Thermal fluctuations of vortex clusters in quasi-two-dimensional Bose-Einstein condensates

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We study the thermal fluctuations of vortex positions in small vortex clusters in a harmonically trapped rotating Bose-Einstein condensate. It is shown that the order-disorder transition of two-shell clusters occurs via the decoupling of shells with respect to each other. The corresponding "melting" temperature depends strongly on the commensurability between numbers of vortices in shells. We show that melting can be achieved at experimentally attainable parameters and very low temperatures. Also studied is the effect of thermal fluctuations on vortices in an anisotropic trap with small quadrupole deformation. We show that thermal fluctuations lead to the decoupling of a vortex cluster from the pinning potential produced by this deformation. The decoupling temperatures are estimated and strong commensurability effects are revealed.

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I. INTRODUCTION

The properties of Bose-Einstein condensates (BEC's) of alkali-metal-atom gases have attracted considerable current interest. Recent progress in this field has allowed for the creation of quasi-two-dimensional atomic gas either using one-dimensional (1D) optical lattices or applying a tight axial trapping [1–4]. It is well known that, according to the Mermin-Wagner-Hohenberg theorem, Bose-Einstein condensation is impossible in 2D homogeneous systems in the thermodynamic limit. However, Bose-Einstein condensation at finite temperature becomes possible in a trapped gas.

Recently, the Berezinskii-Kosterlitz-Thouless (BKT) transition associated with the creation of vortex-antivortex pairs was studied theoretically in 2D BEC clouds [5–7] and it was shown that this transition can occur in the experimentally attainable range of parameters. For instance, according to Ref. [6], the BKT transition can happen at $T \approx 0.5 T_c$ for the number of particles, $N \sim 10^3 \div 10^4$, and realistic values of other parameters. These results demonstrate the importance of temperature effects in 2D BEC's even at temperatures well below the critical one. At the same time, the effect of temperature on vortex lattices in BEC's has not been studied yet, although the fluctuations of positions of vortices should become considerable even at lower temperatures than those corresponding to the BKT transition. Finally, experimental evidence for the BKT transition in trapped condensates was reported in Ref. [8]. Recently, the effect of temperature on vortex matter was analyzed in Ref. [9], but in the strongly fluctuative regime at relatively high temperatures, when the positions of the vortices are random.

It is well known from the theory of superconductivity that thermal fluctuations can lead to the melting of flux line lattice. However, in real superconductors this usually happens only in the vicinity of the critical temperature. For the case of atomic BEC's, the critical temperature depends on the number of particles in the trap. Therefore, melting can occur at temperatures much lower than the critical one. In finite systems, fluctuations of vortex positions depend also on the number of vortices. In such systems, the melting temperature is not a strictly defined quantity. In this case, a characteristic temperature of the order-disorder transition ("melting") can be defined using the Lindemann criterion; see the discussion in [10]. With increasing of the number of vortices, the fluctuations of the vortex positions are determined by elastic shear modulus of the system—i.e., by the Tkachenko modes studied in Ref. [11]. However, when a vortex number is not large, quantization effects start to play a very important role and the "melting" temperatures in this case can be much smaller than that for a larger system. Thermal fluctuations of the system of interacting point particles trapped by external potential were studied before in Refs. [10,12–15] mostly by using Monte Carlo simulations. If there are not many particles (or vortices) in the system, in the ground state, they form a cluster consisting of shells. It was shown in Refs. [12–15] that with increasing the temperature, first, the order between different shells is destroyed and these shells become decoupled with respect to each other. Only after this, with a sufficient increase of temperature, does a radial disordering of the cluster occur. This leads to a hierarchy of melting temperatures, which depends dramatically on the symmetry of cluster and number of particles.

In addition to thermal fluctuations, quantum ones can be significant in atomic condensates. In recent works [16,17], quantum and thermal fluctuations in finite vortex arrays in a one-dimensional optical lattice were considered. See also Refs. [18,19] for thermal fluctuations in spinor condensates.

In the present paper, we study the intershell melting of small vortex clusters in quasi-2D BEC's at different numbers of vortices in the system. We consider the situation when a cluster consists of only two shells. First, we find the ground-state configurations of vortices and then calculate the deviations of vortices from their equilibrium positions in a harmonic approximation. We show that, if the numbers of vortices in the inner and outer shells are not commensurate, deviations of the shells with respect to each other can be very significant even at low temperatures, $T \ll T_c$, and a large number of particles in the system and shells become decoupled with respect to each other, thus leading to a disordering of the vortex cluster. Also studied is the role of thermal fluctuations on the small cluster, consisting of two, three, four vortices, in a trap with a small *quadrupole deformation*, which acts as a source of orientational pinning for the cluster.

The paper is organized as follows. In Sec. II we present our model, which allows one to find an energetically favorable vortex configuration in the 2D case and also to calculate semi-quantitatively deviations of vortex positions due to the thermal fluctuations. In Sec. III we study the intershell melting process in different two-shell vortex clusters and obtain a order-disorder transition temperature. In Sec. IV we analyze the effect of thermal fluctuations on vortices in the trap with a small quadrupole deformation. We conclude in Sec. V.

