Optical implementation of the encoding of two qubits to a single qutrit

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We have devised an optical scheme for the recently proposed protocol for encoding two qubits into one qutrit. In this protocol, Alice encodes an arbitrary pure product state of two qubits into a state of one qutrit. Bob can then restore either of the two encoded qubit states error-free but not both of them simultaneously. We have successfully realized this scheme experimentally using spatial-mode encoding. Each qubit (qutrit) was represented by a single photon that could propagate through two (three) separate fibers. We theoretically propose two generalizations of the original protocol. We have found a probabilistic operation that enables us to retrieve both qubits simultaneously with average fidelity above 90% and we have proposed an extension of the original encoding transformation to encode N qubits into one $(N+1)$-dimensional system.

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I. INTRODUCTION

A full description of the most elementary two-level quantum system (a qubit) in a pure state requires two real numbers—the angular coordinates of the point on the surface of the Bloch sphere that specifies the state. Although an infinite amount of classical information is required for its full characterization, the qubit can be used to transmit only a finite amount of classical information—a single bit at most. This is an example of the celebrated Holevo bound [1] which states that no more than $\log_2 d$ bits of information can be extracted from a single copy of a d-level quantum system. The deep reason for this lies in the process of quantum measurement, which cannot perfectly discriminate among nonorthogonal quantum states, and consequently the optimal strategy to encode information is to prepare the system in one out of d orthogonal states.

Going beyond qubits, a qutrit (three-level system) in a pure state is specified by four real numbers (up to an irrelevant overall phase). This is the same number of parameters that is necessary to specify two uncorrelated qubits in a pure product state. Based on this observation, Grudka and Wojcik (GW) investigated whether a product state of two qubits could be somehow encoded onto a single qutrit [2]. Interestingly, they showed that this is indeed possible, at least to a certain extent. In particular they proposed an encoding and decoding strategy which allows perfect extraction of either the first or the second qubit from the qutrit with average probability 2/3.

In the present paper we report on the experimental demonstration of the GW protocol for optical qubits represented by single photons propagating in two different optical fibers. This way of encoding has been used quite rarely so far but it is very suitable for our present purpose. In particular, it allows for a very natural transition from qubits to a qutrit encoded into a path of a single photon which can propagate in three different optical fibers. This approach might thus be advantageous in all situations where one wishes to exploit higher-dimensional Hilbert spaces.

In addition to the experimental realization of the original GW protocol, we have also extended it in several ways. First, we have theoretically demonstrated that it is possible, with a certain probability, to extract from the qutrit *both* qubits simultaneously. Of course, this decoding is imperfect but the average fidelity of the decoded qubits, $F\approx 0.9024$, is surprisingly high. We have also shown that the protocol can be easily generalized for several qubits or even qudits.

The rest of the paper is organized as follows. In Sec. II we briefly review the GW protocol and present the optimal procedure for probabilistic simultaneous decoding of both qubits. We also describe here the extension of the protocol to more than two qubits and to qudits. The experimental setup is described in Sec. III and the experimental results are presented in Sec. IV. The paper ends with a brief summary and conclusions in Sec. V.

II. THEORY

Let us begin by briefly reviewing the GW protocol. Consider two qubits labeled 1 and 2 prepared in pure states,

$$|\psi_1\rangle = \alpha_1|0\rangle_1 + \beta_1|1\rangle_1, \quad |\psi_2\rangle = \alpha_2|0\rangle_2 + \beta_2|1\rangle_2. \tag{1}$$

Let $|0\rangle$, $|1\rangle$, $|2\rangle$ be the basis in the Hilbert space of the qutrit into which we will encode the qubits. The probabilistic encoding operation then consists of the following mapping:

$$|0\rangle_1|0\rangle_2 \to |0\rangle,$$
$$|0\rangle_1|1\rangle_2 \to |1\rangle,$$
$$|1\rangle_1|1\rangle_2 \to |2\rangle. \tag{2}$$

