

# Engineering of arbitrary $U(N)$ transformations by quantum Householder reflections

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We propose a simple physical implementation of the quantum Householder reflection (QHR)  $M(v)=I-2|v\rangle\langle v|$  in a quantum system of  $N$  degenerate states (forming a qunit) coupled simultaneously to an ancillary (excited) state by  $N$  resonant or nearly resonant pulsed external fields. We also introduce the generalized QHR  $M(v; \varphi)=I+(e^{i\varphi}-1)|v\rangle\langle v|$ , which can be produced in the same  $N$ -pod system when the fields are appropriately detuned from resonance with the excited state. We use these two operators as building blocks in constructing arbitrary preselected unitary transformations. We show that the most general  $U(N)$  transformation can be factorized (and thereby produced) by either  $N-1$  standard QHRs and an  $N$ -dimensional phase gate, or  $N-1$  generalized QHRs and a one-dimensional phase gate. Viewed mathematically, these QHR factorizations provide parametrizations of the  $U(N)$  group. As an example, we propose a recipe for constructing the quantum Fourier transform (QFT) by at most  $N$  interaction steps. For example, the QFT requires a single QHR for  $N=2$ , and only two QHRs for  $N=3$  and 4.

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## I. INTRODUCTION

Coherent control of quantum dynamics traditionally involves scenarios for transfer of population, complete or partial, from one bound initial energy state to another, single or superposition state, or a continuum of states. Such techniques are well developed, particularly for two- and three-state systems, e.g.,  $\pi$  pulses [1], adiabatic passage using one or more level crossings [2], or stimulated Raman adiabatic passage and its extensions [3]. Essentially all these techniques start from a single initial state; such a state can be prepared experimentally, e.g., by optical pumping.

At the same time, in contemporary quantum physics implementations of specific propagators are often demanded; for example, some fields in quantum information lean heavily on the quantum Fourier transform [4]. Another example is quantum state engineering when a system starts in a coherent superposition of states; then one must construct the entire propagator, while the above techniques provide only some transition probabilities.

The implementation of such propagators is well understood and used for qubits, i.e., two-state quantum systems, upon which the theory of quantum information is primarily built [4]. On the other hand, qunits— $N$ -state quantum systems—offer some advantages. For example, a qubit can encode two continuous parameters: the population ratio of the two qubit states and the relative phase of their amplitudes. A qunit in a pure state can encode  $2(N-1)$  parameters ( $N-1$  populations and  $N-1$  relative phases), i.e., by using qunits, information can be encoded in significantly fewer particles than with qubits. This is beneficial for storing quantum information, which can be particularly important if the number of particles that can be used is restricted, e.g., due to decoherence [4]. Furthermore, there are indications that using qunits can improve error thresholds in fault-tolerant computation.

Physical realizations of qunit operations in the existing proposals [5], however, are difficult to implement. These

implementations use sequences of  $U(2)$  operations, i.e., transformations acting at each instance of time upon only two of the  $N$  states of the qunit. The general  $U(N)$  transformation of a qunit requires  $O(N^2)$  such  $U(2)$  operations [5]; hence the complexity increases rapidly with the qunit dimension  $N$ , which makes qunit manipulations challenging, even for qutrits ( $N=3$ ).

In this paper, we show that a general  $U(N)$  transformation can be implemented physically in a quantum system with only  $N$  interaction steps. For this purpose we introduce a compact quantum implementation, in a single interaction step, of the Householder reflection [6]. The latter is a powerful and numerically very robust unitary transformation, which has many applications in classical data analysis, e.g., in solving systems of linear algebraic equations, finding eigenvalues of high-dimensional matrices, least-squares optimization, QR decomposition, etc. [7]. The Householder transformation, acting upon an arbitrary  $N$ -dimensional matrix, produces an upper (or lower) triangular matrix by  $N-1$  operations. When the initial matrix is unitary, the resulting final matrix is diagonal, i.e., a phase gate or a unit matrix. We use this property to decompose an arbitrary  $U(N)$  matrix into Householder matrices and hence design a recipe for physical realization of a general  $U(N)$  transformation.

