Time-modulated type-II optical parametric oscillator: Quantum dynamics and strong Einstein-Podolsky-Rosen entanglement

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We investigate semiclassical dynamics and quantum properties of light beams generated in time-modulated nondegenerate optical parametric oscillator (NOPO). Having in view production of continuous-variable (CV) entangled states of light beams we propose two experimentally feasible schemes of NOPO: (i) driven by continuously modulated pump field; (ii) under action of a periodic sequence of identical laser pulses. It is shown that the time modulation of pump field amplitude essentially improves the degree of CV entanglement in NOPO. On the whole the level of integral two-mode squeezing, which characterizes the degree of CV entanglement, goes below the standard limit established in an ordinary NOPO with monochromatic pumping. We develop semiclassical and quantum theories of these devices for both below- and above-threshold regimes of generation. Properties of CV entanglement for various operational regimes of the devices are discussed in the time domain in application to time-resolved quantum information technologies. Our analytical results are in well agreement with the results of numerical simulation and support a concept of CV entangled states of time-modulated light beams.

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I. INTRODUCTION

Continuous-variable (CV) entangled states of light beams provide excellent tools for testing the foundations of quantum physics and arouse growing interest due to apparent usefulness as a promising technology in quantum information and communication protocols [1,2]. The efficiency of quantum information schemes significantly depends on the degree of entanglement. On the other hand, in the majority of real applications bright light beams are required. It is therefore highly desirable to elaborate reliable sources of light beams having the mentioned properties. The recent development of CV quantum information is stipulated mainly by preparation of EPR (Einstein-Podolsky-Rosen) entangled states, which particularly can be generated by a nondegenerate parametric amplifier [3,4]. A type-II optical parametric oscillator (OPO) pumped above threshold has also been theoretically predicted to be a very efficient source of bright entangled beams. EPR entanglement in NOPO above threshold was proposed in Ref. [3] and its strong consideration has recently been given in Ref. [5]. This means that, in addition to the already demonstrated intensity quantum correlations above threshold [6], phase anticorrelations exist in the system. Nevertheless, to our knowledge, up to now no direct evidence of such phase anticorrelations has been observed, and the above-threshold generation of bright light beams with high degree of CV entanglement meets serious problems [7].

In this direction, the ultrastable, phase-locked type-II NOPOs operated above threshold have been recently described and investigated experimentally in the area of quantum optics [8–14]. The simplest scheme realized in the experiments is the NOPO with additional intracavity quarter-wave plate to provide polarization mixing between two orthogonally polarized modes of the subharmonics [8–10]. Recently, a full quantum mechanical treatment of this system in application to generation of CV entangled states of light beams under mode phase-locked condition has been presented [11,12] where the regimes below, near, and above threshold were considered. Quantum optical effects have been also demonstrated in the series of experiments [13]. The intensity-difference squeezing in electronically phase-locked NOPO above threshold as well as the Hong-Ou-Mandel interferometry using twin beams have also been experimentally demonstrated [14].

In the CV regime, a wide variety of quantum communication applications has been demonstrated in different physical systems—either in the pulsed or cw regime. In this field, the usual way to measure CV entanglement is by homodyne detection in the spectral domain. Nevertheless, it seems, that analysis of quantum communication protocols should be very easy in terms of information transfers which can be naturally performed for communication schemes operating mainly in time-modulated or pulsed regimes. In these regimes will be possible to manipulate individually each quantum state involved in the information exchange. This statement has emerged recently and efficient setups have been proposed for generation and characterization of quadrature-squeezed pulses [15] as well as quadrature-entangled pulses [16] in time-domain in addition to many other experiments performed in the frequency domain [17]. For this goal the method of time-resolved homodyne measurement has been developed following the pioneering experiment on quantum tomography and quantum correlations [18] (see, also the papers [19] in this area). Pulsed homodyne detection differs from the usually used method of spectral analysis; in the pulsed method a single measurement on the quadrature amplitude of the signal pulse is performed for each pulse. This approach opens a possibility for homodyne measurement of quadrature variance in the time domain that is important for

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elaboration of time-resolved quantum information protocols. Such protocols are now at the first stage of development (see, for example, Ref. [20]). In spite of these developments, an important issue for time-resolved communication protocols is to investigate CV entanglement for various time-modulated regimes of generation including pulsed regimes.

As a realization of this program, in this paper we propose and investigate entangled states of time-modulated light field generated in two schemes of type-II OPO: (i) driven by continuously modulated pump field; (ii) under action of a periodic sequence of identical laser pulses. There are various questions that emerge in the study of these problems. What is the degree of CV entanglement in nonstationary time-dependent regimes, particularly, in the different regimes of amplitude modulation or pulsed regimes? Will CV entanglement take place in the time-dependent regime of lasing or how far can it be extended into the high intensity domain? We stress that the schemes proposed here are experimentally feasible, operate in both operational regimes (below and above threshold) and, what is very remarkable, provide highly effective mechanism for improvement of the degree of CV entanglement, even in the presence of dissipation and cavity induced feedback.

For systems with CV the number of available criteria for analyzing entanglement is very limited even for two-mode states [3,21–24]. For non-Gaussian states, such as light field states generated in above-threshold NOPO, these criteria often provide only sufficient conditions for inseparability. In particular, a well-established approach is to consider CV entangling resources as two-mode squeezing through the variances of the quadrature amplitudes. In this case, quadrature entanglement wave-function inseparability criterion is often formulated as $V<1$ for the half-sum of the squeezed variances [21,22] (see below). As is shown theoretically, in NOPO under a continuous, monochromatic pump, the maximal level of integral intracavity two-mode squeezing, which is only 50% relative to the level of vacuum fluctuations, is realized if the pump field intensity is close to the generation threshold [5,25,26]. However, most of the experiments relying on entangled quantum variables have been performed for fields outside a cavity in the spectral domain. Moreover, the spectral squeezing significantly lower than the integral intracavity squeezing has been achieved at definite low-frequency spectral ranges [7,13]. Thus, it is an established standard to describe squeezing with the spectra of quantum fluctuations, as has been done even for some pulsed squeezing experiments [27]. Unlike that, we follow the ideology of the cited papers [15,16,19,20] and analyze the time-dependent characteristics of CV entanglement for time modulated light beams.

As we show below, application of pump laser fields with periodically varying amplitudes allows to qualitatively improve the situation, i.e., to go beyond the limit 50%. It indicates a high degree of quadrature entanglement obeying the condition of strong integral entanglement, $V<0.5$. A similar conclusion holds for the output measured integral quadrature variance V^{out} of output fields that are external to the cavity.

It seems intuitively clear that such achievement is due to the control of quantum dissipative dynamics as well as diffusion processes and cavity induced feedback through the application of suitable tailored, time-dependent driving field.

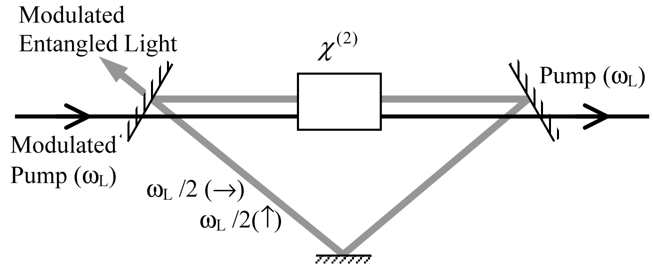

FIG. 1. The principal scheme of NOPO in a cavity that supports the pump mode at frequency ω_L and subharmonic modes of orthogonal polarizations at frequency $\omega_L/2$.

Indeed, some interesting examples of suppression of quantum decoherence by modulation of system parameters have been considered in Ref. [28]. Improvement of both sub-Poissonian statistics of an anharmonic oscillator and quadrature squeezing in a resonance fluorescence by application of amplitude-modulated pump fields have been demonstrated in Refs. [29,30]. So, our analysis also offers insights into crossovers between two phenomena, generation of CV entangled light beams, and the control of quantum dissipative dynamics.

We develop quantum theories of these devices for below- and above-threshold regimes concluding that such achievement takes place for both operational regimes of NOPO. The paper is organized as follows. In Sec. II we formulate the model of time-modulated NOPO, and we present a semiclassical analysis of the system as well as analysis of quantum stochastic equations of motion. In Sec. III we calculate time-dependent quadrature variances of two-mode squeezing on the basis of a perturbation theory. In Sec. IV we investigate the CV entangling resources of NOPO driven by harmonically modulated pump field. In Sec. V we consider NOPO under action of a periodic sequence of laser pulses. We summarize our results in Sec. VI.

