initial conditions.

Since we are interested in quantum entanglement between A and B, it is more convenient to rewrite the solution (2) in terms of basis vectors of individual systems. With the help of Clebsch-Gordan coefficients, we have

$$|\Psi(\tau)\rangle = e^{is(s+1)\tau}\sum_{m=-s}^{s}\sum_{m'=-s}^{s} F_{mm'}(\tau)|s,m\rangle_A|s,m'\rangle_B \tag{3}$$

with

$$F_{mm'}(\tau) \equiv \sum_{J=0}^{2s}\sum_{M=-J}^{J} C_{JM}e^{-iJ(J+1)\tau/2}G(s,s,m,m';J,M). \tag{4}$$

In writing Eq. (3), $|s,m\rangle_i$ $(i=A,B)$ are common eigenvectors of J_i^2 and J_{i_z}, i.e., $J_i^2|s,m\rangle_i=s(s+1)|s,m\rangle_i$ and $J_{i_z}|s,m\rangle_i=m|s,m\rangle_i$. In addition, $G(s,s,m,m';J,M)=(\langle s,m'|\langle s,m|)|JM\rangle$ are Clebsch-Gordan coefficients.

Let the initial state of the system be a disentangled state:

$$|\Psi(0)\rangle = \sum_{m=-s}^{s} A_m|s,m\rangle_A \otimes \sum_{m'=-s}^{s} B_{m'}|s,m'\rangle_B. \tag{5}$$

Then C_{JM} is related to the initial amplitudes A_m and $B_{m'}$ by

$$C_{JM} = \sum_{m=-s}^{s}\sum_{m'=-s}^{s} A_m B_{m'} G^*(s,s,m,m';J,M). \tag{6}$$

Quite generally, $F_{mm'}(\tau)$ cannot be factorized into a product of functions of m and m', except at full revival times $\tau=2k\pi$ (k=integer). Therefore the two particles are entangled most of the time.

To characterize the time dependence of quantum entanglement, we employ the purity function of subsystem A (or B) as an entanglement measure,

$$P(\tau) = \mathrm{Tr}[\rho_A(\tau)^2] = \mathrm{Tr}[\rho_B(\tau)^2] \tag{7}$$

where $\rho_A(\tau)=\mathrm{Tr}_B[\rho(\tau)]$ is the reduced density of the particle A, and $\rho(\tau)$ is the whole $(A+B)$ density matrix. A disentangled (product) state corresponds to $P=1$. The smaller the value of P, the higher the entanglement. Comparing with the usual measure based on entanglement entropy $-\mathrm{Tr}(\rho_A \ln \rho_A)$, the calculation of purity is more convenient as it does not involve diagonalization of density matrices.

From Eq. (3), P takes the explicit form

$$P(\tau) = \sum_{m=-s}^{s}\sum_{n=-s}^{s}\sum_{p=-s}^{s}\sum_{q=-s}^{s} F_{mn}F^*_{pn}F_{pq}F^*_{mq}. \tag{8}$$

Together with Eqs. (4) and (6), the time dependence of entanglement can be determined. In this paper, we will restrict our study to initial states described by spin-coherent states

$$\sum_{m=-s}^{s} A_m|s,m\rangle_A = |\theta_1,\phi_1\rangle_A \tag{9}$$

$$\sum_{m=-s}^{s} B_m|s,m\rangle_B = |\theta_2,\phi_2\rangle_B. \tag{10}$$

Here the spin-coherent states $|\theta,\phi\rangle$ are defined by [6]

$$|\theta,\phi\rangle \equiv \sum_{m=-s}^{s}\left(\frac{(2s)!}{(s+m)!(s-m)!}\right)^{1/2}\frac{\xi^{s+m}}{(1+|\xi|^2)^s}|s,m\rangle \tag{11}$$

with $\xi\equiv\cot(\theta/2)e^{-i\phi}$. For convenience our notation of θ is the same as that in the usual spherical coordinates, which is different from that used in [6]. Physically, $|\theta,\phi\rangle$ is a wave packet localized at the angular coordinate (θ,ϕ). Since the interaction (1) is spherically symmetric, we have the freedom to choose axes such that the centers of wave packets are symmetrically placed about the z axis in the positive z region, i.e., $\theta_2=\theta_1\equiv\theta_0\leqslant\pi/2$. In addition, $\phi_1=0$ and $\phi_2=\pi$ can be taken by the choice of axes.