II. MODEL

Consider a quasi-two-dimensional condensate with N particles confined by the radial harmonic trapping potential

$$U(r) = \frac{m\omega_\perp^2 r^2}{2}, \tag{1}$$

where $\omega_\perp$ is a trapping frequency, m is the mass of the atom, and r is the radial coordinate. The system is rotated with the angular velocity ω. In this paper, we restrict ourselves to a range of temperatures much smaller than T_c. Therefore, we can neglect the noncondensate contribution to the free energy of the system. Thus, the energy functional reads

$$F = \hbar\,\omega_\perp N(T) \int r dr \int d\varphi \left(\frac{1}{2}|\nabla\psi|^2 + \frac{r^2}{2}|\psi|^2 + 2\pi g_N |\psi|^4 - i\omega\psi^* \frac{\partial\psi}{\partial\phi} \right), \tag{2}$$

where the integration is performed over the area of the system, φ is the polar angle, $N(T)$ is the number of condensed atoms, ω is the rotation frequency, $g_N = N\sqrt{\frac{2}{\pi}}\frac{a}{a_z}$ is the interaction parameter, and a and a_z are the scattering length and oscillator length $\left(a_z = \sqrt{\frac{\hbar}{m\omega_z}}\right)$ in the z direction, which is kinematically frozen. Distances and rotation frequencies are measured in units of the radial oscillator length and the trapping frequency, respectively. The normalization condition for the order parameter reads $\int r dr \int d\varphi |\psi|^2 = 1$. In this paper, we analyze the case of dilute BEC's and take $g_N = 5$, which corresponds to $\omega_z/2\pi = 1.05$ kHz at $N = 1000$ for ^{87}Rb($a \simeq 5.3$ nm). Since we consider the range of low temperatures $T \lesssim 0.1 T_c$, we can assume that $N(T) \simeq N$. For the dependence of T_c on N, we use the ideal gas result for the 2D case:

$$\frac{\hbar\omega_\perp}{kT_c} = \sqrt{\frac{\zeta(2)}{N}}, \tag{3}$$

where $\zeta(2)$ is a Riemann zeta function, $\sqrt{\zeta(2)} \approx 1.28$. Equation (3) remains accurate even for the case of interacting particles [20].

A. Ground state

Now we present a method allowing one to find a ground state of the system, which corresponds to the certain vortex cluster, and deviations of vortices from their equilibrium positions due to thermal fluctuations.

In the general case, ψ can be represented as a Fourier expansion

$$\psi(r,\varphi) = \sum_l f_l(r) \exp(-il\varphi). \tag{4}$$

Let us denote the number of vortices in the system as v. If the superfluid phase in BEC's has a q-fold symmetry, then only terms with l's divisible by q survive in the expansion (4). For instance, a vortex cluster consisting of a single ring of v vortices corresponds to the expansion (4) with $l=0$, v, $2v$, $3v$,…. A two-shell cluster with v_1 and v_2 vortices in the shells ($v_1+v_2=v$), where v_2 is divisible by v_1, corresponds to the expansion (4) with $l=0$, v_1, $2v_1$, $3v_1$,…. If v_2 is not divisible by v_1, then, in the general case, the expansion (4) contains all harmonics. Typically, the main contribution to the energy is given by just a few harmonics, and by taking into account approximately ten of them, one can find the energy of the system with a very high accuracy provided that the number of vortices in the cloud is not too large, $v \lesssim 10$–20.

In the limit of noninteracting gas ($g_N=0$), it follows from the Gross-Pitaevskii equation that each function f_l coincides with the eigenfunction of the harmonic oscillator corresponding to the angular momentum l. These functions have the Gaussian profile $\sim r^l \exp\left(-\frac{r^2}{2}\right)$. Therefore, one can assume that this Gaussian approximation remains accurate in the case of weakly interacting dilute gas. The accuracy can be improved if we introduce a variational parameter R_l characterizing the spatial extent of f_l. Finally, our ansatz for f_l has the form

$$f_l(r, C_l, R_l, \phi_l) = C_l \left(\frac{r}{R_l}\right)^l \exp\left(-\frac{r^2}{2R_l^2} - i\phi_l\right), \tag{5}$$

where C_l, R_l, and ϕ_l can be found from the condition of the minimum of the energy (2) and C_l is a real number. This approach was used for the first time in Ref. [21] to evaluate energies and density plots of different vortex configurations. In Ref. [22], a simplified version of this method with fixed values of $R_l=1$ was applied to the limit of weakly interacting gas with taking into account up to nine terms in the expansion (4). In Ref. [23], the results for such approximate solutions to the Gross-Pitaevskii equation were compared with some known results of numerical solutions. A good accuracy of the ansatz was revealed. See also Ref. [24] for a related approach. In Ref. [25], a version of this method was also used to calculate the energy of axially symmetric vortex phases in spinor condensates with a comparison of the obtained results with numerical solutions, and a good agreement was found. Therefore, this method can be also applied to our problem and we expect that the results must be semi-quantitatively accurate and with the help of this model one can reveal the effect of symmetry of vortex clusters on the melting temperatures and estimate the values of those temperatures.