II. TIME-MODULATED NOPO IN THE STOCHASTIC VARIABLES

We consider a type-II phase-matched NOPO with triply resonant optical ring cavity under action of pump field with periodically varying amplitude (see Fig. 1) [12]. Below we provide two concrete examples (i) and (ii) mentioned above. The interaction Hamiltonian describing both cases within the framework of rotating wave approximation and in the interaction picture is

$$H = i\hbar f(t)(e^{i(\Phi_L-\omega_L t)}a_3^\dagger - e^{-i(\Phi_L-\omega_L t)}a_3) + i\hbar k(e^{i\Phi_k}a_3 a_1^\dagger a_2^\dagger - e^{-i\Phi_k}a_3^\dagger a_1 a_2), \quad (1)$$

where a_i are the boson operators for cavity modes at the frequencies ω_i. The pump mode a_3 is driven by an amplitude-modulated external field at the frequency $\omega_L=\omega_3$ with time-periodic, real valued amplitude $f(t+T)=f(t)$. The constant $ke^{i\Phi_k}$ determines an efficiency of the down-conversion process $\omega_L \to \frac{\omega_L}{2}(\uparrow)+\frac{\omega_L}{2}(\to)$ in $\chi^{(2)}$ medium. We take into account the cavity damping rates γ_i of the modes and consider the case of high cavity losses for the pump mode ($\gamma_3 \gg \gamma$, $\gamma_1=\gamma_2=\gamma$) when the pump mode is eliminated adiabatically (see Fig. 1). However, in our analysis we allow for the pump depletion effects.

Following the standard procedure we derive in the positive P repesentation the stochastic equations for complex c-number variables $\alpha_{1,2}$ and $\beta_{1,2}$ corresponding to operators $a_{1,2}$ and $a^{\dagger}_{1,2}$ for the case of zero detunings,

$$\frac{d\alpha_1}{dt} = -(\gamma + \lambda\alpha_2\beta_2)\alpha_1 + \varepsilon(t)\beta_2 + W_{\alpha_1}(t), \tag{2}$$

$$\frac{d\beta_1}{dt} = -(\gamma + \lambda\alpha_2\beta_2)\beta_1 + \varepsilon(t)\alpha_2 + W_{\beta_1}(t). \tag{3}$$

Here, $\varepsilon(t)=f(t)k/\gamma_3$, $\lambda=k^2/\gamma_3$, equations for α_2,β_2 are obtained from (2) and (3) by exchanging the subscripts $(1)\leftrightarrows(2)$ and behavior of the pump mode is described as

$$\alpha_3(t) = [f(t) - k\alpha_1\alpha_2]/\gamma_3, \tag{4}$$

$$\beta_3(t) = [f(t) - k\beta_1\beta_2]/\gamma_3. \tag{5}$$

Our derivation is based on the Ito stochastic calculus, and the nonzero stochastic correlations are

$$\langle W_{\alpha_1}(t)W_{\alpha_2}(t')\rangle = [\varepsilon(t) - \lambda\alpha_1\alpha_2]\delta(t-t'), \tag{6}$$

$$\langle W_{\beta_1}(t)W_{\beta_2}(t')\rangle = [\varepsilon(t) - \lambda\beta_1\beta_2]\delta(t-t'). \tag{7}$$

Note, that while obtaining these equations we used the transformed boson operators $a_i \to a_i\exp(-i\Phi_i)$ with Φ_i being $\Phi_3=\Phi_L$, $\Phi_1=\Phi_2=\frac{1}{2}(\Phi_L+\Phi_k)$. This leads to cancellation of phases at intermediate stages of calculation. The equations of motion (2) and (3) are with time-dependent coefficients and for the case of constant amplitude ε coincide with well-known equations of motion for an ordinary NOPO.

A. Semiclassical dynamics

First, we shall study in general the solution of stochastic equations in semiclassical treatment, neglecting the noise terms and assuming $\beta_i=\alpha_i^*$, for mean photon numbers n_j and phases φ_j of the modes $[n_j=\alpha_j\beta_j,\ \varphi_j=\frac{1}{2i}\ln(\alpha_j/\beta_j)]$. An analysis shows that similar to the standard NOPO, the considered system also exhibits threshold behavior, which is easily described through the period-averaged pump field amplitude $\overline{f(t)}=\frac{1}{T}\int_0^T f(t)dt$. The below-threshold regime with a stable trivial zero-amplitude solution is realized for $\bar{f}<f_{\mathrm{th}}$, where $f_{\mathrm{th}}=\gamma\gamma_3/k$ is the threshold value. When $\bar{f}>f_{\mathrm{th}}$, the stable nontrivial solution exists with the following properties. First, as for a usual NOPO, the phase difference is undefined due to the phase diffusion, while the sum of phases is equal to $\varphi_1+\varphi_2=2\pi m$. The mean photon numbers for subharmonic modes $n_i=\langle a_i^{\dagger}a_i\rangle=|\alpha_i|^2$ are equal one to the other $(n_1=n_2=n)$ due to the symmetry of the system, $\gamma_1=\gamma_2=\gamma$.

Let us explain these statements in details. We present now steady state solution of Eqs. (2) and (3) in the semiclassical approximation and carry out the standard linear stability analysis.

The trivial zero-amplitude solution $\alpha_1=\alpha_2=0$ is stable in the region $\bar{f}<f_{\mathrm{th}}$ and describes the below-threshold regime of both subharmonics. In this regime the solution for the pump mode reads as $\alpha_3(t)=f(t)/\gamma_3$. To check the stability we turn to the linearized on the small deviations $\delta\alpha_i$, $\delta\alpha_i^*$ equations which we rewrite in the following form:

$$\frac{d}{dt}\delta X_{\pm} = [-\gamma \pm \varepsilon(t)]\delta X_{\pm}, \tag{8}$$

$$\frac{d}{dt}\delta Y_{\pm} = [-\gamma \pm \varepsilon(t)]\delta Y_{\pm}, \tag{9}$$

where the quadrature field variables are defined as $\delta\alpha_{\pm}=\frac{1}{\sqrt{2}}(\delta\alpha_1\pm\delta\alpha_2)$ and $\delta\alpha_{\pm}=\delta X_{\pm}+i\delta Y_{\pm}$. In these variables the time evolution has the simple form

$$\delta X_{\pm}(t) = \exp\left(\pm\int_{t_0}^{t}\varepsilon(t')dt' - \gamma(t-t_0)\right)\delta X_{\pm}(t_0), \tag{10}$$

$$\delta Y_{\pm}(t) = \exp\left(\mp\int_{t_0}^{t}\varepsilon(t')dt' - \gamma(t-t_0)\right)\delta Y_{\pm}(t_0). \tag{11}$$

Since the function $\varepsilon(t)$ is periodic on time, we see from Eqs. (10) and (11) that the solution $\alpha_i=0$ is stable if the following relation holds $\bar{\varepsilon}<\gamma$ and thus $\bar{f}<f_{\mathrm{th}}$. Note, that in obtaining this result it is useful to use the following formula:

$$\int_{t_1}^{t_2}\varepsilon(t)dt = (t_2-t_1)\bar{\varepsilon} + \epsilon(t_2) - \epsilon(t_1), \tag{12}$$

where $\bar{\varepsilon}=\frac{1}{T}\int_{t_0}^{T}\varepsilon(t)dt$ is the period-averaged amplitude and $\epsilon(t)$ is a periodic function, $\epsilon(t+T)=\epsilon(t)$.