The quantum Householder reflection (QHR) consists of a single interaction step involving  $N$  simultaneous pulsed fields. In contrast to the existing  $U(2)$  realizations of qunit transformations, here each Householder reflection acts simultaneously upon many states:  $N$  states in the first step,  $N-1$  states in the second, etc. This allows us to greatly reduce the number of physical steps, from  $O(N^2)$  in  $U(2)$  realizations to only  $O(N)$  in our proposal.

We introduce two types of QHR: standard and generalized QHR; the latter involves an additional phase factor. The physical realizations of both use simultaneous pulses of precise areas in a system with an  $N$ -pod linkage pattern, the difference being that the standard QHR operates on exact resonance, whereas the generalized QHR requires specific

detunings. Any unitary matrix can be decomposed into  $N-1$  standard QHRs and a phase gate, or into  $N$  generalized QHRs, without a phase gate. This advantage of the generalized QHR implementation derives from the additional phase in each step, which delivers  $N$  additional phases in the end, thereby making the phase gate unnecessary.

This paper is organized as follows. In Sec. II we define the standard and generalized QHR gates and propose physical implementations. In Sec. III we describe the decompositions of a general  $U(N)$  matrix by means of standard and generalized QHRs, which provide the routes for realization of an arbitrary  $U(N)$  transformation. In Sec. IV we apply these decompositions to quantum Fourier transforms. The conclusions are summarized in Sec. V.

## II. QUANTUM HOUSEHOLDER REFLECTION

### A. Standard QHR

An  $N$ -dimensional quantum Householder reflection is defined as the operator

$$\mathbf{M}(v) = \mathbf{I} - 2|v\rangle\langle v|, \quad (1)$$

where  $|v\rangle$  is an  $N$ -dimensional normalized complex column vector and  $\mathbf{I}$  is the identity operator. The QHR (1) is Hermitian and unitary,  $\mathbf{M}(v) = \mathbf{M}(v)^\dagger = \mathbf{M}(v)^{-1}$ , which means that  $\mathbf{M}(v)$  is involutory,  $\mathbf{M}^2(v) = \mathbf{I}$ ; in addition,  $\det \mathbf{M}(v) = -1$ . If the vector  $|v\rangle$  is real,  $\mathbf{M}(v)$  has a simple geometric interpretation: reflection with respect to an  $(N-1)$ -dimensional plane with a normal vector  $|v\rangle$ ; in the complex case the interpretation is more involved. In general, the Householder vector  $|v\rangle$  is complex, which implies that it contains  $2(N-1)$  real parameters (taking into account the normalization condition and the unimportant global phase).

### B. Generalized QHR

We define the generalized QHR as

$$\mathbf{M}(v; \varphi) = \mathbf{I} + (e^{i\varphi} - 1)|v\rangle\langle v|, \quad (2)$$

where  $|v\rangle$  is again an  $N$ -dimensional normalized complex column vector and  $\varphi$  is an arbitrary phase. The standard QHR (1) is a special case of the generalized QHR (2) for  $\varphi = \pi$ :  $\mathbf{M}(v; \pi) \equiv \mathbf{M}(v)$ . The generalized QHR is unitary,

$$\mathbf{M}(v; \varphi)^{-1} = \mathbf{M}(v; \varphi)^\dagger = \mathbf{M}(v; -\varphi), \quad (3)$$

and its determinant is  $\det \mathbf{M} = e^{i\varphi}$ .

### C. Physical implementations

#### 1. Coherently driven $N$ -pod system

The standard and generalized QHRs have simple physical realizations. Consider the  $(N+1)$ -state system with  $N$  degenerate [in the rotating-wave approximation (RWA) sense [1]] ground states  $|n\rangle$  ( $n=1, 2, \dots, N$ ), which represent the qunit, coupled coherently and simultaneously by  $N$  external fields to an ancillary excited state  $|e\rangle \equiv |N+1\rangle$ , as shown in Fig. 1 [8]. Such an  $N$ -pod system can be formed, e.g., by coupling the magnetic sublevels of several  $J=1$  levels to a single  $J$