Note that the state $|1\rangle_1|0\rangle_2$ is filtered out in the mapping, which is necessary in order to accommodate the two qubits into a single qutrit. Consequently, the normalized state of the qutrit after the encoding reads

$$|\Psi\rangle = \frac{1}{\sqrt{\mathcal{N}}}(\alpha_1\alpha_2|0\rangle + \alpha_1\beta_2|1\rangle + \beta_1\beta_2|2\rangle), \tag{3}$$

where $\mathcal{N}=1-|\beta_1|^2|\alpha_2|^2$ is the probability of successful encoding. In the decoding procedure, the qutrit is projected onto a two-dimensional subspace. The projectors corresponding to the decoding of the first or the second qubit read

$$\Pi_{1+} = |1\rangle\langle 1| + |2\rangle\langle 2|, \quad \Pi_{1-} = |0\rangle\langle 0|,$$

$$\Pi_{2+} = |0\rangle\langle 0| + |1\rangle\langle 1|, \quad \Pi_{2-} = |2\rangle\langle 2|. \tag{4}$$

Projection onto Π_{j+} indicates successful decoding of the jth qubit while Π_{j-} signals failure. Considering the decoding of the first qubit, the resulting state is given by

$$|\psi_{\text{out}}\rangle = \Pi_{1+}|\Psi\rangle = \frac{1}{\sqrt{\mathcal{N}}}\beta_2(\alpha_1|1\rangle + \beta_1|2\rangle); \tag{5}$$

hence $|\psi_{\text{out}}\rangle \propto |\psi_1\rangle$ and the decoding is perfect. The probability of the successful decoding of the first qubit is $|\beta_2|^2/\mathcal{N}$. Assuming uniform distribution of the qubits on the surface of the Bloch sphere, the average probability of successful encoding and decoding is $1/2$. The same holds for the extraction of the second qubit.

The above described procedure allows one to perfectly extract one qubit from the qutrit. We have investigated also an alternative decoding strategy, where both qubits are retrieved simultaneously. This unavoidably introduces some noise. Let ρ_j denote the (generally mixed) state of the jth retrieved qubit; then we can quantify the quality of the decoding procedure by the fidelities $F_1=\overline{F_1(\psi_1)}$ and $F_2=\overline{F_2(\psi_2)}$ averaged over all possible input states, where $F_1(\psi_1)=\langle\psi_1|\rho_1|\psi_1\rangle$ and $F_2(\psi_2)=\langle\psi_2|\rho_2|\psi_2\rangle$. We have concentrated on a symmetric retrieval where $F_1=F_2=F$ and with the help of the techniques introduced in Ref. [3] we have determined the optimal probabilistic decoding operation that maximizes the average fidelity F. This operation explicitly reads

$$|0\rangle \to \frac{1}{\sqrt{2}}|0\rangle_1|0\rangle_2,$$

$$|1\rangle \to |0\rangle_1|1\rangle_2,$$

$$|2\rangle \to \frac{1}{\sqrt{2}}|1\rangle_1|1\rangle_2, \tag{6}$$

and the corresponding average fidelity is

$$F = \frac{4+\sqrt{2}}{6} \approx 0.9024. \tag{7}$$

Note that the operation (6) is not a direct inversion of the encoding transformation (2) since it involves the prefactors $1/\sqrt{2}$. The average probability of success of encoding and decoding operations (2) and (6) reads $P=1/2$. Note also that the procedure is not covariant and various states are encoded and decoded with different probabilities and fidelities. For a given fixed state of the first qubit $|\psi_1\rangle=\cos\frac{\vartheta}{2}|0\rangle_1+e^{i\varphi}\sin\frac{\vartheta}{2}|1\rangle_1$ the probability of success of the joint encoding-decoding procedure averaged over all possible states of the second qubit is given by

$$P_1(\vartheta) = \frac{1}{4} + \frac{1}{2}\cos^2\frac{\vartheta}{2}, \tag{8}$$

and the corresponding normalized fidelity of the retrieved state reads

$$F_1(\vartheta) = \frac{1+2\cos^4\frac{\vartheta}{2} + \frac{\sqrt{2}-1}{2}\sin^2\vartheta}{1+2\cos^2\frac{\vartheta}{2}}. \tag{9}$$

Note that P_1 and F_1 do not depend on the phase φ.