Considering the above-threshold regime we transform the semiclassical counterpart of Eqs. (2) and (3) to the photon number and phase variables of the modes $\alpha_i=\sqrt{n_i}\exp(i\varphi_i)$. This yields:

$$\frac{d}{dt}n_{1c} = 2\varepsilon(t)(n_{1c}n_{2c})^{1/2}\cos(\varphi_1+\varphi_2) - 2\lambda n_{1c}n_{2c} - 2\gamma n_{1c}, \tag{13}$$

$$\frac{d}{dt}n_{2c} = 2\varepsilon(t)(n_{1c}n_{2c})^{1/2}\cos(\varphi_1+\varphi_2) - 2\lambda n_{1c}n_{2c} - 2\gamma n_{2c}, \tag{14}$$

$$\frac{d}{dt}\varphi_1 = -(n_{2c}/n_{1c})^{1/2}\varepsilon(t)\sin(\varphi_1+\varphi_2), \tag{15}$$

$$\frac{d}{dt}\varphi_2 = -(n_{2c}/n_{1c})^{1/2}\varepsilon(t)\sin(\varphi_1+\varphi_2). \tag{16}$$

Combining (13) with (14), we see that for an over transient regime, $t\gg\gamma^{-1}$, the mean photon numbers of subharmonics are equal one to other, $n_{1c}(t)=n_{2c}(t)=n_c(t)$, which is a consequence of the symmetry, $\gamma_1=\gamma_2=\gamma$. After simplification the equations read as

$$\frac{d}{dt}n_c(t) = 2n_c(t)\{\varepsilon(t)\cos[\varphi_1(t)+\varphi_2(t)]-\gamma\} - 2\lambda n_c^2(t), \tag{17}$$

$$\frac{d}{dt}\varphi_i = -\varepsilon(t)\sin[\varphi_1(t)+\varphi_2(t)], (i=1,2). \tag{18}$$

It can be immediately seen on Eq. (18) that

$$\cos[\varphi_1(t)+\varphi_2(t)] = \tanh(2\int_{t_0}^{t}\varepsilon(t')dt' + c_0), \tag{19}$$

where $c_0=\tanh^{-1}\{\cos[\varphi_1(t_0)+\varphi_2(t_0)]\}$ is an integration constant. Then, in the limit $t-t_0 \gg \gamma^{-1}$, we get $\varphi_1+\varphi_2=2\pi m$ $(m=0,\pm1,\pm2,\ldots)$. Thus, the Eq. (17) is reduced to the following form for $Z=1/n_c$:

$$\frac{d}{dt}Z = -2[\varepsilon(t)-\gamma]Z + 2\lambda. \tag{20}$$

The photon number is calculated in the following form:

$$n_c^{-1}(t) = n_h^{-1}(t) + 2\lambda\int_{t_0}^{t}\exp(-2\int_{\tau}^{t}[\varepsilon(t')-\gamma]dt')d\tau, \tag{21}$$

where $n_h^{-1}(t)=n_h^{-1}(t_0)\exp\{-2\int_{t_0}^{t}[\varepsilon(t')-\gamma]dt'\}$ is the solution of the homogeneous Eqs. (17). In the asymptotic $t-t_0\to\infty$ and for $\bar{\varepsilon}>\gamma$ we have $n_h^{-1}\to 0$ and hence we get the following result for over transient regime:

$$n_c^{-1}(t) = 2\lambda\int_{-\infty}^{0}\exp(2\int_{0}^{\tau}[\varepsilon(t'+t)-\gamma]dt')d\tau. \tag{22}$$

We conclude, that the mean photon number is a periodic function of time. Indeed, using the formula (12) we get the following result

$$n_c^{-1}(t) = 2\lambda\int_{-\infty}^{0}\exp[2(\bar{\varepsilon}-\gamma)\tau - 2\epsilon(t) + 2\epsilon(t+\tau)]d\tau \tag{23}$$

which is obviously the periodic function due to periodicity of the function $\epsilon(t)$. Using $n_c(t+T)=n_c(t)$ we can transform Eq. (23) to the analogous form which is more convenient in practical calculations

$$n_c^{-1}(t) = \frac{2\lambda}{\exp[2T(\bar{\varepsilon}-\gamma)]-1}\int_0^T \exp[2(\bar{\varepsilon}-\gamma)\tau']\exp[-2\epsilon(t) + 2\epsilon(t+\tau')]d\tau. \tag{24}$$

Particularly, in the near-threshold regime, if $\bar{\varepsilon}-\varepsilon_{th}\ll\delta$, we obtain the approximative result

$$n_c(t) \approx \bar{n}_c\frac{e^{2\epsilon(t)}}{\langle e^{2\epsilon(t)}\rangle}, \tag{25}$$

where $\overline{e^{2\epsilon(t)}}=\frac{1}{T}\int_0^T e^{2\epsilon(\tau)}d\tau$ and $\bar{n}_c=(\bar{\varepsilon}-\gamma)/\lambda$ is period-averaged photon number (see below).

We see that the time-dependent photon number is zero at the threshold, because $\bar{n}_c=0$ at $\bar{\varepsilon}=\gamma$. Besides this, the amplitude of oscillations of the photon number increases with $\bar{\varepsilon}$ linearly. It is not difficult to derive also the boundary conditions for mean photon number. From Eq. (24) we get

$$\frac{\bar{\varepsilon}-\gamma}{\lambda}\exp(-2\Delta\epsilon) \leqslant n_c(t) \leqslant \frac{\bar{\varepsilon}-\gamma}{\lambda}\exp(+2\Delta\epsilon), \tag{26}$$

where $\Delta\epsilon=\epsilon_{max}-\epsilon_{min}$, or

$$\overline{n_c}\exp(-2\Delta\epsilon) \leqslant n_c(t) \leqslant \overline{n_c}\exp(+2\Delta\epsilon). \tag{27}$$

For the case of a monochromatic wave pump field, $f(t)=f=\text{const}$, and hence $\varepsilon(t)=\varepsilon=fk/\gamma_3$, we obtain the well-known result for the photon number of an ordinary NOPO in steady state

$$n_{st} = \frac{\varepsilon-\gamma}{\lambda} = \frac{f-f_{th}}{k}. \tag{28}$$

It is interesting to consider period-averaged mean photon number $\bar{n}_c=\frac{1}{T}\int_0^T n_c(t)dt$. As we see from Eq. (22), $\bar{n}$ depends from period-averaged amplitude $\bar{f}$ and coincides on the form with the analogous result (28) for nonmodulated NOPO,

$$\bar{n}_c = (\bar{f}-f_{th})/k. \tag{29}$$

Thus, time-modulation leads to oscillations of the mean photon number but the averaged photon number remains to be the same as in an ordinary NOPO.

In the limit $\lambda/\gamma=k^2/\gamma\gamma_3\ll 1$ of a weak nonlinearity the mean photon number is proportional to λ^{-1} and hence infinitely increases. However, in this limit the threshold of generation $f_{th}=\gamma\gamma_3/k$ also increases and hence infinitely strong pump fields are needed for realization of this regime.

B. Quantum dynamics

In this section we develop a systematic perturbation procedure to study quantum properties of time-modulated NOPO in a fully quantum mechanical approach for both regimes of generation. For this goal we introduce an approach in addition to the standard ones proposed for quantum analysis of nonlinear systems using P representation. One of the advantages of this approach is the possibility to perform analysis of quantum fluctuations, as well as calculations of the physical quantities, in a general and very effective form. In particular, following this approach, we are able to calculate variances of quadrature amplitudes and field correlators in the most general form when the results for the below-threshold regime are obtained as a special case of the above-threshold regime. Note, that in its previous form this approach is shortly described in Ref. [12] in application to harmonically modulated NOPO.

Our intermediate goal is to formulate a closed system of equations for four bilinear stochastic variables $\alpha_1\beta_1$, $\alpha_2\beta_2$, $\alpha_1\alpha_2$, $\beta_1\beta_2$. After averaging over P distribution these quantities $\langle\alpha_1\beta_1\rangle$, $\langle\alpha_2\beta_2\rangle$, $\langle\alpha_1\alpha_2\rangle$, $\langle\beta_1\beta_2\rangle$, are elements of the so-called covariance matrix, which for an ordinary NOPO below threshold completely describes the quantum properties of the generated modes. The straightforward calculations using the Ito rules for changing variables lead to the following stochastic equations:

$$d\alpha_1\beta_1 = -2\gamma\alpha_1\beta_1(1+\alpha_2\beta_2)dt + \varepsilon(t)(\alpha_1\alpha_2+\beta_1\beta_2)dt + dW_{\alpha_1}dW_{\beta_1} + \alpha_1 dW_{\beta_1} + \beta_1 dW_{\alpha_1}, \tag{30}$$

$$d\alpha_2\beta_2 = -2\gamma\alpha_2\beta_2(1+\alpha_1\beta_1)dt + \varepsilon(t)(\alpha_1\alpha_2+\beta_1\beta_2)dt + dW_{\alpha_2}dW_{\beta_2} + \alpha_2 dW_{\beta_2} + \beta_2 dW_{\alpha_2}, \tag{31}$$

$$d\alpha_1\alpha_2 = -\gamma\alpha_1\alpha_2(2+\alpha_1\beta_1+\alpha_2\beta_2)dt + \varepsilon(t)(\alpha_1\beta_1+\alpha_2\beta_2)dt + dW_{\alpha_1}dW_{\alpha_2} + \alpha_1 dW_{\alpha_2} + \alpha_2 dW_{\alpha_1}, \tag{32}$$

$$d\beta_1\beta_2 = -\gamma\beta_1\beta_2(2+\alpha_1\beta_1+\alpha_2\beta_2)dt + \varepsilon(t)(\alpha_1\beta_1+\alpha_2\beta_2)dt + dW_{\beta_1}dW_{\beta_2} + \beta_1 dW_{\beta_2} + \beta_2 dW_{\beta_1}. \tag{33}$$