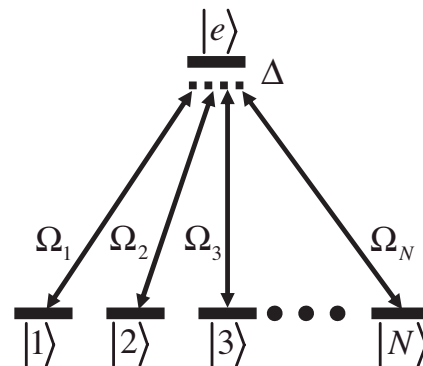


FIG. 1. Physical realization of the quantum Householder reflection:  $N$  degenerate (in the RWA sense) ground states, forming the qunit, coherently coupled via a common excited state by pulsed external fields of the same time dependence and the same detuning, but possibly different amplitudes and phases.

$=0$  level by polarized laser pulses [8]; for a qutrit only one  $J=1$  level suffices. The propagator  $U_{N+1}(t, t_0)$  of this system obeys the Schrödinger equation

$$i\hbar \frac{d}{dt} U_{N+1}(t, t_0) = \mathbf{H}(t) U_{N+1}(t, t_0), \quad (4)$$

with the RWA Hamiltonian [1]

$$\mathbf{H}(t) = \frac{\hbar}{2} \begin{bmatrix} 0 & 0 & \cdots & 0 & \Omega_1(t) \\ 0 & 0 & \cdots & 0 & \Omega_2(t) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \Omega_N(t) \\ \Omega_1^*(t) & \Omega_2^*(t) & \cdots & \Omega_N^*(t) & 2\Delta(t) \end{bmatrix}, \quad (5)$$

and the initial condition  $U_{N+1}(t_0, t_0) = \mathbf{I}$ . The excited state  $|e\rangle$  can be generally off resonance by a detuning  $\Delta(t)$  [8], which, however, must be the same for all fields. The functions  $\Omega_1(t), \dots, \Omega_N(t)$  are the Rabi frequencies of the couplings between the ground states and the excited state; we require that they have the same time dependence, described by the envelope function  $f(t)$ , but we allow for different amplitudes  $\chi_n$  and phases  $\beta_n$ ,

$$\Omega_n(t) = \chi_n f(t) e^{i\beta_n} \quad (n=1, 2, \dots, N). \quad (6)$$

By using the Morris-Shore transformation [9] the coupled  $(N+1)$ -state system can be decomposed into a set of  $N-1$  dark ground states, which are superpositions of qunit states, and a two-state system, consisting of a bright ground state and the excited state  $|e\rangle$  [8]. This two-state system is driven by a Hamiltonian involving the same detuning  $\Delta(t)$  as in Eq. (5), and the coupling is the root-mean-square (rms) Rabi frequency  $\Omega(t) = \sqrt{\sum_{n=1}^N \Omega_n^2(t)} = \chi f(t)$ .

The exact solution for the propagator reads [8]

$$\mathbf{U}_{N+1} = \begin{bmatrix} 1 + (a-1)\frac{\chi_1^2}{\chi^2} & (a-1)\frac{\chi_1\chi_2e^{i\beta_{12}}}{\chi^2} & \cdots & (a-1)\frac{\chi_1\chi_Ne^{i\beta_{1N}}}{\chi^2} & b\frac{\chi_1e^{i\beta_1}}{\chi} \\ (a-1)\frac{\chi_1\chi_2e^{-i\beta_{12}}}{\chi^2} & 1 + (a-1)\frac{\chi_2^2}{\chi^2} & \cdots & (a-1)\frac{\chi_2\chi_Ne^{i\beta_{2N}}}{\chi^2} & b\frac{\chi_2e^{i\beta_2}}{\chi} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (a-1)\frac{\chi_1\chi_Ne^{-i\beta_{1N}}}{\chi^2} & (a-1)\frac{\chi_2\chi_Ne^{-i\beta_{2N}}}{\chi^2} & \cdots & 1 + (a-1)\frac{\chi_N^2}{\chi^2} & b\frac{\chi_Ne^{i\beta_N}}{\chi} \\ -b^*\frac{\chi_1e^{-i\beta_1}}{\chi} & -b^*\frac{\chi_2e^{-i\beta_2}}{\chi} & \cdots & -b^*\frac{\chi_Ne^{-i\beta_N}}{\chi} & a^* \end{bmatrix}. \quad (7)$$

Here  $\chi = \sqrt{\sum_{n=1}^N \chi_n^2}$  is the rms peak Rabi frequency and  $\beta_{km} = \beta_k - \beta_m$  ( $k, m = 1, 2, \dots, N$ ) are the relative phases of the external fields. The complex parameters  $a$  and  $b$  (with  $|b|^2 = 1 - |a|^2$ ) are the Cayley-Klein parameters of the  $SU(2)$  propagator for the Morris-Shore bright-excited two-state system.