In Fig. 1, we present an example of $P(\tau)$ with $s=100$ and $\theta_0=\pi/4$. At early times, the generation of entanglement is signaled by a rapid decrease of P. As time increases, we see a sequence of peak structures. The most prominent feature is that P peaks at almost fractional values $1/n$, where $n=1,2,3,\ldots$ are integers. Specifically, we find that

$$P\left(\tau=\frac{2\pi l}{n}\right)\approx\frac{1}{n} \tag{12}$$

where l is an integer, and l and n are relatively prime. Some of these fractional values are indicated in Fig. 1, and agree with the exact numerical peak values of P with errors less than or equal to 4%. For the parameters used in Fig. 1, up to $n=15$ can be observed.

To explain Eq. (12), we follow the theory of fractional revivals by Averbukh and Perelman [8]. For angular-momentum-coherent states, Rozmej and Arvieu have also provided a detailed account of revivals in single systems [9].

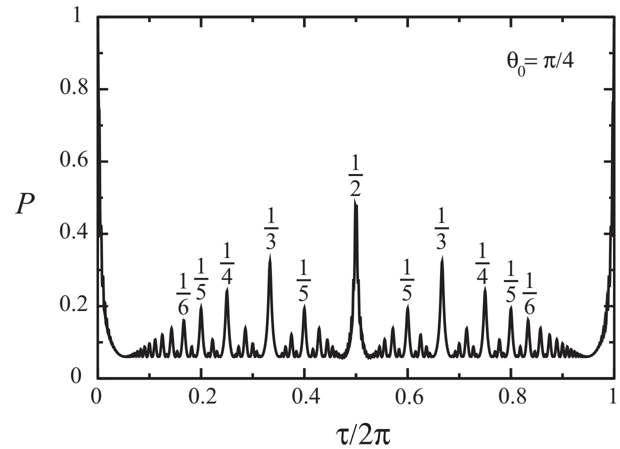

FIG. 1. Time evolution of the purity P of subsystem A (or B) with $s=100$. The initial state is a spin-coherent state specified by the parameter $\theta_0=\pi/4$ (see text). At revival times $\tau=2\pi l/n$, the purity attains nearly the value $1/n$ as indicated in the figure.

Here we extend these previous studies to address entanglement in composite systems. The strategy is based on the fact that the phase factor $e^{-iJ(J+1)\tau/2}$ is periodic in J when $\tau=2\pi l/n$, i.e., $e^{-i\pi(J+r)(J+r+1)l/n}=e^{-i\pi J(J+1)l/n}$ with an integer period r. It can be shown that $r=n$ when n is odd, and $r=2n$ when n is even.

By exploiting the periodicity of the phase factor, a discrete Fourier transform gives

$$e^{-i\pi J(J+1)l/n} = \frac{1}{r}\sum_{k=1}^{r} a_k e^{-2\pi iJk/r} \tag{13}$$

where a_k are Fourier amplitudes that can be obtained by inverse discrete Fourier transform of (13). For odd n's, $a_k=e^{i\gamma_{kn}}/\sqrt{n}$, and for even n's, $a_{2k}=0$ and $a_{2k-1}=e^{i\gamma_{kn}}/\sqrt{n}$, where γ_{kn} are real angles. Hence the number of nonzero amplitudes in Eq. (13) is always n, and all the nonzero a_k have the same absolute value. Therefore the state vector (2) can be decomposed into a superposition state,

$$\left|\Psi\left(\frac{2\pi l}{n}\right)\right\rangle = \frac{e^{i2\pi s(s+1)l/n}}{\sqrt{n}}\sum_{k=1}^{n} e^{i\gamma_{kn}}|\psi_k\rangle \tag{14}$$

where

$$|\psi_k\rangle = \sum_{J=0}^{2s}\sum_{M=-J}^{J} C_{JM}e^{-2\pi iJ(k+\delta)/n}|J,M\rangle \tag{15}$$

with $\delta=\mathrm{mod}(n+1,2)/2$.