These equations can be rewritten in the more compact form. Introducing the denotations, $n_1=\alpha_1\beta_1$, $n_2=\alpha_2\beta_2$, $z_\alpha=\alpha_1\alpha_2$, $z_\beta=\beta_1\beta_2$, $dW_{n_i}=\alpha_i dW_{\beta_i}+\beta_i dW_{\alpha_i}$, $dW_{z_\alpha}=\alpha_1 dW_{\alpha_2}+\alpha_2 dW_{\alpha_1}$, $dW_{z_\beta}=\beta_1 dW_{\beta_2}+\beta_2 dW_{\beta_1}$, and also applying the relations (6) and (7) and $\langle dW_{\alpha_1}(t)dW_{\beta_1}(t)\rangle=\langle dW_{\alpha_2}(t)dW_{\beta_2}(t)\rangle=0$, we obtain

$$dn_1 = -2\gamma n_1 dt - 2\gamma n_1 n_2 dt + \varepsilon(t)(z_\alpha+z_\beta)dt + dW_{n_1}, \tag{34}$$

$$dn_2 = -2\gamma n_2 dt - 2\gamma n_1 n_2 dt + \varepsilon(t)(z_\alpha+z_\beta)dt + dW_{n_2}, \tag{35}$$

$$dz_\alpha = -\gamma z_\alpha(2+n_1+n_2)dt + \varepsilon(t)(n_1+n_2)dt + [\varepsilon(t)-\gamma z_\alpha]dt + dW_{z_\alpha}, \tag{36}$$

$$dz_\beta = -\gamma z_\beta(2+n_1+n_2)dt + \varepsilon(t)(n_1+n_2)dt + [\varepsilon(t)-\gamma z_\beta]dt + dW_{z_\beta}. \tag{37}$$

We have introduced Gaussian noise terms with zero means and the following correlators:

$$\langle dW_{n_i}(t)\rangle = \langle dW_{z_\alpha}(t)\rangle = \langle dW_{z_\beta}(t)\rangle = 0, \tag{38}$$

$$\langle dW_{n_1}(t)dW_{n_1}(t)\rangle = \langle dW_{n_2}(t)dW_{n_2}(t)\rangle = \langle dW_{z_\alpha}(t)dW_{z_\beta}(t)\rangle = 0, \tag{39}$$

$$\langle dW_{n_1}(t)dW_{n_2}(t)\rangle = [\varepsilon(t)(z_\alpha+z_\beta) - 2\gamma n_1 n_2]dt, \tag{40}$$

$$\langle dW_{z_\alpha}(t)dW_{z_\alpha}(t)\rangle = z_\alpha[2\varepsilon(t) - 2\gamma z_\alpha]dt, \tag{41}$$

$$\langle dW_{z_\beta}(t)dW_{z_\beta}(t)\rangle = z_\beta[2\varepsilon(t) - 2\gamma z_\beta]dt, \tag{42}$$

$$\langle dW_{n_{1,2}}(t)dW_{z_{\alpha,\beta}}(t)\rangle = n_{1,2}[\varepsilon(t) - \gamma z_{\alpha,\beta}]dt. \tag{43}$$

Obtained, Eqs. (34)–(43) are sufficient for calculations of the mean photon numbers and the variances. Nevertheless, for reasons which will become clear below, we use the following combinations of the stochastic variables: $n_+=\alpha_1\beta_1+\alpha_2\beta_2$, $n_-^2=(\alpha_1\beta_1-\alpha_2\beta_2)^2$, $R=(\alpha_1-\beta_2)(\beta_1-\alpha_2)$. The equations for the stochastic variables can be obtained from Eqs. (2) and (3) by using the Ito rules for changing variables,

$$dn_+ = [2\varepsilon(t) - 2\gamma - \lambda n_+]n_+ dt - [2\varepsilon(t)R + \lambda n_-^2]dt + dW_{n_+}, \tag{44}$$

$$dR = -[2\varepsilon(t) + 2\gamma + \lambda n_+]R dt - \lambda n_-^2 dt - dW_{\alpha_1}dW_{\alpha_2} - dW_{\beta_1}dW_{\beta_2} + dW_R, \tag{45}$$

$$dn_-^2 = -4\gamma n_-^2 dt + dW_{n_-}dW_{n_-} + dW_{n_-^2}. \tag{46}$$

These equations also contain terms generated from the noise correlations. This set of coupled stochastic equations is full and has nontrivial noise terms satisfying the following correlators:

$$\langle dW_{n_+}(t)dW_{n_+}(t)\rangle = [2\varepsilon(t)(n_+ - R) - \lambda n_+^2 + \lambda n_-^2]dt, \tag{47}$$

$$\langle dW_{n_-^2}(t)dW_{n_-^2}(t)\rangle = n_-^2\langle dW_{n_-}(t)dW_{n_-}(t)\rangle = -n_-^2\langle dW_{n_+}(t)dW_{n_+}(t)\rangle, \tag{48}$$

$$\langle dW_R(t)dW_R(t)\rangle = -2R[2\varepsilon(t) - \lambda n_+ + \lambda R]dt, \tag{49}$$

$$\langle dW_R(t)dW_{n_+}(t)\rangle = -R[2\varepsilon(t) + \lambda n_+]dt, \tag{50}$$

$$\langle dW_R(t)dW_{n_-^2}(t)\rangle = -n_-^2[2\varepsilon(t) - \lambda n_+ + \lambda R]dt. \tag{51}$$

We complete this section by deriving equations for the averaged quantities, namely $\langle n_+\rangle=\langle a_1^\dagger a_1\rangle+\langle a_2^\dagger a_2\rangle=\langle\alpha_1\beta_1\rangle+\langle\alpha_2\beta_2\rangle$, $\langle R\rangle=\langle a_1^\dagger a_1\rangle+\langle a_2^\dagger a_2\rangle-\langle a_1 a_2\rangle-\langle a_1^\dagger a_2^\dagger\rangle=\langle(\alpha_1-\beta_2)(\beta_1-\alpha_2)\rangle$, $\Delta=\langle(a_1^\dagger a_1-a_2^\dagger a_2)^2\rangle=\langle n_-^2\rangle+\langle n_+\rangle$. The results are found to be

$$\frac{d}{dt}\langle n_+\rangle = [2\varepsilon(t) - 2\gamma - \lambda]\langle n_+\rangle - \lambda\langle n_+^2\rangle - 2\varepsilon(t)\langle R\rangle + \lambda\Delta, \tag{52}$$

$$\frac{d}{dt}\langle R\rangle = -[2\varepsilon(t) + 2\gamma + \lambda]\langle R\rangle - \lambda\langle n_+ R\rangle - 2\varepsilon(t) + \lambda\Delta, \tag{53}$$

$$\frac{d}{dt}\Delta = -4\gamma\Delta + 2\gamma\langle n_+\rangle. \tag{54}$$

From Eq. (54) the variance of photon-number difference Δ can be expressed as a function of $\langle n_+\rangle$,

$$\Delta(t) = 2\gamma\int_{-\infty}^{t} e^{4\gamma(t'-t)}\langle n_+(t')\rangle dt'. \tag{55}$$

A significant point about this result is that the variance $\Delta=\langle(a_1^\dagger a_1-a_2^\dagger a_2)^2\rangle$ is expressed in a simple enough form through the mean photon number and, that is essential, in the framework of "exact" quantum theory without a linear treatment of quantum fluctuations. Substituting this expression into (52) and (53) we get the following equations which are convenient for the perturbative analysis of quantum fluctuations:

$$\frac{d}{dt}\langle n_+\rangle = [2\varepsilon(t) - 2\gamma - \lambda]\langle n_+\rangle - \lambda\langle n_+^2\rangle - 2\varepsilon(t)\langle R\rangle + 2\gamma\lambda\int_{-\infty}^{t} e^{4\gamma(\tau-t)}\langle n_+(\tau)\rangle d\tau, \quad (56)$$

$$\frac{d}{dt}\langle R\rangle = -[2\varepsilon(t) + 2\gamma + \lambda]\langle R\rangle - \lambda\langle n_+R\rangle - 2\varepsilon(t) + 2\gamma\lambda\int_{-\infty}^{t} e^{4\gamma(\tau-t)}\langle n_+(\tau)\rangle d\tau. \quad (57)$$

III. QUANTUM FLUCTUATIONS AND VARIANCES

We are now in the position to study quantum effects in time-modulated NOPO and will state the main results of the paper concerning CV entanglement.

