Wigner-Araki-Yanase theorem on distinguishability

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The presence of an additive-conserved quantity imposes a limitation on the measurement process. According to the Wigner-Araki-Yanase theorem, perfect repeatability and distinguishability of the apparatus cannot be attained simultaneously. Instead of repeatability, in this paper, the distinguishability in both systems is examined. We derive a trade-off inequality between the distinguishability of the final states on the system and the one on the apparatus. An inequality shows that perfect distinguishability of both systems cannot be attained simultaneously.

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According to the Wigner-Araki-Yanase theorem, the presence of an additive-conserved quantity imposes a limitation on the measurement process. Wigner, and later Araki and Yanase, showed [1–3] that in the sense of von Neumann's ideal measurement one cannot precisely measure observables which do not commute with the conserved quantity. That is, repeatability of the measurements and perfect distinguishability of the final states of the measuring apparatus cannot be realized simultaneously. On the other hand, if we abandon the repeatability condition, perfect distinguishability of the final states on the apparatus can be attained [4,5]. Ozawa [6,7] has derived a quantitative relation between the noise operator and the disturbance operator by a Robertson-type inequality to discuss the trade-off between repeatability and distinguishability of the apparatus. We, in this paper, relax the condition. We do not impose repeatability on the measurement process; instead, we treat the distinguishability of the final states also of the system. We ask for the quantitative trade-off between the distinguishability of the final states of the measured system and the one of the measuring apparatus. According to our result, there is no interaction that achieves perfect distinguishability in both systems. Since our result is quantitative, it enables us to discuss the dependence on the size of the apparatus and the environment.

Let us consider two quantum systems: a *system* and an *apparatus*. Each system is described by a Hilbert space $\mathcal{H}_S$ and $\mathcal{H}_A$, respectively. Suppose that there exists an additive conservative quantity. That is, there exists an observable L_S in the system and an observable L_A in the apparatus such that their summation L_S+L_A is conserved by any physical dynamics for the closed system. Let us consider a pair of orthogonal vector states, $|\psi_0\rangle,|\psi_1\rangle \in \mathcal{H}_S$. The goal of the measurement process is to make them distinguishable in the apparatus by choosing an initial state of the apparatus and the interaction between the system and the apparatus. In the case of the ideal measurement, the repeatability of the measurements is also imposed. That is, the states $|\psi_0\rangle$ and $|\psi_1\rangle$ should be invariant with the interaction. We, in this paper, do not employ this repeatability condition. We relax the condition to the distinguishability condition in the system. That is, we ask if it is possible for the final states to be distinguishable in both systems. The distinguishability is characterized by a quantity called *fidelity*. The fidelity [8,9] between two states ρ_0 and ρ_1 is defined by $F(\rho_0,\rho_1) \coloneqq \mathrm{tr}(\sqrt{\rho_0^{1/2}\rho_1\rho_0^{1/2}})$. It takes a non-negative value less than 1 and becomes smaller if the states are more distinguishable. The perfect distinguishability corresponds to the vanishing fidelity. The following lemma [10] justifies that the fidelity indeed represents the distinguishability.

Lemma 1. The fidelity equals the square root of minimum overlap coefficient between two probability distributions p_0 and p_1:

$$F(\rho_0,\rho_1) = \min_{\{E_\alpha\}:POVM} \sum_\alpha \sqrt{p_0(\alpha)p_1(\alpha)},$$

where p_0 and p_1 are defined by $p_0(\alpha)=tr(\rho_0 E_\alpha)$ and $p_1(\alpha)=tr(\rho_1 E_\alpha)$. The minimum is taken over all the possible positive-operator-valued measures (POVM's), where a POVM $\{E_\alpha\}$ is a family of the positive operators satisfying $\Sigma_\alpha E_\alpha = \mathbf{1}$. Moreover, the minimum is attained by a projection-valued measure (PVM), where a PVM $\{E_\alpha\}$ is a family of the projection operators satisfying $\Sigma_\alpha E_\alpha=\mathbf{1}$.

This lemma plays an essential role in the proof of our theorem. In the presence of the additive conserved quantity, we have the following theorem.