The scheme can also be extended to probabilistic encoding and decoding of N qubits $|\psi_j\rangle=\alpha_j|0\rangle_j+\beta_j|1\rangle_j$, $j=1,\ldots,N$, into an $(N+1)$-dimensional state. The encoding strategy is a straightforward generalization of the two-qubit procedure (2),

$$\bigotimes_{k=1}^{j}|1\rangle_k \bigotimes_{l=j+1}^{N}|0\rangle_l \to |j\rangle, \quad j=0,\ldots,N. \tag{10}$$

This transformation produces the following state of the $(N+1)$-dimensional system:

$$|\Psi_{N+1}\rangle = \sum_{j=0}^{N}\prod_{k=1}^{j}\beta_k\prod_{l=j+1}^{N}\alpha_l|j\rangle. \tag{11}$$

The positive-operator-valued measure (POVM) which decodes the nth qubit has the form

$$\Pi_{n+} = |n-1\rangle\langle n-1| + |n\rangle\langle n|,$$

$$\Pi_{n-} = I - \Pi_{n+}, \tag{12}$$

where I denotes the identity operator.

The protocol can be further generalized to qudits. If we possess N qudit states then we can encode them to one $[N(d-1)+1]$-dimensional system in such a way that an arbitrary single qudit can be perfectly extracted from that state with a certain probability. The principle is the same as in the previous cases. To illustrate it, let us consider encoding of two qutrits

$$|\phi_1\rangle = \alpha_1|0\rangle_1 + \beta_1|1\rangle_1 + \gamma_1|2\rangle_1,$$

$$|\phi_2\rangle = \alpha_2|0\rangle_2 + \beta_2|1\rangle_2 + \gamma_2|2\rangle_2 \tag{13}$$

into the state of a five-dimensional system. The encoding strategy can be expressed as follows:

$$|0\rangle_1|j\rangle_2 \to |j\rangle, \quad j=0,1,2,$$

$$|1\rangle_1|2\rangle_2 \to |3\rangle,$$

$$|2\rangle_1|2\rangle_2 \to |4\rangle. \tag{14}$$

This results in a five-dimensional state which carries both qutrits,

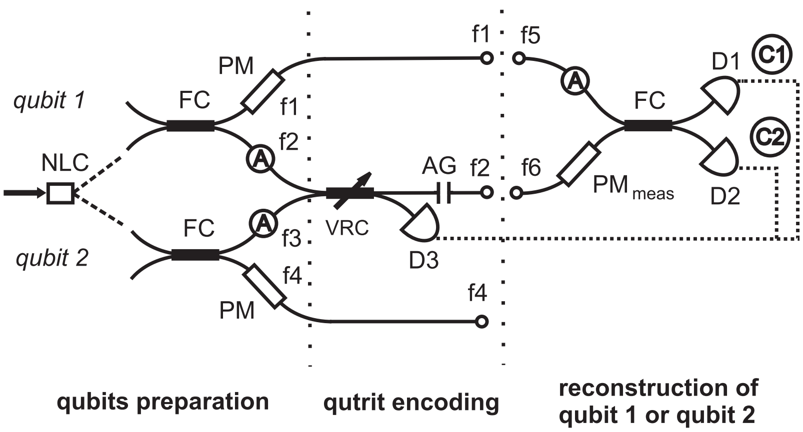

FIG. 1. Setup of the experiment. NLC denotes a nonlinear crystal, A an attenuator, FC a fiber coupler, PM a phase modulator, AG an adjustable air gap, VRC a variable-ratio coupler, and D1–D3 denote detectors.