Now we substitute Eqs. (4) and (5) into Eq. (2) and after integration we obtain

$$\frac{F}{\hbar\omega_\perp N} = \sum_l \alpha_l C_l^2 + \sum_l I_{llll} C_l^4 + 4\sum_{l>k} I_{llkk} C_l^2 C_k^2 + 4\sum_{l>k>m} I_{lkkm} C_l C_k^2 C_m \delta_{l+m,2k} \times \cos(\phi_l + \phi_m - 2\phi_k)$$

$$+8\sum_{l>k>m>n} I_{lkmn}C_lC_kC_mC_n\delta_{l+k,m+n} \times\cos(\phi_l+\phi_k-\phi_m-\phi_n), \tag{6}$$

where

$$\alpha_l=\frac{\pi}{2}\Gamma(l+2)(1+R_l^4)+\pi R_l^2\Gamma(l+1)\omega l, \tag{7}$$

$$I_{lkmn}=2\pi^2 g_N\Gamma\left(\frac{l+m+n+k}{2}+1\right)R_{lkmn}^2\times\left(\frac{R_{lkmn}}{R_l}\right)^l\left(\frac{R_{lkmn}}{R_k}\right)^k \times\left(\frac{R_{lkmn}}{R_m}\right)^m\left(\frac{R_{lkmn}}{R_n}\right)^n, \tag{8}$$

$$R_{lkmn}=\sqrt{2}(R_l^{-2}+R_k^{-2}+R_m^{-2}+R_n^{-2})^{-1/2}, \tag{9}$$

where $\Gamma(l)$ is a gamma function. The normalization condition is now given by

$$\pi\sum_l C_l^2R_l^2\Gamma(l+1)=1. \tag{10}$$

The values of the parameters R_l, C_l, and ϕ_l can be found from the minimum of the energy (6) taking into account Eq. (10). For instance, for the axially symmetric vortex-free state, $C_0=\sqrt{1/\pi R_0^2}$, $R_0=(1+2g)^{1/4}$, and $C_l=0$ at $l\geqslant 1$. Note that the energy is proportional to $\hbar\omega_\perp N$ at given values of ω and g_N.

B. Thermal fluctuations: Harmonic approximation

After finding the ground state values of the variational parameters, one can calculate the equilibrium positions of the vortices $\{r_0^{(j)},\varphi_0^{(j)}\}$, $j=1,\ldots,v$, by numerical solution of the equation

$$\psi(r_0^{(j)},\varphi_0^{(j)},p_n^{(0)})=0, \tag{11}$$

where we introduced the notation $\{p_n\}$ for the set of all variational parameters (R_l, C_l, and ϕ_l) and $\{p_n^{(0)}\}$ denotes the ground-state values of these parameters. Fluctuations of p_n, which are the degrees of freedom for the system in this model, lead to fluctuations of the vortex positions. We denote the deviations of the variational parameters from their equilibrium values as δp_n and express the deviations of the vortices $\delta r^{(j)}$ and $\delta\varphi^{(j)}$ through the deviations of the variational parameters in a linear approximation. The perturbed positions of the vortices are determined by the equation

$$\psi(r_0^{(j)}+\delta r^{(j)},\varphi_0^{(j)}+\delta\varphi^{(j)},p_n^{(0)}+\delta p_n)=0. \tag{12}$$

Finally, the deviation of the position of a given vortex from equilibrium is

$$\delta r^{(j)}=\frac{A_n^{(j)}}{D^{(j)}}\delta p_n, \tag{13}$$

$$\delta\varphi^{(j)}=\frac{B_n^{(j)}}{D^{(j)}}\delta p_n. \tag{14}$$

Here and below repeated indices are summed; $A_n^{(j)}$, $B_n^{(j)}$, and $D^{(j)}$ are given by

$$A_n^{(j)}=\mathrm{Im}\left(\frac{\partial\psi}{\partial p_n}\right)\mathrm{Re}\left(\frac{\partial\psi}{\partial r}\right)-\mathrm{Re}\left(\frac{\partial\psi}{\partial p_n}\right)\mathrm{Im}\left(\frac{\partial\psi}{\partial r}\right), \tag{15}$$

$$B_n^{(j)}=\mathrm{Im}\left(\frac{\partial\psi}{\partial p_n}\right)\mathrm{Re}\left(\frac{\partial\psi}{\partial\varphi}\right)-\mathrm{Re}\left(\frac{\partial\psi}{\partial p_n}\right)\mathrm{Im}\left(\frac{\partial\psi}{\partial\varphi}\right), \tag{16}$$

$$D^{(j)}=\mathrm{Im}\left(\frac{\partial\psi}{\partial r}\right)\mathrm{Re}\left(\frac{\partial\psi}{\partial\varphi}\right)-\mathrm{Re}\left(\frac{\partial\psi}{\partial r}\right)\mathrm{Im}\left(\frac{\partial\psi}{\partial\varphi}\right). \tag{17}$$

All the derivatives on p_n, r, and φ in Eqs. (15)–(17) are taken at ground-state values of parameters $p_n=p_n^{(0)}$ and space coordinates, corresponding to the equilibrium position of a given vortex, $r=r_0^{(j)}$, $\varphi=\varphi_0^{(j)}$. The squared deviation of the radial and polar coordinates of the two vortices labeled as j_1 and j_2 with respect to each other is given by

$$\delta r_{(j_1j_2)}^2=G_{mn}^{(j_1j_2)}\delta p_m\delta p_n, \tag{18}$$

$$\delta\varphi_{(j_1j_2)}^2=J_{mn}^{(j_1j_2)}\delta p_m\delta p_n, \tag{19}$$

where

$$G_{mn}^{(j_1j_2)}=\left(\frac{A_n^{(j1)}}{D^{(j1)}}-\frac{A_n^{(j2)}}{D^{(j2)}}\right)\left(\frac{A_m^{(j1)}}{D^{(j1)}}-\frac{A_m^{(j2)}}{D^{(j2)}}\right), \tag{20}$$

$$J_{mn}^{(j_1j_2)}=\left(\frac{B_n^{(j1)}}{D^{(j1)}}-\frac{B_n^{(j2)}}{D^{(j2)}}\right)\left(\frac{B_m^{(j1)}}{D^{(j1)}}-\frac{B_m^{(j2)}}{D^{(j2)}}\right). \tag{21}$$

In the same manner, we can express the deviations of the energy from the ground-state value as a quadratic function in terms of the deviations of the variational parameters:

$$\delta F=E_{st}\delta p_s\delta p_t, \tag{22}$$

where

$$E_{st}=\frac{\partial^2 F}{\partial p_s\,\partial p_t}. \tag{23}$$

The derivatives here are also calculated at $p_n=p_n^{(0)}$.