### 2. Standard QHR: Exact resonance

In the case of exact resonance ( $\Delta=0$ ) the Cayley-Klein parameters for any pulse shape  $f(t)$  are

$$a = \cos \frac{A}{2}, \quad b = -i \sin \frac{A}{2}, \quad (8)$$

where  $A$  is the rms pulse area,

$$A = \chi \int_{t_i}^{t_f} f(t) dt. \quad (9)$$

If

$$A = 2(2k+1)\pi \quad (k=0, 1, 2, \dots), \quad (10)$$

then  $a=-1$ ,  $b=0$ , and the last row and column of the propagator (7) vanish, except for the diagonal element, which is  $-1$ ; the propagator (7) reduces to

$$\mathbf{U}_{N+1} = \begin{bmatrix} \Gamma & \Upsilon & 0 \\ & \mathbf{U}^\pi & \vdots \\ \mathbb{L} & \mathbb{J} & 0 \\ 0 & \cdots & 0 & -1 \end{bmatrix}. \quad (11)$$

Here  $\mathbf{U}^\pi$  is an  $N$ -dimensional unitary matrix (with  $\det \mathbf{U}^\pi = -1$ ), which represents the propagator within the  $N$ -state degenerate manifold; it has exactly the QHR form (1),  $\mathbf{U}^\pi = \mathbf{M}(v; \pi) = \mathbf{M}(v)$ . The components of the  $N$ -dimensional QHR vector  $|v\rangle$  are the normalized Rabi frequencies, with the accompanying phases

$$|v\rangle = \frac{1}{\chi} [\chi_1 e^{i\beta_1}, \chi_2 e^{i\beta_2}, \dots, \chi_N e^{i\beta_N}]^T. \quad (12)$$

Hence the propagator  $\mathbf{U}^\pi$  within the degenerate  $N$ -state manifold of the  $N$ -pod system driven by the Hamiltonian (5), with  $\Delta=0$  and rms pulse area (9), represents indeed a physi-

cal realization of QHR in a single interaction step. Any QHR vector (12) can be produced by appropriately selecting the peak couplings  $\chi_n$  and the phases  $\beta_n$ , while obeying Eq. (10) (e.g., by adjusting the pulse duration).

### 3. Generalized QHR

The unitary propagator (7) for  $a=e^{i\varphi}$  ( $b=0$ ) reduces to

$$\mathbf{U}_{N+1} = \begin{bmatrix} \Gamma & \Upsilon & 0 \\ & \mathbf{U}^\varphi & \vdots \\ \mathbb{L} & \mathbb{J} & 0 \\ 0 & \cdots & 0 & e^{-i\varphi} \end{bmatrix}, \quad (13)$$

where, as is easily verified, we have  $\mathbf{U}^\varphi = \mathbf{M}(v; \varphi)$ , and hence the propagator  $\mathbf{U}^\varphi$  represents a physical realization of the generalized QHR (2). The vector  $|v\rangle$  is again given by Eq. (12). The condition  $a=e^{i\varphi}$  for  $\varphi \neq 0, \pi$  can only be realized off resonance ( $\Delta \neq 0$ ). There is a beautiful off-resonant solution to the Schrödinger equation—the Rosen-Zener model—which we shall use here to exemplify the generalized QHR.