$$|\Phi\rangle \propto \alpha_1\alpha_2|0\rangle + \alpha_1\beta_2|1\rangle + \alpha_1\gamma_2|2\rangle + \beta_1\gamma_2|3\rangle + \gamma_1\gamma_2|4\rangle. \tag{15}$$

The decoding POVM elements can be easily inferred from the structure of the state (15) and we obtain

$$\Pi_{1+} = \sum_{j=2}^{4} |j\rangle\langle j|, \quad \Pi_{2+} = \sum_{j=0}^{2} |j\rangle\langle j|. \tag{16}$$

III. DESCRIPTION OF THE EXPERIMENT

Our experimental setup is shown in Fig. 1. The experiment is based on the interplay of the second-order and fourth-order interference. To prepare the two qubits we utilize photon pairs generated by the process of type-I spontaneous parametric down-conversion in a 10-mm-long $LiIO_3$ nonlinear crystal (NLC) pumped by a krypton-ion cw laser (413.1 nm, 180 mW). Down-converted beams filtered by cutoff filters and circular apertures are coupled into single-mode optical fibers. The photons in each generated pair are tightly time correlated. Each of the photons enters a fiber coupler (FC) and splits into two channels. The first qubit is represented by a single photon in fibers f1 and f2 while the second qubit corresponds to a single photon in fibers f3 and f4. The final state of each qubit is determined by the adjustment of intensity ratio and phase shift between these channels. To control the intensity ratio we use a balanced fiber coupler followed by an attenuator (A) in one of the output arms (instead of a fiber coupler with a variable splitting ratio). That is, we use a conditional preparation procedure. The phase difference is set by a phase modulator (PM). Equatorial states of qubits are realized simply by changing just the phase difference whereas the losses in both fibers are balanced.

The compression of two qubits into one qutrit is realized by a variable-ratio coupler (VRC) and a subsequent measurement. This operation is probabilistic and it succeeds only if detector D3 fires. The VRC mixes modes f2 and f3. If a signal on detector D3 is detected there cannot be more than one photon in the remaining fibers f1, f2, and f4. These three modes constitute a qutrit. If the described operation is successful the (non-normalized) state of the qutrit reads

$$\sqrt{T}\alpha_1\alpha_2|001\rangle_{\text{f1f2f4}} + (R-T)\alpha_1\beta_2|010\rangle_{\text{f1f2f4}} + \sqrt{R}\beta_1\beta_2|100\rangle_{\text{f1f2f4}}, \tag{17}$$

where R denotes the reflectance and $T=1-R$ the transmittance of the VRC (R applies between ports f2-f2 and f3-f3 and T between f2-f3 and f3-f2), $|001\rangle_{\text{f1f2f4}}$ denotes the state with one photon in mode f4, etc. To acquire the state of the qutrit in the form given by Eq. (3), suitable for the reverse decoding, a filtration procedure represented by additional attenuations has to be applied. The damping factor $\eta_1=(R-T)^2/R$ must be applied in mode f1, whereas $\eta_4=(R-T)^2/T$ in mode f4. Thus the probability of success of the whole encoding transformation (including the filtration) is $P=\eta_{\text{det}}(T-R)^2(1-|\beta_1|^2|\alpha_2|^2)$, where the efficiency of detector D3, η_{det}, is also taken into account. The average probability of successful encoding (assuming the uniform distribution of the states of qubits on the Bloch sphere) reads $\bar{P}=\frac{3}{4}\eta_{\text{det}}(R-T)^2$. Now we want to find a splitting ratio of the VRC that maximizes this probability. Of course, it is restricted by the conditions $\eta_1, \eta_4 \leqslant 1$. The optimal value of the splitting ratio is $R=1/4$, $T=3/4$. The corresponding damping factors read $\eta_1=1$ (i.e., no attenuation) and $\eta_4=1/3$ (in the experiment η_4 is set by increasing the attenuation of the attenuator in fiber f5). For $R=1/4$, $T=3/4$ the average probability of success reads $\bar{P}=3\eta_{\text{det}}/16$.