We now make use of the spin-coherent states assumed in the initial condition. Owing to the complicated forms of Clebsch-Gordan coefficients, C_{JM} do not have simple closed forms. However, by numerical calculations, we observe that $M=J$ terms constitute most of the system state vector (2), as long as θ_0 is not too close to $\pi/2$. Such an observation can be quantified by $\sum_{J=0}^{2s}|C_{JJ}|^2$, which corresponds to the probability of finding the system in the subspace formed by $|J,J\rangle$ eigenvectors. We find that for $\theta_0\leqslant\pi/4$ and $s\geqslant 2$, $\sum_{J=0}^{2s}|C_{JJ}|^2$ is greater than 98% [10]. Such a high probability can be understood classically by adding the angular momentum vectors of A and B within the widths associated with their corresponding spin coherent states. Such a classical picture gives $J_z/J=1-O(1/s)$, i.e., J_z and J are nearly the same up to a correction term of order $O(1/s)$ due to fluctuations. However, we remark that the classical picture is valid when $J_z \gg 1$. It fails to describe situations when $2s\cos\theta_0$ is small. This happens when θ_0 is close to $\pi/2$, and in this case the fluctuations become important; the set of $|J,J\rangle$ states is not sufficient to expand the system state vector.

Let us focus on initial conditions with $\theta_0 \leq \pi/4$ so that Eq. (15) is well approximated by keeping $M=J$ terms only. In this way $|\psi_k\rangle$ can be thought of as a rotation generated by the J_z operator:

$$|\psi_k\rangle \approx e^{-iJ_z\chi_k}\sum_{J=0}^{2s} C_{JJ}|J,J\rangle \approx e^{-iJ_z\chi_k}|\Psi(0)\rangle \tag{16}$$

where $\chi_k=2\pi(k+\delta)/n$ is the rotation angle dependent on the index k. Since $J_z=J_{A_z}+J_{B_z}$, the state vector (14) at fractional revival times becomes

$$\left|\Psi\left(\frac{2\pi l}{n}\right)\right\rangle \approx \frac{e^{i2\pi s(s+1)l/n}}{\sqrt{n}}\sum_{k=1}^{n} e^{i\gamma_{kn}}|\theta_0,\chi_k\rangle_A|\theta_0,\pi+\chi_k\rangle_B. \tag{17}$$

Equation (17) is a kind of entangled spin-coherent state in which the initial packet pair is now split into n identical and equally spaced pairs. If these wave packets are well separated such that $\langle\theta,\chi_k|\theta,\chi_{k'}\rangle \approx \delta_{kk'}$, then Eq. (17) is an approximate form of Schmidt decomposition in which the Schmidt modes are spin-coherent states, and all the n Schmidt eigenvalues are of the same absolute value. Consequently, the purity is approximately $1/n$. Note that the $n=1$ case corresponds to full revivals, i.e., complete disentanglement $P=1$. Hence the fractional revivals of P may be considered as "fractional disentanglement."

We remark that the survival probability $|\langle\Psi(0)|\Psi(\tau)\rangle|^2$ also displays peaks with fractional values, according to the state obtained in Eq. (17). However, one needs to distinguish that the survival probability itself is not a measure of quantum entanglement. In our system, the time dependence of the survival probability is highly oscillatory due to the fast precession of wave packets. As such fast oscillations are not related to entanglement, they do not appear in the purity function.