The averaged squared deviations of the radial and polar coordinates of the two vortices with respect to each other due to thermal fluctuations are given by

$$\langle\delta r_{(j_1j_2)}^2\rangle_T=\frac{\int d(\delta p)G_{mn}^{(j_1j_2)}\delta p_m\delta p_n\exp\left(-\frac{1}{kT}E_{st}\delta p_s\delta p_t\right)}{\int d(\delta p)\exp\left(-\frac{1}{kT}E_{st}\delta p_s\delta p_t\right)}, \tag{24}$$

$$\langle \delta\varphi^2_{(j_1j_2)}\rangle_T = \frac{\int d(\delta p) J^{(j_1j_2)}_{mn}\delta p_m \delta p_n \exp\left(-\frac{1}{kT}E_{st}\delta p_s \delta p_t\right)}{\int d(\delta p)\exp\left(-\frac{1}{kT}E_{st}\delta p_s \delta p_t\right)}. \quad (25)$$

In the general case, the integrals in Eqs. (24) and (25) cannot be calculated analytically, since the matrix E_{st} is not necessarily diagonal. Therefore, we have to switch to a new basis $\delta t = M\delta t$, where M is a matrix, which diagonalizes the quadratic form (22). Here and below we will use a matrix form for the equations. The quadratic forms (18), (19), and (22) in the new basis can be written as

$$\delta r^2_{(j_1j_2)} = (\delta t)^T P^{(j_1j_2)}\delta t, \quad (26)$$

$$\delta\varphi^2_{(j_1j_2)} = (\delta t)^T R^{(j_1j_2)}\delta t, \quad (27)$$

$$\delta F = (\delta t)^T Q\delta t, \quad (28)$$

where $Q = M^T EM$, $P^{(j_1j_2)} = M^T G^{(j_1j_2)} M$, and $R^{(j_1j_2)} = M^T J^{(j_1j_2)} M$. The matrix Q must be diagonal, and from this condition one can find the matrix M numerically and then calculate $P^{(j_1j_2)}$ and $R^{(j_1j_2)}$. In the new basis, the integrals in Eqs. (24) and (25) can be found analytically and finally we get

$$\langle \delta r^2_{(j_1j_2)}\rangle_T = kT\frac{P^{(j_1j_2)}_{nn}}{Q_{nn}}, \quad (29)$$

$$\langle \delta\varphi^2_{(j_1j_2)}\rangle_T = kT\frac{R^{(j_1j_2)}_{nn}}{Q_{nn}}. \quad (30)$$

As usual in a harmonic approximation, average squares of deviations are proportional to the temperature.

For the vortex cluster consisting of two shells we also introduce a quantity $\Delta\varphi$, which has a sense of averaged displacement of vortex shells with respect to each other. It can be defined as a root of the square displacement of a pair of vortices from different shells averaged over all possible pairs of vortices:

$$\Delta\varphi = \left[\frac{1}{v_1 v_2}\sum_{j_1,j_2}\langle \delta\varphi^2_{(j_1j_2)}\rangle_T\right]^{1/2}. \quad (31)$$

Now, if we take into account Eq. (31) and the fact that the energy in the ground state is proportional to $\hbar\omega_\perp N$, we obtain the following relation:

$$\Delta\varphi = \frac{t^{1/2}}{N^{1/4}} d(g_N,\omega), \quad (32)$$

where t is the reduced temperature, $t = T/T_c$; the function $d(g_N,\omega)$ depends on the interaction constant g_N and rotation speed ω. Of course, $d(g_N,\omega)$ is also very strongly dependent on the vortex cluster symmetry and in the next section we will calculate it for some values of g_N and ω and vortex configurations.

Note that the harmonic approximation remains accurate only if the deviations of the positions of the vortices are much smaller than the characteristic distance between two neighboring vortices. The melting temperature can be defined through the Lindemann criterion.

III. INTERSHELL MELTING OF VORTEX CLUSTERS

If there are not many vortices in the system, they are situated in concentric shells. In the single-vortex state, a vortex occupies the center of the cloud. If the number of vortices v is more than 1, but less than 6, vortices are arranged in one shell. With further increasing of the number of vortices, one of the vortices jumps to the center of the cloud, whereas the others are still situated in the single shell [26]. Thermal fluctuations in these cases can lead only to radial displacements of the vortex positions, since there is only one shell in the system. However, when the number of vortices is increased, they are arranged in two shells. For instance, it was shown in Ref. [21] that in a phase with ten vortices, two of them are situated in the inner shell and eight are arranged in the outer shell. For phases with a larger amount of vortices, their number in the inner shell can increase.