Since there is a nonzero probability that both photons impinge on detector D3 and a nonzero probability that the "qutrit" photon is lost due to the damping factor η_4, the situation may occur when detector D3 clicks but the qutrit is not created. The conditional probability of successful encoding provided that detector D3 has fired reads

$$P_{\text{cond}} = (R-T)^2(1-|\beta_1|^2|\alpha_2|^2)\,[T|\alpha_1|^2|\alpha_2|^2 + (R-T)^2|\alpha_1|^2|\beta_2|^2 + R|\beta_1|^2|\beta_2|^2 + RT(2-\eta_{\text{det}})|\alpha_1|^2|\beta_2|^2]^{-1}. \tag{18}$$

We can see that an unbalanced coupler lies at the heart of the encoding procedure. In this context it is worth noting that unbalanced beam splitters find several applications in optical quantum-information processing ranging from the realization of the quantum logic gates [4–6] to optimal universal [7,8] and phase-covariant [9] cloning of single-photon states.

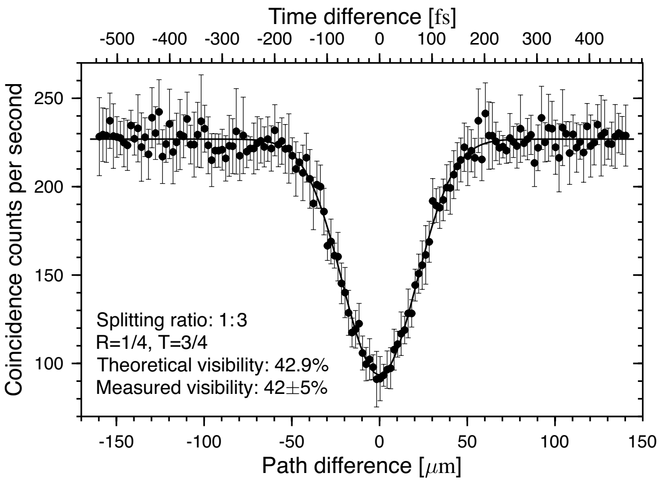

FIG. 2. Hong-Ou-Mandel dip for the splitting ratio 25:75. The full line shows a Gaussian fit.

The state of the created qutrit can be, in principle, checked by a tomographic reconstruction combining direct and interferometric measurements on the three modes; namely, by measuring in bases $\{|0\rangle,|1\rangle,|2\rangle\}$, $\{(|0\rangle\pm|1\rangle)/\sqrt{2},|2\rangle\}$, $\{(|0\rangle\pm i|1\rangle)/\sqrt{2},|2\rangle\}$, $\{(|0\rangle\pm|2\rangle)/\sqrt{2},|1\rangle\}$, $\{(|0\rangle\pm i|2\rangle)/\sqrt{2},|1\rangle\}$, $\{(|1\rangle\pm|2\rangle)/\sqrt{2},|0\rangle\}$, and $\{(|1\rangle\pm i|2\rangle)/\sqrt{2},|0\rangle\}$ one could fully reconstruct the qutrit state from the recorded data. We are not primarily interested in this task, as our aim is to show that after the successful encoding procedure it is possible to recover a state of either of the two encoded qubits. However, knowing the fidelities of the decoded qubits, we are also able to estimate a lower bound on the fidelity of the intermediate qutrit as described in the next section.

From the qutrit one can extract error-free either of the two original qubits but not both of them. Choosing fibers f1 and f2 one can decode qubit 1, whereas selecting f2 and f4 qubit 2 can be obtained. This procedure is also probabilistic as there is nonzero probability that the photon is in the remaining fiber. To check the states of decoded qubits we use an interferometric measurement. Connecting fiber f1 with f5 and f2 with f6 the decoded qubit 1 is verified whereas by connecting f2-f6 and f4-f5 qubit 2 can be checked. Of course, only the cases when the encoding procedure successfully occurred are taken into account.