To characterize CV entanglement we address both the inseparability criterion [21,22] and the EPR paradox criterion [3,23]. These criteria could be quantified by analyzing the variances $V_-=V(X_1-X_2)$ and $V_+=V(Y_1+Y_2)$ in the terms of the quadrature amplitudes of two modes $X_k=X_k(\Theta_k)=\frac{1}{\sqrt{2}}(a_k^\dagger e^{-i\Theta_k}+a_k e^{i\Theta_k})$, $Y_k=X_k(\Theta_k-\frac{\pi}{2})$ $(k=1,2)$, where $V(x)=\langle x^2\rangle-\langle x\rangle^2$ is a denotation of the variance. The inseparability criterion, or weak entanglement criterion reads as $V=\frac{1}{2}(V_+ + V_-)<1$, and due to the mentioned symmetries is reduced to the following form $V=V_+=V_-<1$. This sum criterion can be transformed into a product criterion $V_+V_-=V^2<1$. The strong CV entanglement criterion for the product of variances shows that when the inequality $V_+V_-<1/4$ is satisfied and hence $V<0.5$, there arises an EPR-like paradox.

We consider here the time-dependent output variances, which can be recorded by time-resolved homodyne detection. These quantities will be expressed through the stochastic variables using the relationships between normally ordered operator averages and stochastic moments with respect to the P function and then will be calculated in a linear treatment of quantum fluctuations. Restoring the previous phase structure of intracavity interaction, we obtain that $V_+=V_-=V$ and

$$V = 1 + \langle\alpha_1\beta_1\rangle + \langle\alpha_2\beta_2\rangle - \langle\alpha_1\alpha_2\rangle e^{i\Theta} - \langle\beta_1\beta_2\rangle e^{-i\Theta}, \quad (58)$$

where $\Theta=\Theta_1+\Theta_2+\Phi_L+\Phi_k$. As can be seen, the possible minimal level of the variance, realized under appropriate selection of the phases $\Theta_1+\Theta_2=-\Phi_L-\Phi_k$ in the formula (58), is expressed as $V(t)=1+\langle R(t)\rangle$.

Then the equations (56) and (57) should be solved within a standard procedure of linearization over the small quantum fluctuations for calculation of $\langle R(t)\rangle$. In our analysis we will use the modified perturbative approach expanding measurable quantities in power series on the small parameter of the theory.

In order to rewrite the above expressions for both intracavity photon numbers and quadrature variance in terms of output fields, which are external to the cavity, the standard method of input-output relations [31] is used. We consider the output behavior of NOPO assuming that all losses occur through the output coupler, so that $\gamma_3^{out}=\gamma_3$ (see Fig. 1). In this case the output fields of subharmonics are $\Phi_i^{out}(t)=\sqrt{2\gamma}a_i(t)$ $(i=1,2)$ while output field of the pump mode is equal to $\Phi_L^{out}(t)=\sqrt{2\gamma_3}a_3(t)-\Phi_L^{in}(t)$, where initial external field is determined as $<\Phi_L^{in}(t)>=f(t)/\sqrt{2\gamma_3}$. Then, the output measured time-dependent variances can be written through the normally ordered moments of the output-quadrature field variables as $V_+^{out}(\theta,t)=<:X^{out}(t)X^{out}(t):>-<X^{out}(t)>^2$, $V_-^{out}(\theta,t)=<:Y^{out}(t)Y^{out}(t):>-<Y^{out}(t)>^2$. We have defined $X^{out}=X_1^{out}-X_2^{out}$, $Y^{out}=Y_1^{out}-Y_2^{out}$, where $X_i^{out}=X_i^{out}(\theta)=\frac{1}{\sqrt{2}}(\Phi_i^{out}e^{i\theta}+\Phi_i^{out}e^{-i\theta})$, $Y_i^{out}(\theta)=X_i^{out}(\theta-\pi/2)$ $(i=1,2)$. Thus, we express in the standard way the variance $V^{out}=\frac{1}{2}(V_+^{out}+V_-^{out})$ through the intracavity variances in the following form $V^{out}=2\gamma(V-1)$. We present below applications of these results to two concrete schemes of time-modulated NOPO.

A. Above-threshold regime

First, we consider the above-threshold regime linearizing quantum fluctuations around the stable semiclassical solutions, $\langle n_+\rangle=n_{1c}+n_{2c}+\langle\delta n_+\rangle=2n_c+\langle\delta n_+\rangle$, $\langle R\rangle=R^0+\langle\delta R\rangle=\langle\delta R\rangle$, $\langle n_+R\rangle=2n_c\langle\delta R\rangle$, $\langle n_+^2\rangle=4n_c\langle\delta n_+\rangle$, where it is assumed that $n_{1c}=n_{2c}=n_c(t)$, $\varphi_1+\varphi_2=2\pi k$, and hence $R^0=0$. Note, that in the current experiments the ratio of nonlinearity to dumping is small, $k/\gamma\ll 1$ (typically 10^{-4} or less), and hence $\lambda/\gamma=k^2/(\gamma\gamma_3)\ll 1$ is the small parameter of the theory. Therefore, the zero order terms in the above expansion correspond to a large classical field of the order γ/λ in accordance with Eq. (22), while the next terms describing the quantum fluctuations are of the order of 1. On the whole, combining the procedure of linearization with $\lambda/\gamma\ll 1$ approximation we get a linear equation for the variance $V(t)=1+\langle\delta R\rangle$,

$$\frac{d}{dt}V(t) = -2[\gamma + \varepsilon(t) + \lambda n_c(t)]V(t) + 2\lambda n_c(t) + 2\gamma + 4\gamma\lambda\int_{-\infty}^{t} e^{4\gamma(\tau-t)}n_c(\tau)d\tau, \quad (59)$$

with the following periodic asymptotic solution in the over-transient regime:

$$V(t) = 2\int_{-\infty}^{t} \exp\left(-2\int_{\tau}^{t}[\gamma + \varepsilon(t') + \lambda n_c(t')]dt'\right) \times \left(\gamma + \lambda n_c(\tau) + 2\gamma\lambda\int_{-\infty}^{\tau} e^{4\gamma(\tau'-\tau)}n_c(\tau')d\tau'\right)d\tau. \quad (60)$$

Indeed, it may be verified that Eq. (60) can be rewritten in the following form in which its periodic dependence becomes more evident:

$$V(t) = 2\gamma \int_{-\infty}^{0} \exp\left(-2\int_{\tau}^{0} [\gamma + \varepsilon(t'+t) + \gamma n_c(t'+t)]dt'\right) \times \left(1 + n_c(\tau+t) + 2\gamma \int_{-\infty}^{\tau} e^{4\gamma(\tau'-\tau)} n_c(\tau'+t)d\tau'\right)d\tau. \quad (61)$$

B. Below-threshold regime

The analysis of the below-threshold regime is more simple and leads to formulas (59)–(61) with $n_c=0$.

In this case, the variance is derived from Eq. (60) as

$$V(t) = 2\gamma \int_{-\infty}^{t} \exp\left(-2\int_{\tau}^{t} [\gamma + \varepsilon(t')]dt'\right)d\tau. \quad (62)$$

Let us turn to the mean photon number of the modes in NOPO below threshold. Because $n_c=0$, Eq. (52) is transformed to

$$\frac{d}{dt}\langle \delta n_+ \rangle = 2[\varepsilon(t) - \gamma]\langle \delta n_+ \rangle - 2\varepsilon(t)[V(t) - 1]. \quad (63)$$

Finally, combining Eqs. (62) and (63) in over transient regimes yields

$$\langle \delta n_1 \rangle = \langle \delta n_2 \rangle = \int_{-\infty}^{t} \exp[-2\gamma(t-\tau)] \times \sinh\left(\int_{\tau}^{t} \varepsilon(t')dt'\right)\varepsilon(\tau)d\tau. \quad (64)$$

This result describes the mean photon number on the level of quantum noise versus time-modulated amplitude of the pump field.