Here, we consider the process of intershell disordering in two-shell clusters containing 10, 11, 12, and 13 vortices, respectively. It would be more convenient for the comparison to calculate the melting temperatures for these configurations at the same value of rotation frequency. However, only one of these states, can be a true ground state, and if the system is not in a ground state, then sooner or later it will switch to the ground state due to thermal fluctuations. Therefore, we find melting temperatures for different vortex configurations at different, but quite close to each other rotation speeds, which correspond to ground states of the given configuration.

We choose the value of the gas parameter $g_N = 5$, as was explained in Sec. II, and find the ground states of the system. We have obtained that two-shell vortex clusters consisting of $v = 10$, 11, 12, and 13 vortices are energetically favorable in the vicinity of the point $\omega = 0.9$. For instance, the ground state of the system is represented by phases with 10, 11, 12, and 13 vortices at $\omega = 0.9$, 0.91, 0.92, and 0.94, respectively. In these cases, the inner shells contain $v_1 = 2$, 3, 3, and 4 vortices, whereas the outer shells have $v_2 = 8$, 8, 9, and 9 vortices, respectively. The density plots for these vortex phases are shown in Fig. 1. Let us calculate the deviations of the vortex positions for these states.

Using a technique presented in the previous section, we found that if we increase the temperature from zero, at first fluctuations of the relative phases ϕ_l of different harmonics of the order parameter become important and deviations of the positions of vortices are almost entirely due to fluctuations of ϕ_l and not due to fluctuations of C_l and R_l. This can be expected, since it is well known that fluctuations of the phase of the order parameter are more pronounced at relatively low temperatures and only at much higher temperatures does an amplitude of the order parameter start to fluctuate. Also, fluctuations of ϕ_l lead predominantly to azimuthal displacements of vortices; displacements in a radial direction are much smaller. This reflects the fact that the temperature of the intershell melting is much lower than that of the radial melting. We define an intershell melting tem-

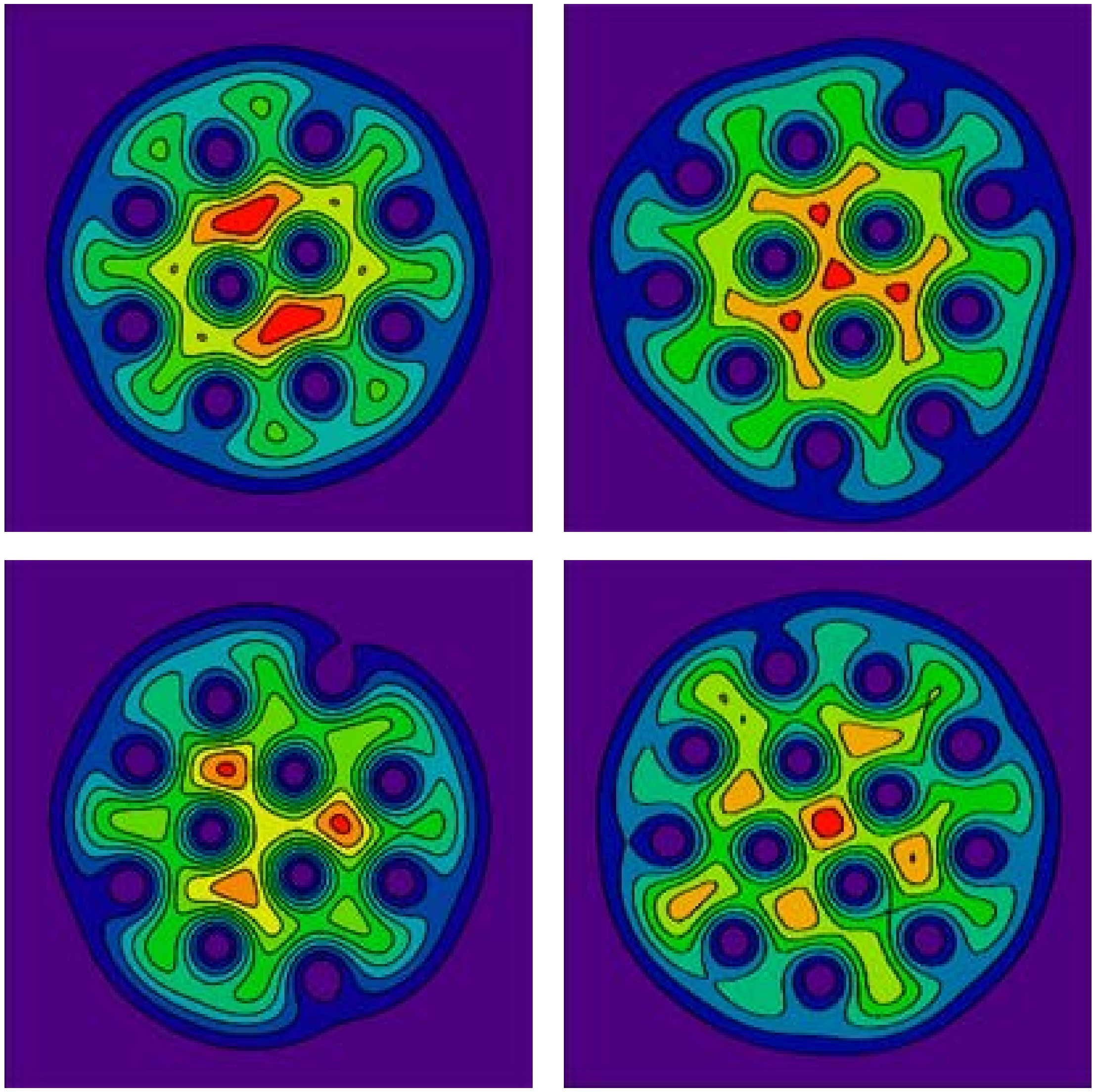

FIG. 1. (Color online). Density plots for the states with 10, 11, 12, and 13 vortices. Dark spots correspond to vortices.