The whole process, including encoding and decoding of information, depends on the quality of the fourth- and second-order interference. Before starting the measurement it is necessary to adjust the Hong-Ou-Mandel (HOM) interferometer [10] formed by the VRC. First we set the VRC splitting ratio to 50:50 and adjusted the precise time overlap of the two photons at the VRC and tuned their polarizations (this was done by mechanical fiber polarization controllers not shown in the scheme). The visibility of the HOM dip was about 98%. Then we changed the VRC splitting ratio to 25:75. Figure 2 shows the HOM dip measured with this splitting ratio. During this measurement the attenuator in fiber f5 was closed and the coincidences between detectors D1 and D3 and between D2 and D3 were counted. In the graph the sum of these two coincidence counts is plotted. Each point was averaged over eight 1 s measurements. For comparison the theoretical value of visibility is 42.9%.

The fiber-based Mach-Zehnder (MZ) interferometer was adjusted by the following procedure: Only one beam from the nonlinear crystal was used; the other one was blocked. To set the same optical lengths of the arms of the interferometer an adjustable air gap (AG) was employed. Its precision is about 0.1 μm (the precise setting of the phase difference was then done by an electro-optical phase modulator $\mathrm{PM_{meas}}$). First, losses in both arms of the interferometer were balanced and the polarization states in both arms were aligned (it was done by mechanical fiber polarization controllers that are not shown in the scheme). In this setting we reached visibilities above 97%. Then we unbalanced the losses to take into account the reflectivity of the VRC ($R=1/4$) and the damping factor η_1 or η_4. Then we opened both inputs again and let both photons from each pair come into the system.

Fluctuations of temperature and temperature gradients cause changes of the refractive indices of fibers. This is the reason for the substantial instability of the interference pattern. Therefore the interferometer must be thermally isolated (we use polystyrene boxes). However, this is not sufficient—the phase difference between the arms of the interferometer still drifts in time by about $\pi/1000$ per second on average. The environmental perturbations may further be limited by means of active stabilization (the active stabilization does not directly decrease the disturbing influence of the environment but it periodically sets the proper value of the phase difference). In the experiment 5 s measurement blocks are alternated with stabilization cycles, each taking also about 5 s on average. In each stabilization cycle the value of the phase drift is estimated and it is compensated by means of the phase modulator $\mathrm{PM_{meas}}$. So the phase-difference drift during the 5 s measurement period usually does not exceed $\pi/200$. During the stabilization only one beam from the crystal is allowed to enter the system; the other one is blocked.

The experiment itself starts with the setting of input qubit states by means of attenuators and phase modulators in fibers f1, f2, f3, and f4. Then the proper measurement basis for the verification of the state of a reconstructed qubit is set by adjusting the attenuator and phase modulator in the measurement part of the setup (fibers f5 and f6). The measurement basis consists of the original state of the corresponding qubit and of the state orthogonal to it. Then the coincidences between detectors D1 and D3 and coincidences between D2 and D3 are counted. We use Perkin-Elmer single-photon-counting modules (employing silicon avalanche photodiodes with quantum efficiency $\eta\approx 60\%$ and dark counts about $50\ \mathrm{s}^{-1}$) and coincidence electronics based on time-to-amplitude converters and single-channel analyzers with a 2 ns coincidence window. In the ideal case the coincidences should be detected only between detectors D1 and D3—these events correspond to the detection of the original state of the qubit; we denote the corresponding coincidence rate as C^{+}. Coincidences between detectors D2 and D3 represent erroneous detections and we denote the corresponding coincidence rate as C^{-}. The fidelity of the reconstructed qubit state is then