It is useful to estimate the maximum number of wave packets that are well separated on the precession orbit. The inverse of this number also provides an estimation of the minimum purity. Since spin-coherent states are minimum wave packets, they have the widths about $\sqrt{s}$. If we let w be the minimum number of widths required for good separation, then $(2\pi s\sin\theta_0)/w\sqrt{s}$ is the maximum n, after taking the precession radius $s\sin\theta_0$ into account. It is quite sufficient to choose $w\approx 3$, and the minimum purity is estimated to be $P_{min}\approx 3/(2\pi\sqrt{s}\sin\theta_0)$. Hence, the larger the value of s, the stronger the entanglement. For the parameters used in Fig. 1, the estimation gives $P_{min}\approx 0.068$, which is close to the actual

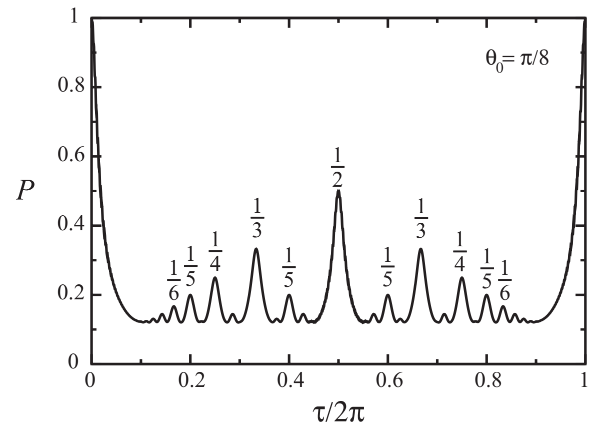

FIG. 2. Time evolution of the purity P of subsystem A (or B) with $s=100$ and $\theta_0=\pi/8$. The approximate values of purity at fractional revival times are shown as in Fig. 1.

minimum purity 0.06 observed in Fig. 1. We have also performed calculations for θ_0 smaller than $\pi/4$ (Fig. 2). This corresponds to a smaller precession radius, and hence fewer well-separated packets can be accommodated. This explains why there are fewer revival peaks of low purity values in Fig. 2.

It is worth noting that, although the systems considered here have the same spin s, a similar analysis can be applied to systems with different spins. This is because our results are based on Eq. (13), which is a property of total spin J, independent of individual spin values. A slight complication is that the z axis should be chosen as the precession axis, such that the classical $\mathbf{J}$ and $\mathbf{J}_z$ are almost the same as before. As long as individual spins are sufficiently large and their initial angles support a large J, then Eq. (12) is expected to hold for some finite numbers of n. This is confirmed also in our numerical calculations (not shown).

To summarize, we investigated the evolution of entanglement driven by angular momentum coupling (1) with near-classical initial conditions. In the large-s limit, the emergence of $1/n$ purity peaks at fractional revival times corresponds to the generation of n pairs of coherent states. Our results could have applications in bipartite systems described by angular momentum coupling (1). One example is atomic wave mixing in a spin-1 Bose condensate proposed by Goldstein and Meystre [5]. In such a process, the effective Hamiltonian can be reduced to (1) and the value of s is of the order of the number of atoms in the condensate. Another example is based on photon-atom interaction. By making use of Schwinger's oscillator representation of angular momentum, one can construct the interaction (1) by coupling atoms with two quantized field modes. Specifically, the interaction (1) describes the collective Raman coupling inside a cavity [11]. In such systems, the state (17) corresponds to a kind of photon-atom entanglement in which the Schmidt modes are coherent states of atoms and photons.

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[10] At smaller angles θ_0, $\Sigma_{J=0}^{2s}|C_{JJ}|^2$ can be much closer to 1. For example, $\Sigma_{J=0}^{2s}|C_{JJ}|^2 > 0.999$ when $\theta_0 = \pi/8$ in the range $2 \leqslant s \leqslant 100$. We also note that $\Sigma_{J=0}^{2s}|C_{JJ}|^2$ begins to decrease appreciably for angles $5\pi/8$. For example, at $\theta_0 = 5\pi/8$, $\Sigma_{J=0}^{2s}|C_{JJ}|^2 \approx 0.85$.
[11] We may consider $J_{B+} = a_s^\dagger a_p$ and $J_{B-} = a_p^\dagger a_s$, where a_p and a_s are annihilation operators of the pump and Stokes quantized field modes. Then Eq. (1) is an effective Hamiltonian of a collection of Λ-type three-level atoms interacting with the two fields (the upper level adiabatically eliminated), with J_A being the collective atomic operators and ac Stark shifts included.