perature t_{melt} of the cluster as a temperature at which $\Delta\varphi$ is equal to $\gamma\frac{360}{v_2}$, where $\gamma \sim 0.1$ is a characteristic number from the Lindemann criterion:

$$t_{melt} = \left(\frac{360}{n_2 d(g_N,\omega)}\gamma\right)^2 \sqrt{N}. \quad (33)$$

The factor $360/v_2$ in Eq. (33) reflects the fact that the two-shell cluster is invariant under the rotation of shells with respect to each other on an angle $360°/v_2$. We have calculated the values of d for clusters with 10, 11, 12, and 13 vortices. Our results are $d(g_N,\omega) \approx 608°$, 3500°, 123°, and 810° for 10-, 11-, 12-, and 13-vortex clusters, respectively. We can see that 10- and 12-vortex clusters are most stable among the analyzed configurations and the average angle between the shells is less than in other cases. This is because the number of vortices in the outer shell v_2 is divisible by v_1. Intuitively, it is clear that the stability of a cluster with v_2 divisible by v_1 depends also on the ratio v_2/v_1, since in the limit $v_2/v_1 \to 1$, each vortex in the inner shell corresponds to one vortex from the outer shell. Probably, this is the reason why the 12-vortex cluster is more stable than the 10-vortex configuration. At the same time, 11- and 13-vortex clusters are the most unstable among those considered here, since v_2 and v_1 are incommensurate and the deviation of shells with respect to each other is the largest. Note that with changing of ω with fixed g_N, $d(g_N,\omega)$ increases, in accordance with calculations [11] for Tkachenko modes. One can see from Eq. (34) and our estimates for $d(g_N,\omega)$ that the 12-vortex cluster is not melted and remains stable at $N=10^3$ and $t \approx 0.1$, whereas in the other cases a displacement angle between different shells is comparable with the angle between the two neighboring vortices in the outer shell and therefore the shells are decoupled. The difference in melting temperatures for 12- and 11-vortex clusters is several orders of magnitude.

Experimentally, melting of vortex clusters can be studied by tuning of ω at fixed g_N and N. After obtaining a desirable vortex configuration, one can also tune T and reach a melting range of temperatures. Vortex positions can be found by the free expansion technique, and after repeating this procedure