The Rosen-Zener model [10] can be seen as an extension of the resonance solution (8) to nonzero detuning for a special pulse shape (hyperbolic secant),

$$f(t) = \text{sech}(t/T), \quad (14a)$$

$$\Delta(t) = \Delta_0. \quad (14b)$$

The Cayley-Klein parameter  $a$  reads [8,10]

$$a = \frac{\Gamma^2\left(\frac{1}{2} + \frac{1}{2}i\Delta_0 T\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{2}\chi T + \frac{1}{2}i\Delta_0 T\right)\Gamma\left(\frac{1}{2} - \frac{1}{2}\chi T + \frac{1}{2}i\Delta_0 T\right)}, \quad (15)$$

where  $\Gamma(z)$  is Euler's gamma function. Using the reflection formula  $\Gamma\left(\frac{1}{2}+z\right)\Gamma\left(\frac{1}{2}-z\right) = \pi/\cos \pi z$ , we find

$$|a|^2 = 1 - \frac{\sin^2\left(\frac{1}{2}\pi\chi T\right)}{\cosh^2\left(\frac{1}{2}\pi\Delta_0 T\right)}. \quad (16)$$

Hence in this model,  $|a|=1$  for  $\chi T=2l$  ( $l=1,2,\dots$ ); then the last row and the last column of the propagator (7) vanish, except the diagonal element. The phase  $\varphi$  of  $a=e^{i\varphi}$  depends on the detuning  $\Delta_0$  and for an arbitrary integer  $l$  we find from Eq. (15)

$$a = e^{i\varphi} = \prod_{k=0}^{l-1} \frac{\Delta_0 T + i(2k+1)}{\Delta_0 T - i(2k+1)}, \quad (17)$$

and hence

$$\varphi = 2 \arg \prod_{k=0}^{l-1} [\Delta_0 T + i(2k+1)]. \quad (18)$$

This can be seen as an algebraic equation for  $\Delta_0$ , which has  $l$  real solutions. For example, for  $l=1$  (which corresponds to the rms pulse area  $A=2\pi$ ), we have  $\Delta_0 T = \cot(\varphi/2)$ . Hence the generalized QHR phase  $\varphi$  can be produced by an appropriate choice of the detuning  $\Delta_0$ .

The use of nonresonant interaction, besides providing an additional phase parameter, has another important advantage over resonant pulses: lower transient population of the intermediate state. This can be crucial if the lifetime of this state is short compared to the interaction duration. Equation (18) provides the opportunity to control this transient population, which is proportional to  $\Delta_0^{-2}$ , by using large peak Rabi frequency (implying larger  $l$ ) and find the largest solution for  $\Delta_0$ . It is important that the standard QHR can also be realized off resonance, by selecting a detuning  $\Delta_0$  for which  $\varphi=\pi$ .

### III. QHR DECOMPOSITION OF $U(N)$

#### A. Standard QHR decomposition

We shall show that QHR is a very efficient tool for constructing a general  $U(N)$  qunit gate. In particular, we shall show that any  $N$ -dimensional unitary matrix  $\mathbf{U}$  ( $\mathbf{U}^{-1}=\mathbf{U}^\dagger$ ) can be expressed as a product of  $N-1$  standard QHRs  $\mathbf{M}(v_n)$  ( $n=1,2,\dots,N-1$ ) and a phase gate  $\Phi(\phi_1, \phi_2, \dots, \phi_N)$ ,

$$\mathbf{U} = \mathbf{M}(v_1)\mathbf{M}(v_2) \cdots \mathbf{M}(v_{N-1})\Phi(\phi_1, \phi_2, \dots, \phi_N), \quad (19)$$

where

$$\Phi(\phi_1, \phi_2, \dots, \phi_N) = \text{diag}(e^{i\phi_1}, e^{i\phi_2}, \dots, e^{i\phi_N}). \quad (20)$$

We shall prove this assertion by explicitly constructing the decomposition (19). The standard QHRs  $\mathbf{M}(v_n)$  involve vectors  $|v_n\rangle$ , which we construct as follows. First we define the normalized vector  $|v_1\rangle$  as

$$|v_1\rangle = \frac{|u_1\rangle - e^{i\phi_1}|e_1\rangle}{\sqrt{2[1 - \text{Re}(u_{11}e^{-i\phi_1})]}}, \quad (21)$$

where the vector  $|u_n\rangle$  denotes the  $n$ th column of  $\mathbf{U}=\{u_{kn}\}$ ,  $\phi_1 = \arg u_{11}$ , and  $|e_1\rangle = [1, 0, \dots, 0]^T$ . We find

$$\mathbf{M}(v_1)|u_1\rangle = e^{i\phi_1}|e_1\rangle, \quad (22a)$$

$$\mathbf{M}(v_1)|u_n\rangle = |u_n\rangle + 2e^{-i\phi_1}u_{1n}|v_1\rangle, \quad (22b)$$

$$\langle e_1 | \mathbf{M}(v_1) | u_n \rangle = 0 \quad (n=2,3,\dots,N). \quad (22c)$$