$$F=\frac{C^{+}}{C^{+}+C^{-}}. \tag{19}$$

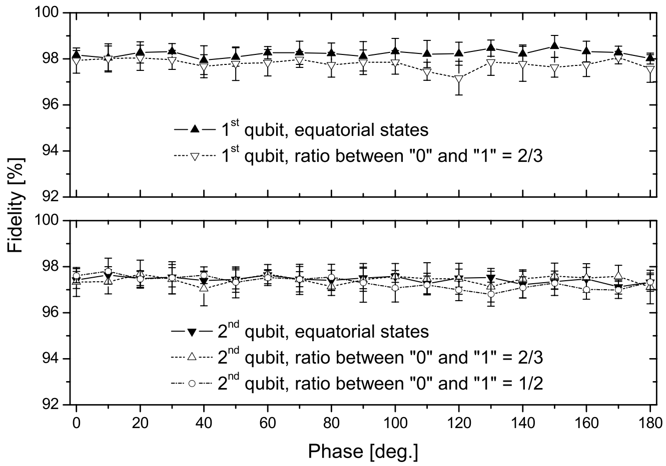

FIG. 3. Observed fidelities of reconstructed qubit states for various input states.

To realize experimentally the encoding of N qubits into one $(N+1)$-dimensional system one can repeatedly use the unit depicted in Fig. 1 (i.e., add other Mach-Zehnder interferometers interconnected by variable-ratio couplers). However, in such a case one would need N time-correlated photons.

IV. EXPERIMENTAL RESULTS

We encoded and decoded the states of qubits,

$$|\psi\rangle = \left(\cos\frac{\vartheta_1}{2}|01\rangle_{f1f2} + e^{i\varphi_1}\sin\frac{\vartheta_1}{2}|10\rangle_{f1f2}\right) \otimes \left(\cos\frac{\vartheta_2}{2}|01\rangle_{f3f4} + e^{i\varphi_2}\sin\frac{\vartheta_2}{2}|10\rangle_{f3f4}\right), \quad (20)$$

with the following parameters. For qubit 1 we checked 38 different states:

(1) $\vartheta_1=90°$ (i.e., $|\alpha_1|^2:|\beta_1|^2=1:1$), $\varphi_1=0°,10°,\ldots,180°$, ϑ_2,φ_2 were arbitrary;

(2) $\vartheta_1=78.46°$ (i.e., $|\alpha_1|^2:|\beta_1|^2=3:2$), $\varphi_1=0°,10°,\ldots,180°$, ϑ_2,φ_2 were arbitrary.

For qubit 2 we checked 57 different states:

(1) $\vartheta_2=90°$ (i.e., $|\alpha_2|^2:|\beta_2|^2=1:1$), $\varphi_2=0°,10°,\ldots,180°$, ϑ_1,φ_1 were arbitrary;

(2) $\vartheta_2=78.46°$ (i.e., $|\alpha_2|^2:|\beta_2|^2=3:2$), $\varphi_2=0°,10°,\ldots,180°$, ϑ_1,φ_1 were arbitrary;

(3) $\vartheta_2=70.53°$ (i.e., $|\alpha_2|^2:|\beta_2|^2=2:1$), $\varphi_2=0°,10°,\ldots,180°$, ϑ_1,φ_1 were arbitrary.

When we verified the state of the reconstructed qubit 1 the qubit 2 was prepared in an arbitrary state, and vice versa.

The observed fidelities of reconstructed qubit states are shown in Fig. 3. Each point was calculated as an average from ten 5 s measurements (we detected hundreds of coincidences C^+ per second—the exact numbers varied according to the attenuations). The fidelities exhibit values around 98% and are independent of the qubit states (i.e., constant within the statistical errors). The deviations from the ideal 100% fidelity are caused by misalignments, inaccuracies in polarization settings, limited precision of parameter settings, and a phase drift (during the measurement period) in the MZ interferometer. Note that our experiment substantially differs from a simple interference measurement with one unbalanced Mach-Zehnder interferometer. Such a one-photon interferometry would lead in our case to visibilities only 80% and 92% in the first (f1, f2) and second (f2, f4) interferometer, respectively. These visibilities would correspond to fidelities of only 90% and 96%.