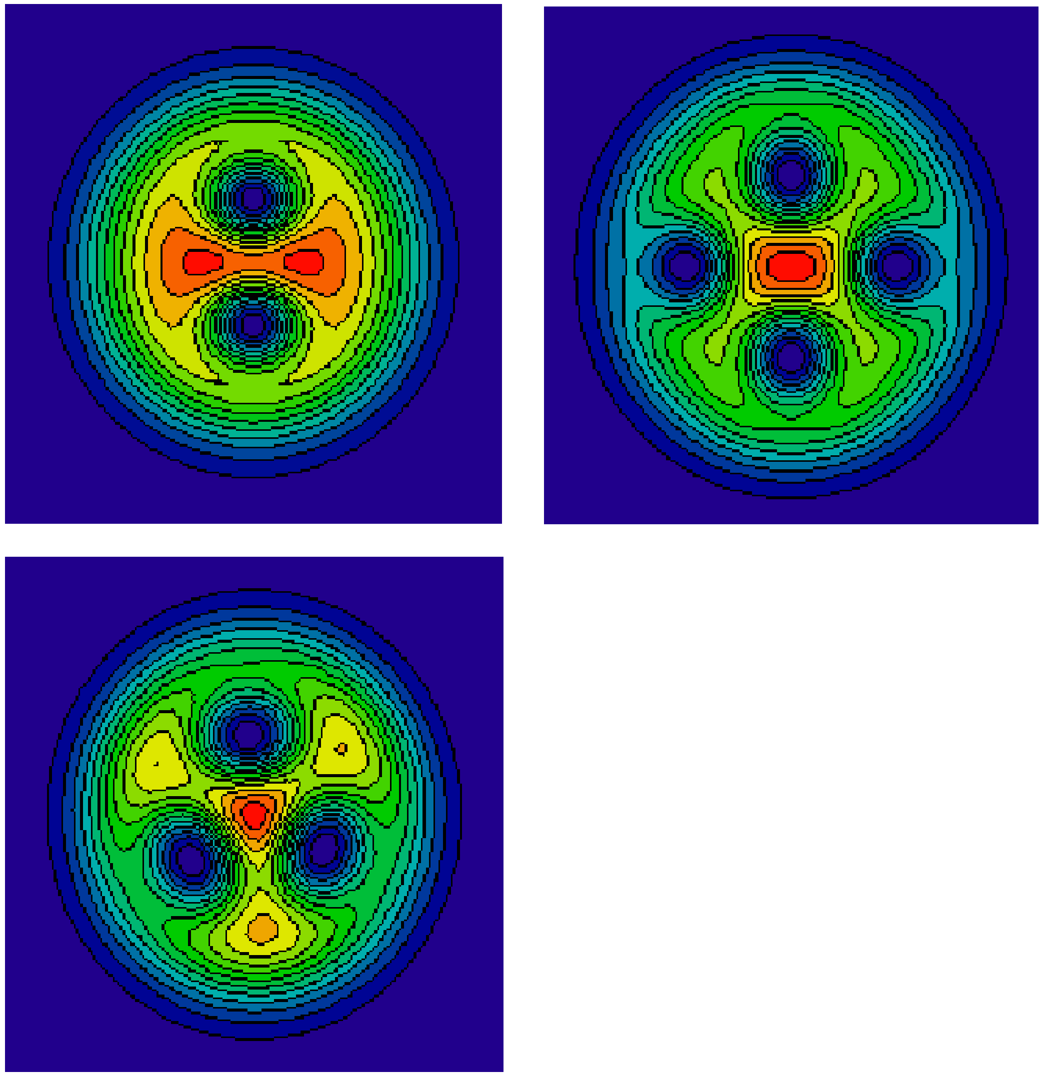

FIG. 2. (Color online). Density plots for the states with two, three, and four vortices in a trap with small quadrupole deformation. Dark spots correspond to vortices.

one can determine the average deviation of the vortex positions from the equilibrium. It is also possible to use a Bragg spectroscopy for the systems containing much larger vortex arrays than those considered here.

IV. VORTICES IN A TRAP WITH QUADRUPOLE DEFORMATION

In this section we consider the effect of thermal fluctuations on vortices in the trap with a quadrupole deformation, which breaks rotational symmetry. Such a deformation is often used in experiments to facilitate the creation of vortices. In fact, it introduces a preferable direction for the arrangements of vortices acting as a source of orientational pinning for a vortex cluster. The additional quadrupolar pinning potential is given by

$$U_{quadr}(r) = \frac{\varepsilon m \omega_{\perp}^2 r^2 \cos 2\varphi}{2}, \qquad (34)$$

where ε is a small coefficient, $\varepsilon \ll 1$. At zero temperature quadrupole deformation of the trap potential fixes the azimuthal positions of the vortices, whereas thermal fluctuations lead to displacements of the vortices, which depend on ε and T. Note that, in the case of a two-shell cluster at $\varepsilon = 0$, considered in the previous section, pinning centers are created by each vortex shell for another shell and the total rotational symmetry is preserved. The additional potential (34) leads to the following contribution to the energy (6):

$$\frac{F_{quadr}}{\hbar \omega_{\perp} N} = 2\pi\varepsilon \sum_m c_m c_{m+2} \frac{R_{m+2}^{m+4} R_m^{m+6}}{(R_{m+2}^2 + R_m^2)^{m+3}} \times \Gamma(m+3)\cos(\phi_{m+2} - \phi_m). \qquad (35)$$

This term relates the phases and amplitudes of different harmonics of the order parameter with angular momenta different by 2 from each other. Here we analyze the situation when there are $v=2$, 3, or 4 vortices in the trap. The density plots for these vortex states are presented in Fig. 2. The direction of the minimum of the trapping potential is vertical. We found that at small quadrupole deformation $\varepsilon \lesssim 0.1$, fluctuations of the relative phases of the order parameter Furrier harmonics are much stronger than fluctuations of their amplitudes, similarly to the case considered in the previous section. Therefore, we apply the same ideas to the present problem. Fluctuations of the relative phases lead mostly to displacements of the vortices in the azimuthal direction. These are so-called scissors modes, which are responsible for such oscillations [27–30].

In the limit of small ε, one can consider F_{quadr} as a perturbation to the energy of the system with $\varepsilon=0$. If there are two vortices in the system, the nonperturbed order parameter contains all harmonics divisible by 2 and the main contribution to the energy is given by harmonics with $l=0$ and 2. The amplitudes of these harmonics are of ε^0 order of magnitude, the relative angles between them are fixed, and the energy is degenerate with respect to ϕ_2, which reflects the fact that vortices can rotate freely (Goldstone mode). Quadrupole deformation connects ϕ_0 and ϕ_2, ϕ_2 and ϕ_4, ϕ_4 and ϕ_6, etc. As a result, $F_{quadr}\sim\varepsilon^1$ and the angle of deviation of the vortex cluster due to thermal fluctuations is given by

$$\Delta\varphi = \frac{t^{1/2}}{\varepsilon^{\tau}N^{1/4}}d(g_N,\omega), \tag{36}$$

where $d(g_N,\omega)$ is a function, independent of t; $\tau=0.5$.

If there are four vortices in the system, then at $\varepsilon=0$ the order parameter contains all the harmonics divisible by 4. However, these harmonics cannot be related through Eq. (35), since their angular momenta should differ by 2 and not by 4. In this case, coupling of vortex cluster to the quadrupole deformations occurs in the next order of ε. Namely, quadrupole deformation induces a harmonic with $l=2$, whose amplitude is of the order of ε^1, and finally $F_{quadr}\sim\varepsilon^2$, and we again arrive at Eq. (36) with $\tau=1$.

Now we consider the situation, when there are three vortices in the system. Again, a nonperturbed order parameter consists of contributions with l's divisible by 3, and their phases are not related by Eq. (35), as in the previous case with four vortices. In the next approximation with respect to ε, quadrupole deformation induces other harmonics with all integer l's and the amplitudes of these harmonics are of the order of ε^1. This is possible because any integer l, which is not divisible by 3, can be represented as $3l\pm2$, and therefore it can be obtained by adding or subtraction 2 from $3l$. However, it turns out that the energy even in the second order of ε is again degenerate and this degeneracy is removed only in the next order of ε. After all, $F_{quadr}\sim\varepsilon^3$ and Eq. (36) is again valid but with $\tau=1.5$.