Hence the action of  $\mathbf{M}(v_1)$  upon  $\mathbf{U}$  nullifies the first row and the first column except for the first element,

$$\mathbf{M}(v_1)\mathbf{U} = \begin{bmatrix} e^{i\phi_1} & 0 & \cdots & 0 \\ 0 & \Gamma & & \Upsilon \\ \vdots & & \mathbf{U}_{N-1} & \\ 0 & \mathbf{L} & & \mathbf{J} \end{bmatrix}, \quad (23)$$

where  $\mathbf{U}_{N-1}$  is a  $U(N-1)$  matrix. We repeat the same procedure on  $\mathbf{M}(v_1)\mathbf{U}$  and construct the vector  $|v_2\rangle$ ,

$$|v_2\rangle = \frac{|u'_2\rangle - e^{i\phi_2}|e_2\rangle}{\sqrt{2[1 - \text{Re}(u'_{22}e^{-i\phi_2})]}}, \quad (24)$$

where the vector  $|u'_2\rangle$  is the second column of  $\mathbf{M}(v_1)\mathbf{U}$ ,  $\phi_2 = \arg[\mathbf{M}(v_1)\mathbf{U}]_{22}$ , and  $|e_2\rangle = [0, 1, 0, \dots, 0]^T$ . The corresponding QHR  $\mathbf{M}(v_2)$ , applied to  $\mathbf{M}(v_1)\mathbf{U}$ , has the following effects: (i) it nullifies the second row and the second column of  $\mathbf{M}(v_1)\mathbf{U}$  except for the diagonal element, which becomes  $e^{i\phi_2}$ , and (ii) it does not change the first row and the first column. By repeating the same procedure  $N-1$  times, we construct  $N-1$  consecutive Householder reflections, which nullify all off-diagonal elements, to produce a diagonal matrix comprising  $N$  phase factors,

$$\mathbf{M}(v_{N-1}) \cdots \mathbf{M}(v_1)\mathbf{U} = \Phi(\phi_1, \phi_2, \dots, \phi_N), \quad (25)$$

which completes the proof of Eq. (19) since  $\mathbf{M}(v) = \mathbf{M}(v)^{-1}$ . If  $\mathbf{U}$  is a  $SU(N)$  matrix then  $\det \Phi = \pm 1$ , meaning  $\sum_{n=1}^N \phi_n = 0$  or  $\pi$ .

We note that the choice of the QHRs  $\mathbf{M}(v_n)$  is not unique; for example, the first QHR  $\mathbf{M}(v_1)$  can be constructed from the first row of  $\mathbf{U}$ , instead of the first column. Furthermore, the final diagonal matrix (20) occurs due to the unitarity of  $\mathbf{U}$ , which leads to Eq. (22c); a QHR sequence produces a triangular matrix in general.

The QHR decomposition (19) of the  $U(N)$  group into  $N-1$  Householder matrices (1) and a phase gate provides a simple and efficient physical realization of a general transformation of a qunit by only  $N-1$  interaction steps and a phase gate; this is a significant advance compared to  $O(N^2)$  operations in existing recipes. Each QHR vector is  $N$  dimensional, but the nonzero elements decrease from  $N$  in  $|v_1\rangle$  to just 2 in  $|v_{N-1}\rangle$ , and so does the number of fields required for each QHR [see Eq. (12)].

The decomposition (19) is also of mathematical interest because it provides a very natural parametrization of the  $U(N)$  group. Indeed, a QHR vector with  $n$  nonzero elements contains  $2(n-1)$  real parameters (because of the normalization and the irrelevant global phase). The phase gate (20) contains  $N$  phases. Hence Eq. (19) involves  $\sum_{n=2}^N 2(n-1) + N = N^2$  real parameters, as should be the case for a general  $U(N)$  matrix.