Theorem 2. As described above, let us consider a pair of orthogonal states, $|\psi_0\rangle,|\psi_1\rangle \in \mathcal{H}_S$, in the presence of the additive conserved quantity, L_S+L_A. For any initial state σ on the apparatus and the unitary dynamics U satisfying conservation law, the final states $\rho_0 \coloneqq U(|\psi_0\rangle\langle\psi_0| \otimes \sigma)U^*$ and $\rho_1 \coloneqq U(|\psi_1\rangle\langle\psi_1| \otimes \sigma)U^*$ satisfy the following:

$$|\langle\psi_0|L_S|\psi_1 \leq \|L_A\| F(\rho_0^S,\rho_1^S) + \|L_S\| F(\rho_0^A,\rho_1^A), \qquad (1)$$

where ρ_i^S is the final state ρ_i restricted to the system and ρ_i^A is the one restricted to the apparatus, $F(\cdot,\cdot)$ is the fidelity, and $\|\cdot\|$ represents the operator norm defined as $\|A\| \coloneqq \sup_{\varphi\neq 0,\varphi\in\mathcal{H}} \frac{\|A\varphi\|}{\|\varphi\|}$ for any operator A on a Hilbert space $\mathcal{H}$.

Proof. By the purification of σ, we obtain a dilated Hilbert space and a vector state for the apparatus. We write the

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dilated Hilbert space as $\mathcal{H}_A$ for simplicity and the vector state as $|\Omega\rangle$. The dilated unitary operator $U\otimes\mathbf{1}$ is also abbreviated as U. Let us define the initial vector states $|\Psi_i\rangle := |\psi_i\rangle\otimes|\Omega\rangle$ for $i=0,1$. As Wigner, Araki, and Yanase's original discussion, we consider the following quantity:

$$\langle\psi_0|L_S|\psi_1\rangle = \langle\Psi_0|L_S+L_A|\Psi_1\rangle = \langle\Psi_0|U^*(L_S+L_A)U|\Psi_1\rangle = \langle\Psi_0|U^*L_SU|\Psi_1\rangle + \langle\Psi_0|U^*L_AU|\Psi_1\rangle, \quad (2)$$

where in the first line we have used $\langle\Psi_0|L_A|\Psi_1\rangle=\langle\psi_0|\psi_1\rangle\times\langle\Omega|L_A|\Omega\rangle=0$. Now we consider an arbitrary (PVM) $\{E_\alpha\}$ on the system and an arbitrary PVM $\{P_j\}$ on the apparatus. Since $\Sigma_\alpha E_\alpha=\Sigma_j P_j=\mathbf{1}$ holds, the right-hand side of Eq. (2) can be written as $\Sigma_j\langle\Psi_0|U^*P_jL_SU|\Psi_1\rangle+\Sigma_\alpha\langle\Psi_0|U^*E_\alpha L_AU|\Psi_1\rangle$. By using commutativity $[P_j,L_S]=[E_\alpha,L_A]=0$, we obtain

$$\langle\psi_0|L_S|\psi_1\rangle = \sum_j\langle\Psi_0|U^*P_jL_SP_jU|\Psi_1\rangle + \sum_\alpha\langle\Psi_0|U^*E_\alpha L_AE_\alpha U|\Psi_1\rangle.$$

Taking absolute value of the both sides, we obtain

$$|\langle\psi_0|L_S|\psi_1\rangle| \le \sum_j|\langle\Psi_0|U^*P_jL_SP_jU|\Psi_1\rangle| + \sum_\alpha|\langle\Psi_0|U^*E_\alpha L_AE_\alpha U|\Psi_1\rangle| \le \|L_S\|\sum_j\sqrt{\langle\Psi_0|U^*P_jU|\Psi_0\rangle\langle\Psi_1|U^*P_jU|\Psi_1\rangle} + \|L_A\|\sum_\alpha\sqrt{\langle\Psi_0|U^*E_\alpha U|\Psi_0\rangle\langle\Psi_1|U^*E_\alpha U|\Psi_1\rangle}.$$

We here choose the particular PVM's $\{E_\alpha\}$ and $\{P_j\}$, which attain the fidelity. Thanks to lemma 1, we finally obtain

$$|\langle\psi_0|L_S|\psi_1\rangle| \le \|L_A\|F(\rho_0^S,\rho_1^S) + \|L_S\|F(\rho_0^A,\rho_1^A).$$

It ends the proof. □

According to this theorem, we obtain the following theorem.