Particularly, when $f(t)=f=$const and hence $\varepsilon(t)=\varepsilon=\frac{fk}{\gamma_3}$, Eqs. (60) and (62) take the following form:

$$V = \frac{\gamma}{\gamma+\varepsilon} = \frac{f_{\text{th}}}{f+f_{\text{th}}}, \quad \varepsilon < \varepsilon_{\text{th}}, \quad (65)$$

$$V = \frac{3}{4} - \frac{\gamma}{4\varepsilon}, \quad \varepsilon > \varepsilon_{\text{th}}, \quad (66)$$

which coincide with analogous results for an ordinary NOPO in below- and above-threshold correspondingly. For the general case the integral in (62) cannot be handled but the lower bound for $V(t)$ can be obtained in the general form as

$$V(t) \geqslant \gamma/(\gamma + \varepsilon_{\max}). \quad (67)$$

IV. EPR ENTANGLEMENT IN HARMONICALLY MODULATED NOPO

In the preceding section we have derived the results for the mean photon number (22) and (64) as well as for the squeezed variance characterized CV entanglement (60) and (62). These equations take place for arbitrary periodically modulated amplitude of pump field and for both below- and above-threshold regimes. As an application of these results we consider in this section NOPO with continuously modulated pump field.

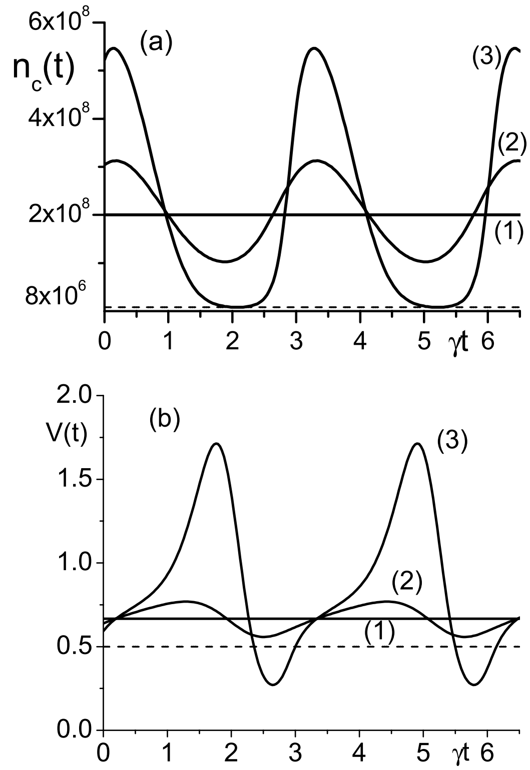

FIG. 2. Mean photon number (a) and the variance $V(t)$ (b) versus dimensionless time for the parameters $k/\gamma=5\times10^{-4}$, $\gamma_3/\gamma=25$, $\delta/\gamma=2$, $\bar{f}=3f_{\text{th}}$: $f_1=0$ (curve 1), $f_1=0.4\bar{f}$ (curve 2), and $f_1=1.2\bar{f}$ (curve 3). The dashed line in (b) corresponds to the stationary limit $V=1/2$.

The corresponding scheme (Fig. 1) involves pump field with the harmonically modulated amplitude $f(t)=f_0+f_1\cos(\delta t)$, where δ is the modulation frequency, $\delta \ll \omega_L$. Such modulation may be realized electronically by using the standard techniques, particularly, by an electro-optic amplitude modulator. Besides, the corresponding scheme can be implemented at least for NOPO driven by a polychromatic pump field with central frequency ω_L and two satellites $\omega_L+\delta$, $\omega_L-\delta$. In the last case the Hamiltonian of this system is indeed given by (1) and f_0 and f_1 are the amplitudes of the central component and the satellites of the pump field.

In above threshold, $\bar{f}=f_0>f_{\text{th}}$, the photon number (22) reads as

$$n_c^{-1}(t) = 2\lambda \int_{-\infty}^{0} \exp\left[2\gamma\tau\left(\frac{\bar{f}}{f_{\text{th}}} - 1\right)\right] \times \exp\left(\frac{2\gamma f_1}{\delta f_{\text{th}}}\{\sin[\delta(t+\tau)] - \sin(\delta t)\}\right)d\tau. \quad (68)$$

This result is illustrated in Fig. 2(a) for the different levels of modulation and for $f_1=0$ reaches to the standard result $n^{\rm out}=2\gamma n_c=2\gamma(f_0-f_{\rm th})/k$.

Let us turn to the study of the entanglement based on the formula (60), which for $f_1=0$ also coincides with an analogous one for the ordinary NOPO. Typical results for the case of harmonic modulation, $\varepsilon(t)=\frac{k}{\gamma_3}[f_0+f_1\cos(\delta t)]$, are presented in Fig. 2(b) for the above-threshold regime. The variance is seen to show a time-dependent modulation with a period $2\pi/\delta$. The drastic difference between the degree of two-mode squeezing and/or entanglement for modulated and stationary dynamics is also clearly seen in Fig. 2(b). The stationary variance (curve 1) near the threshold having a limiting squeezing of 0.5 is bounded by quantum inseparability criterion $V<1$. This variance, however, is still above the stationary limit of $1/2$ for the chosen parameters. It is clearly seen (curve 3) that the variance for the case of modulated dynamics obeys the EPR criterion $V^2<1/4$ of strong CV entanglement for the definite time intervals. To indicate this effect the stationary limit $V=1/2$ is shown in Fig. 2(b) as a dashed line. The minimum values of the variance $V_{\rm min}=V(t_m)$ for both regimes of generation and corresponding photon numbers $n_{\rm min}=n_c(t_m)$ for above-threshold regime at fixed time intervals $t_m=t_0+2\pi m/\delta$, $(m=0,1,2,\ldots)$ are shown in Fig. 3. As it is expected, the degree of EPR entanglement increases with ratio $f_1/\bar{f}$, i.e., with level of the modulation.

Similar conclusions hold for the output measured integral two-mode squeezing which is realized, if $V^{\rm out}=2\gamma(V-1)<0$ [32]. The lower bound for $V^{\rm out}$ is determined by the stationary limit and reads as $V^{\rm out}/2\gamma>-1/2$. The above results indicate that for time-modulated NOPO the normalized output variance becomes less than $-1/2$, i.e., $V^{\rm out}/2\gamma<-1/2$ for the definite time intervals.

The case of a weak modulation: We now illustrate these results analytically for the case of a weak level of modulation, $\varepsilon_1=\frac{k}{\gamma_3}f_1\ll\delta$. In order to this end, we consider below-threshold regime rewriting Eq. (62) in the following form:

$$V(t)=2\gamma\int_{-\infty}^{t}\exp\left(-2(\gamma+\bar{\varepsilon})(t-\tau)+\frac{2\varepsilon_1}{\delta}[\sin(\delta t)-\sin(\delta\tau)]\right)d\tau. \tag{69}$$

Expanding in Eq. (69) the exponent in power series of the ratio ε_1/δ we obtain for the variance up to the first order,

$$V(t)\approx\frac{\gamma}{\gamma+\bar{\varepsilon}}\left(1-\frac{2\varepsilon_1}{\delta}\frac{\sin(\delta t)+\left(\frac{2(\bar{\varepsilon}+\gamma)}{\delta}\right)\cos(\delta t)}{1+\left(\frac{2(\bar{\varepsilon}+\gamma)}{\delta}\right)^2}\right). \tag{70}$$

As we see, the maximal degree of two-mode squeezing in this approximation reads as