We see that the symmetry of the vortex configuration dictates the asymptotic behavior of the pinning energy and the average deviation of vortex cluster at $\varepsilon\to0$. The cluster with two vortices is most strongly pinned, whereas the cluster with three vortices is the most unstable. This effect reflects the commensurability of the angular momenta of the quadrupole deformation ($l=2$) and of the order parameter harmonics, responsible for vortices ($l=2, 3, 4$). The strongest pinning is observed when these momenta are equal to each other (two-vortex state), less stronger pinning when these momenta are commensurate, but not equal (four-vortex phase), and the most weak pinning for the incommensurate case (three-vortex state). Note that similar scaling relations for the frequencies of scissors modes in two- and three- vortex states were obtained recently in Ref. [30].

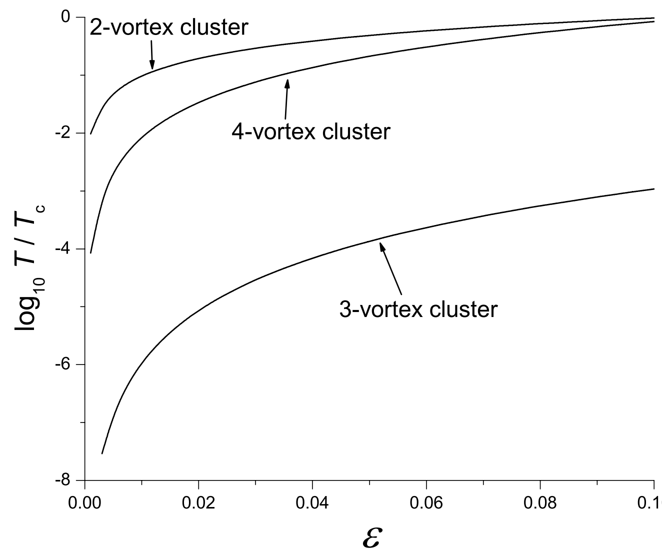

FIG. 3. The phase diagram of the two-, three-, and four-vortex clusters in the trap with the quadrupole deformation at $N=1000$. Above these lines the cluster is decoupled from the deformation and below it is coupled.

Next we calculate values of $d(g_N,\omega)$ for two-, three-, and four-vortex clusters at $g_N=5$ and $\omega=0.68$, 0.75, and 0.78, respectively, where these vortex configurations are energetically favorable, according to our calculations. Our results are $d(g_N,\omega)\approx65°$, 130°, and 11° for two-, three-, and four-vortex states, respectively. If the deviation angle $\Delta\varphi$ becomes of the same order as the angle between two neighboring vortices in the cluster, $2\pi/v$, we will treat this vortex cluster as being depinned from the quadrupole deformation. By using Eq. (36) and this condition, one can easily obtain a phase diagram in (t,ε) space. An example of the phase diagram is presented in Fig. 3 for $g_N=5$ and $N=1000$. Each line determines the boundary between the pinned and unpinned vortex clusters for a given vortex configuration. Below these lines, the vortex cluster is pinned and above it is unpinned. One can see that the region of stability of a two-vortex cluster is much broader than that for the two other configurations.

V. CONCLUSIONS

In this paper, we studied the effect of thermal fluctuations on small vortex clusters in harmonically trapped rotating Bose-Einstein condensates at temperatures much lower than the critical temperature. First, we considered the clusters consisting of two concentric shells of vortices. These were 10-, 11-, 12-, and 13-vortex structures. We obtained that with increasing the temperature from zero, first an order between the positions of vortices from different shells is destroyed, whereas the order within each shell is preserved. By using a Lindemann criterion, we defined the temperature, corresponding to the decoupling of two shells of vortices with respect to each other, which determines an order-disorder transition. This "melting" temperature is strongly dependent on the commensurability of the number of vortices in shells; less commensurate clusters have a lower melting temperature. For instance, the melting temperatures for the 11-vortex cluster consisting of two shells with 3 and 8 vortices and for the commensurate 12-vortex cluster with 3 and 9 vortices in shells differ in several orders of magnitude. An intershell order-disorder transition can be observed at experimentally attainable ranges of parameters. *We have shown that intershell melting in atomic condensates can occur at very low temperatures, especially for incommensurate clusters.*

Also studied are vortex clusters in the trap with small quadrupole deformation of the trapping potential, which acts as an orientational pinning center for vortices. We have analyzed the case of two, three, and four vortices in the system. We have demonstrated that the pinning energy depends very strongly on the number of vortices in the system. With the quadrupole deformation ε tending to zero, the pinning energy becomes proportional to ε^{υ}, where the coefficient $\upsilon=1$, 3, and 2 for two-, three-, and four-vortex configurations, respectively. This is due to the commensurability between the angular momenta $l=2$ transferred to the system by the quadrupole deformation and $l=2$, 3, and 4, responsible for the creation of two, three, and four vortices, respectively. Average deviation angles between the vortex cluster and the trap anisotropy direction diverge in different power laws with tending ε to zero for different vortex configurations, $\Delta\varphi \sim \varepsilon^{\tau}$, with $\tau=0.5$, 1.5, and 1 for two-, three-, and four-vortex clusters, respectively.

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