Theorem 3. Under the setting of theorem 2, the perfect distinguishability for both systems cannot be attained simultaneously.

Proof. The vanishing fidelities in (1) contradict with the nonvanishing left-hand side. □

Let us consider the simplest example. The system is a spin-1/2 system. The conserved quantity is the z component of the spin, S_z+L_A, where L_A is the z component of the spin in the apparatus. S_z is written with the eigenvectors $|1\rangle$ and $|-1\rangle$ as $S_z=\frac{\hbar}{2}(|1\rangle\langle 1|-|-1\rangle\langle -1|)$. The observable to be measured S_k is a component of spin in another direction. That is, the states to be distinguished by the measurement process are $|\psi_1\rangle:=\alpha|1\rangle+\beta|-1\rangle$ and $|\psi_0\rangle:=\bar{\beta}|1\rangle-\bar{\alpha}|-1\rangle$, where $|\alpha|^2+|\beta|^2=1$ with $\alpha\neq 0$, $\beta\neq 0$. The observables S_z and S_k do not commute with each other. In fact,

$$\langle\psi_0|S_z|\psi_1\rangle = \hbar\,\alpha\beta$$

holds. If we assume rigorous repeatability as in the original Wigner-Araki-Yanase theorem, the state change for the dilated Hilbert space should be written as

$$|\psi_j\rangle\otimes|\Omega\rangle \mapsto |\psi_j\rangle\otimes|\phi_j\rangle$$

for $j=0,1$. It gives

$$\langle\psi_0|S_z|\psi_1\rangle = \langle\psi_0|S_z|\psi_1\rangle\langle\phi_0|\phi_1\rangle,$$

and thus $|\phi_0\rangle=|\phi_1\rangle$ holds. Therefore there is no distinguishability on the apparatus side in this case. On the other hand, if we do not impose repeatability, distinguishability in both systems is partially attained. In particular, even perfect distinguishability in the apparatus allows partial distinguishability in the system. Ohira and Pearle [5] have constructed the following interaction between the system and the spin-1/2 apparatus:

$$|\psi_1\rangle\otimes\sqrt{\frac{1}{2}}(|1\rangle+|-1\rangle) \mapsto (\alpha|1\rangle+\beta|-1\rangle)\otimes\sqrt{\frac{1}{2}}(|1\rangle+|-1\rangle),$$

$$|\psi_0\rangle\otimes\sqrt{\frac{1}{2}}(|1\rangle+|-1\rangle) \mapsto (\bar{\beta}|1\rangle+\bar{\alpha}|-1\rangle)\otimes\sqrt{\frac{1}{2}}(|1\rangle-|-1\rangle).$$

It gives the fidelity $F(\rho_0^A,\rho_1^A)=0$ and $F(\rho_0^S,\rho_1^S)=2|\alpha\beta|$. Since $\|L_A\|=\hbar/2$ holds, this interaction satisfies

$$|\langle\psi_0|S_z|\psi_1\rangle| = \|L_A\|F(\rho_0^S,\rho_1^S),$$

which is the equality version of our theorem.

In the following we consider the effect of the environment. We treat a tripartite system which consists of the system, the apparatus, and the environment. The Hilbert space of the environment is written as $\mathcal{H}_E$. In the environment an operator L_E is defined and the conserved quantity is $L_S+L_A+L_E$. We divide the whole system into $\mathcal{H}_S$ and $\mathcal{H}_A\otimes\mathcal{H}_E$. Application of theorem 2 to it derives

$$|\langle\psi_0|L_S|\psi_1\rangle| \le (\|L_A\|+\|L_E\|)F(\rho_0^S,\rho_1^S) + \|L_S\|F(\rho_0^{AE},\rho_1^{AE}),$$

where ρ_j^{AE} is a state over the apparatus and the environment. Since the partial trace does not reduce the fidelity [10], we obtain

$$|\langle\psi_0|L_S|\psi_1\rangle| \le (\|L_A\|+\|L_E\|)F(\rho_0^S,\rho_1^S) + \|L_S\|F(\rho_0^A,\rho_1^A).$$

This inequality shows that to attain high distinguishability in both systems simultaneously a large apparatus or environment is necessary.

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