The measured fidelities F_1 and F_2 of the two retrieved qubits provide information on the state of the intermediate qutrit represented by the single photon in modes f1, f2, and f4. Consider first the ideal case $F_1=F_2=1$. It follows that the density matrix of the qutrit ρ_q had to have the structure $\rho_q = |\Psi\rangle\langle\Psi| + \tilde{\rho}_{q,02}|0\rangle\langle 2| + \tilde{\rho}^*_{q,02}|2\rangle\langle 0|$, where $|\Psi\rangle$ is the ideal pure state of the qutrit given in Eq. (3). The density matrix ρ_q must be positive semidefinite, $\rho_q \geqslant 0$. It can be shown that if $\alpha_1\beta_2 \neq 0$ then this can hold only if $\tilde{\rho}_{q,02}=0$. Thus, if $F_1=F_2=1$ and if the first and second input qubits do not lie on the southern or northern pole of the Bloch sphere, respectively, then we know that the qutrit has been prepared in the pure state $|\Psi\rangle$.

The argument given above can be extended to the practically relevant case of $F_1,F_2<1$ and we can obtain a lower bound on the fidelity $\mathcal{F}=\langle\Psi|\rho_q|\Psi\rangle$ of the actually prepared qutrit state with respect to the ideal pure state $|\Psi\rangle$. The determination of the minimum qutrit fidelity $\mathcal{F}$ compatible with the experimentally observed fidelities F_1 and F_2 can be formulated as the minimization of $\mathcal{F}$ under the constraints

$$\rho_{q,11}|\alpha_1|^2 + \rho_{q,22}|\beta_1|^2 + \rho_{q,12}\alpha_1^*\beta_1 + \rho_{q,21}\alpha_1\beta_1^* = F_1(\rho_{q,11}+\rho_{q,22}),$$

$$\rho_{q,00}|\alpha_2|^2 + \rho_{q,11}|\beta_2|^2 + \rho_{q,01}\alpha_2^*\beta_2 + \rho_{q,10}\alpha_2\beta_2^* = F_2(\rho_{q,00}+\rho_{q,11}),$$

$$\mathrm{Tr}[\rho_q]=1, \quad \rho_q \geqslant 0. \quad (21)$$

The first two constraints fix the fidelity of the first and second retrieved qubits to F_1 and F_2, respectively, and the last two constraints guarantee that ρ_q is normalized and positive semidefinite. This minimization task is an instance of a convex optimization problem, namely, the so-called semidefinite program [15]. This problem can be very efficiently solved numerically [16] and the optimality of the obtained solution can be confirmed with the help of the duality lemma [15]. We performed the numerical minimization for the equatorial qubits. For fidelities $F_1=F_2=98\%$ we found that $\mathcal{F}_{\min}=94.7\%$ which confirms that the experimentally prepared qutrit state is indeed very close to the ideal pure state $|\Psi\rangle$. It is worth stressing that the lower bound on $\mathcal{F}$ was derived solely from the knowledge of the two fidelities F_1 and F_2.

V. CONCLUSIONS

We have experimentally realized the transformation for the encoding of two single-photon qubits into one qutrit and demonstrated that it is possible to restore (probabilistically but error-free) either of the two encoded qubit states. The principle of the experiment is based on the interplay of the second-order and fourth-order interference. We have reached fidelities around 98%. The deviations from the ideal 100% are caused by imperfections of the experimental setup. We have employed encoding to spatial modes, which has been used quite rarely so far but was very suitable for our purpose, since it allows for very natural transition from qubits to a qutrit encoded into path of a single photon propagating in three different optical fibers. Note that, in principle, the scheme could also work with other encodings that admit higher-dimensional Hilbert spaces, such as time-bin [11,12] or orbital angular momentum [13,14] encodings.

Further, we have proposed some generalizations of the encoding and decoding protocol. We have found a probabilistic operation that allows us to retrieve both qubits simultaneously with the average fidelity above 90%. We have also proposed an extension of the original encoding transformation to encode N d-dimensional systems into one $[N(d-1)+1]$-dimensional system. Implementation of the encoding of N qubits into one $(N+1)$-dimensional system can be done by repeatedly using Mach-Zehnder interferometers interconnected by variable-ratio couplers.

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