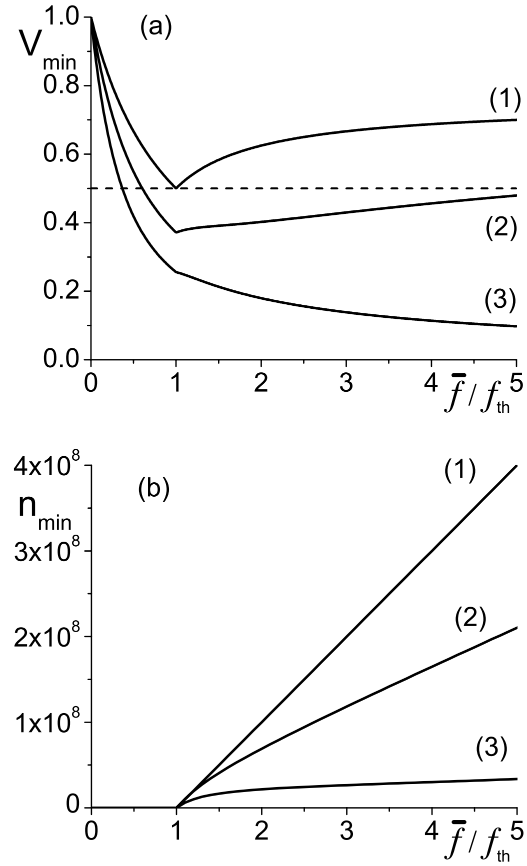

FIG. 3. The minimum level of the variance (a) and the mean photon number at the points of minima of the variance (b) versus $\bar{f}/f_{\rm th}$ for three levels of modulation: $f_1=0$ (curve 1), $f_1=0.75\bar{f}$ (curve 2), and $f_1=2\bar{f}$ (curve 3). The parameters are $k/\gamma=5\times10^{-4}$, $\gamma_3/\gamma=25$, $\delta/\gamma=2$. The dashed line in (a) corresponds to the stationary limit $V=1/2$.

$$V_{\rm min}=\frac{\gamma}{\gamma+\bar{\varepsilon}}\left(1-\frac{2\varepsilon_1}{\delta}\frac{1}{\sqrt{1+\left(\frac{2(\bar{\varepsilon}+\gamma)}{\delta}\right)^2}}\right) \tag{71}$$

and is achieved for the following time intervals $\delta t=\arctan\frac{\delta}{2(\gamma+\bar{\varepsilon})}+2\pi k$. We conclude that near to the threshold, $\bar{\varepsilon}\approx\varepsilon_{\rm th}=\gamma$, $V_{\rm min}<1/2$ and hence the EPR criterion $V^2<1/4$ of strong entanglement is realized for the definite time intervals. Using the formula (69) we obtain, in general, the lower bound for the variance

$$V(t)\geqslant\frac{\gamma}{\gamma+\bar{\varepsilon}+\varepsilon_1}. \tag{72}$$

This value for any level of modulation is less than the minimum degree of two-mode squeezing, $V=1/2$ for an ordinary NOPO near to the threshold.

As the numerical analysis shows, the production of strong entanglement occurs for the period of modulation comparable with the characteristic time of dissipation, $\delta\approx\gamma$, and dissapears for asymptotic cases of slow $(\delta\ll\gamma)$ and fast $(\delta\gg\gamma)$ modulations.

We note that analogous conclusion about improvement of the degree of squeezing by harmonic time modulation has been made also for one-mode squeezing states generated in OPO [33]. Nevertheless the levels of squeezing of both devices are different. Another essential difference relies on the behavior of the corresponding variances in the above-threshold regimes.

V. EPR ENTANGLEMENT INDUCED BY A SEQUENCE OF LASER PULSES

It seems that such improvement of the degree of CV entanglement in NOPO is realized due to the control of quantum dissipative dynamics of subharmonic modes. In one of the standard techniques [28], control of the optical quantum system is achieved through the application of suitable tailored, time-dependent external fields including application of synchronized laser pulses. Thus, in this section we want to extend the pulses control strategy by considering a generation of EPR entangled states of light in a dissipative system.

We turn now to the scheme of Fig. 1 subjected by a periodic sequence of identical laser pulses. We consider a rectangular form of the pulses of the duration T_1 separated by the time intervals with duration T_2. The amplitude of pump field is periodic on the time interval $T=T_1+T_2$ and can be described as

$$f(t) = f_L \sum_{n=0}^{\infty} \Theta(t - t_0 - nT)\Theta(t_0 + nT + T_1 - t), \quad (73)$$

where Θ is the usual step function, t_0 is the initial time interval, and f_L is the amplitude of the laser pulses. This amplitude describes a periodic chain of square pulses of duration T_1 separated by the time interval T_2. Period averaged pump field amplitude is equal to

$$\bar{f} = \frac{1}{T}\int_{t_0+nT}^{t_0+(n+1)T} f(t)dt = \frac{f_L T_1}{T_1 + T_2}, \quad (74)$$

and hence the above-threshold regime is realized if $f_L T_1 > \frac{\gamma\gamma_3}{k}(T_1+T_2)$. The above approach can be easily applied to the pulsed regime. The mean photon numbers and the variance $V(t)$ are calculated based on formulas (22) and (60). The predictions of the numerical calculations are shown in Fig. 4 for one of the preferable regimes (for typical $\gamma = 10^6\ \mathrm{s}^{-1}$, $T_1 = 10^{-8}$ s and the repetition rate $T_2^{-1} = 1$ MHz) assuming that T_1 is much less than T_2. It is clearly evident from Fig. 4(a) that the mean photon number increases during laser pulses and decays during the interval T_2 between pulses due to dissipation in the cavity. One can conclude from Fig. 4(b) that the weak entanglement criterion $V<1$, is fulfilled for any time intervals. However, we have also found remarkable result that the variance goes below the inseparability level of 0.5 in the ranges of maximal photon numbers, for appropriate chosen parameters. We have illustrated this effect for the nonstationary regime, if T_1 is sufficiently shorter than the relaxation time and hence the dissipative effects in modes dynamics for time intervals $t_n < t < t_n + T_1$ are still unessential. However, it can be seen that the improvement of the degree of entanglement is realized for wide ranges of the parameters including $\gamma T_1 \gtrsim 1$. We demonstrate this point by an analytical calculation of the minimum values $V_{\min}$ as well as $n_{\max}$ for the definite time-intervals.

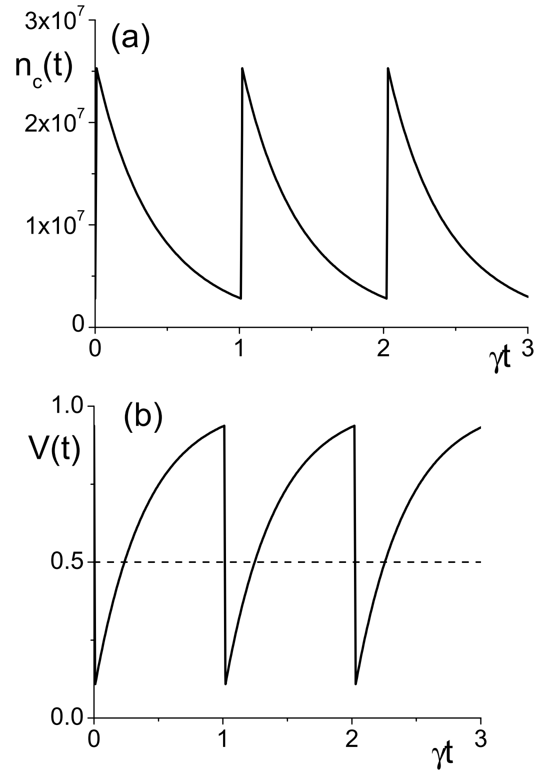

FIG. 4. Mean photon numbers (a) and the variance (b) versus dimensionless time for the parameters $k/\gamma = 5\times10^{-4}$, $\gamma_3/\gamma = 25$, $T_1 = 0.01\gamma^{-1}$, $T_2 = \gamma^{-1}$, $\bar{f} = 1.1 f_{\mathrm{th}}$. The dashed line in (b) indicates the stationary limit.

At first, we consider the equation (20) for the chain of square pulses where the time-dependent parameter $\varepsilon(t)$ is chosen as

$$\varepsilon(t) = \begin{cases} \varepsilon_L, & \text{if } t_n < t < t_n + T_1, \\ 0, & \text{if } t_n + T_1 < t < t_{n+1}. \end{cases} \quad (75)$$

Here $t_n = t_0 + nT$ and $\varepsilon_L = f_L k/\gamma_3$. It is immediately seen that in the presence of pulses, for the time intervals within the range $t_n < t < t_n + T_1$, evolution of the photon number $n_c(t) = Z_L^{-1}$ is given by the following equation:

$$\frac{dZ_L}{dt} = 2\lambda - 2(\varepsilon_L - \gamma)Z_L. \quad (76)$$

Thus, we get

$$Z_L(t) = \frac{\lambda}{\varepsilon_L - \gamma} + \left(Z_L(t_n) - \frac{\lambda}{\varepsilon_L - \gamma}\right)e^{-2(\varepsilon_L - \gamma)(t - t_n)}. \quad (77)$$

The solution of Eq. (20) in the absence of pulses for the time intervals $t_n + T_1 < t_{n+1}$ can be written as

$$Z_0(t) = -\frac{\lambda}{\gamma} + \left(Z_0(t_n + T_1) + \frac{\lambda}{\gamma}\right)e^{2\gamma(t-t_n-T_1)}. \quad (78)$$

It is clear that these solutions should satisfy the following boundary conditions:

$$Z_L(t_n) = Z_0(t_n), \quad Z_L(t_n + T_1) = Z_0(t_n + T_1). \quad (79)$$

At the same time it is easy to understand that for over transient regime the conditions of the periodicity should also be satisfied,

$$Z_0(t_n) = Z_0(t_{n+1}), \quad Z_L(t_n) = Z_L(t_{n+1}). \quad (80)$$

Now, by considering Eqs. (77) and (78) and the conditions (79) and (80) it is immediate to get the following simple relations between $Z_L(t_n)$ and $Z_L(t_n+T_1)$:

$$Z_L(t_n) = -\frac{\lambda}{\gamma} + \left(Z_L(t_n + T_1) + \frac{\lambda}{\gamma}\right)e^{2\gamma T_2}, \quad (81)$$

$$Z_L(t_n + T_1) = \frac{\lambda}{\varepsilon_L - \gamma} + \left(Z_L(t_n) - \frac{\lambda}{\varepsilon_L - \gamma}\right)e^{-2(\varepsilon_L-\gamma)T_1}. \quad (82)$$

These formulas allow us to determine the maximal $n_{\max} = Z_L^{-1}(t_n+T_1)$ and minimal $n_{\min}=Z_L^{-1}(t_n)$ values of the photon number. In this way the results are obtained in the following form:

$$n_{\max} = \frac{1 - e^{-2(\bar{\varepsilon}-\gamma)T}}{\frac{\lambda}{\varepsilon_L - \gamma}(1 - e^{-2(\bar{\varepsilon}-\gamma)T}e^{-2\gamma T_2}) + \frac{\lambda}{\gamma}e^{-2(\bar{\varepsilon}-\gamma)T}(1 - e^{-2\gamma T_2})}, \quad (83)$$

$$n_{\min} = \frac{1 - e^{-2(\bar{\varepsilon}-\gamma)T}}{\frac{\lambda}{\gamma}(e^{2\gamma T_2} - 1) + \frac{\lambda}{\varepsilon_L - \gamma}(e^{2\gamma T_2} - e^{-2(\bar{\varepsilon}-\gamma)T})}. \quad (84)$$

In obtaining these formulas we have used the equality $\varepsilon_L T_1 = \bar{\varepsilon} T$. Thus, we have obtained the extremal values of the photon number for the above-threshold regime in the simple analytical form.

Similar formulas can be derived for the quadrature squeezed variance. We shall obtain these results considering for simplicity NOPO below the threshold. In this operational regime, because $n_c=0$, Eq. (60) is transformed to

$$\frac{d}{dt}V(t) = -2[\gamma + \varepsilon(t)]V(t) + 2\gamma. \quad (85)$$

Since Eqs. (60) and (85) have similar forms, the further calculations of the minimal values of the variance $V_{\min}$ are analogous to the previous calculation performed for the photon number. As shows analysis (see also numerical results in Fig. 4), the minimal values $V_{\min}$ are reached at the time intervals $t=t_n+T_1$ corresponding to the maximal values of the photon number. On the whole, using Eqs. (75) and (85) we get

$$V_{\min} = \frac{\frac{\gamma}{\varepsilon_L + \gamma}(1 - e^{-2(\varepsilon_L+\gamma)T_1}) + e^{-2(\varepsilon_L+\gamma)T_1}(1 - e^{-2\gamma T_2})}{1 - e^{-2\gamma T_2}e^{-2(\varepsilon_L+\gamma)T_1}} \quad (86)$$

and in the limit $T_1 \ll T_2$ of short duration of pulses,

$$V_{\min} = \frac{e^{-2\varepsilon_L T_1}(1 - e^{-2\gamma T_2})}{1 - e^{-2\gamma T_2}e^{-2\varepsilon_L T_1}}. \quad (87)$$

The calculations can again be carried out for the maximal values $V_{\max}$ of the quadrature variance at $t=t_n$, and by using Eqs. (75) and (85) as previously, one obtains in the limit $T_1 \ll T_2$,

$$V_{\min} \simeq e^{-2\varepsilon_L T_1}V_{\max}. \quad (88)$$

For the ranges close to the generation threshold, since $\varepsilon_L T_1 \lesssim \gamma T_2$, the variance can be written as

$$V_{\min}\frac{1}{e^{2\varepsilon_L T_1} + 1}. \quad (89)$$

These formulas are in accordance with the data of Fig. 4(b). As we see the degree of EPR entanglement increases with $\varepsilon_L T_1$.

The minimum values of the variance of output fields at the time intervals $t=t_n+T_1$ are obtained as

$$V_{\min}^{\text{out}}/2\gamma = -\frac{1 - e^{-2\varepsilon_L T_1}}{1 - e^{-2\gamma T_2}e^{-2\varepsilon_L T_1}} \quad (90)$$

in the limit of short duration of pulses. In the near-threshold domain $\varepsilon_L T_1 \lesssim \gamma T_2$ the result above yields the following simple expression:

$$V_{\min}^{\text{out}}/2\gamma = -\frac{1}{1 + e^{-2\varepsilon_L T_1}}. \quad (91)$$

One can conclude from Eqs. (90) and (91) that the output variance goes below the stationary limit $V_{\min}^{\text{out}}/2\gamma = -\frac{1}{2}$ in the pulsed regime too. The perfect CV entanglement $V_{\min}^{\text{out}}/2\gamma = -1$ is realized in the range $\varepsilon_L T_1 \gg 1$.

Note, that considering the pulsed regime we do not take into account any incoming pulse distribution. This topic is currently being explored and will be the subject of forthcoming work.

Comparison of the analytical and numerical results: It is well known that the linearized theory is applicable only outside the critical region, although the variance (60) is surprisingly well defined also at the threshold. As our analysis shows, the condition of the validity of linear results for the near-threshold regimes reads as $|\bar{f}/f_{\text{th}} - 1| \gg (\lambda/\gamma)\exp[2(f_1/f_{\text{th}})(\gamma/\delta)]$ for NOPO driven by harmonically modulated pump field, while for NOPO under sequence of square laser pulses the condition takes the more simple form $|\bar{f}/f_{\text{th}} - 1| \gg (\lambda/\gamma)$. For typical $\lambda/\gamma \ll 1$, both conditions are fairly easy to satisfy even for narrow critical ranges. Note, that the accuracy of our analytical calculations has been verified by the numerical simulations based on the quantum state diffusion method.

VI. CONCLUSION

In conclusion, we point out that, indeed, the class of currently proposed schemes generating intensive light beams with high degree of entanglement may be significantly extended if instead of monochromatic pumping we consider pump fields with periodically varying amplitudes. In this direction, we have presented and studied theoretically two schemes of time-modulated NOPO. More importantly, we have seen that both schemes operate under nonstationary conditions that has a significant impact on formation of high-degree CV entanglement even in the presence of dissipation and cavity induced feedback. It is known that the sum of intracavity two-mode squeezing quadratures of an ordinary NOPO is not perfectly squeezed, leading to a limiting CV entanglement of 0.5. We have demonstrated that time modulation of twin beams generated in NOPO essentially improves the degree of CV entanglement by going beyond the standard limit established in an ordinary NOPO. The properties of periodically pulsed entanglement can be widely controlled via the modulation parameters. The important point is that this improvement relates to the integral or total squeezing of intracavity modes as well as output photon fields of subharmonic rather than the spectral squeezing. Thus, CV entanglement has been analyzed in the time domain in addition to many analogous investigations of squeezed variances performed in the spectral domain. We believe that time-dependent output variances could be observed by means of time-resolved homodyne measurements. We hope that the results obtained may be also applicable to a general class of quantum dissipative systems and can serve as a guide for further studies of entanglement physics in application to time-resolved quantum information protocols.

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