#### B. Generalized QHR decomposition

We shall show now that any unitary matrix  $\mathbf{U}$  can be expressed as a product of  $N$  generalized QHRs  $\mathbf{M}(v_n; \varphi_n)$

( $n=1, 2, \dots, N$ ) defined by Eq. (2), without a phase gate, that is,

$$\mathbf{U} = \prod_{n=1}^N \mathbf{M}(v_n; \varphi_n). \quad (26)$$

We first define the normalized vector

$$|v_1\rangle = \frac{1}{e^{-i\varphi_1} - 1} \sqrt{\frac{2 \sin(\varphi_1/2)}{|1 - u_{11}|}} (|u_1\rangle - |e_1\rangle), \quad (27)$$

where the vector  $|u_n\rangle$  denotes again the  $n$ th column of  $\mathbf{U}$  and  $\varphi_1 = 2 \arg(1 - u_{11}) - \pi$ . It is readily seen that

$$\mathbf{M}(v_1; -\varphi_1)|u_1\rangle = |e_1\rangle, \quad (28a)$$

$$\langle e_1 | \mathbf{M}(v_1; -\varphi_1) | u_n \rangle = 0 \quad (n = 2, 3, \dots, N). \quad (28b)$$

Therefore, the action of  $\mathbf{M}(v_1; -\varphi_1)$  upon  $\mathbf{U}$  nullifies the first row and the first column except for the first element, which is turned into unity,

$$\mathbf{M}(v_1; -\varphi_1)\mathbf{U} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \Gamma & & \Upsilon \\ \vdots & & \mathbf{U}_{N-1} & \\ 0 & \mathbf{L} & & \mathbf{J} \end{bmatrix}, \quad (29)$$

where  $\mathbf{U}_{N-1}$  is a  $U(N-1)$  matrix. We repeat the same procedure on  $\mathbf{U}_{N-1}$  and construct the vector

$$|v_2\rangle = \frac{1}{e^{-i\varphi_2} - 1} \sqrt{\frac{2 \sin(\varphi_2/2)}{|1 - u'_{22}|}} (|u'_2\rangle - |e_2\rangle), \quad (30)$$

where the vector  $|u'_2\rangle$  is the second column of  $\mathbf{M}(v_1; -\varphi_1)\mathbf{U}$  and  $\varphi_2 = 2 \arg(1 - u'_{22}) - \pi$ . The action of  $\mathbf{M}(v_2; -\varphi_2)$  upon  $\mathbf{M}(v_1; -\varphi_1)\mathbf{U}$  has the following effects: (i) it nullifies the second row and the second column of  $\mathbf{M}(v_1; -\varphi_1)\mathbf{U}$  except for the diagonal element which is turned into unity, and (ii) it does not change the first row and the first column of  $\mathbf{M}(v_1; -\varphi_1)\mathbf{U}$ . By repeating the same procedure  $N$  times, we construct  $N$  consecutive generalized Householder reflections, which nullify all off-diagonal elements to produce the identity matrix

$$\prod_{n=1}^N \mathbf{M}(v_n; -\varphi_n)\mathbf{U} = \mathbf{I}. \quad (31)$$

By recalling Eq. (3) we obtain Eq. (26) immediately. Note that the last QHR  $\mathbf{M}(v_N; \varphi_N) = \Phi(0, \dots, 0, \varphi_N)$  is actually a one-dimensional phase gate.

Therefore the use of generalized QHRs replaces the  $N$ -dimensional phase gate needed in the standard QHR implementation (19) by a one-dimensional phase gate  $\Phi(0, \dots, 0, \varphi_N)$ . We point out that again, as for the standard QHRs  $\mathbf{M}(v_n)$ , the choice of any of the generalized QHRs  $\mathbf{M}(v_n; \varphi_n)$  is not unique because it can be constructed from the respective row, rather than the column, of the corresponding matrix.

## C. Examples

### 1. Qubit

As an example of the QHR decomposition we first consider the qubit, which is the conventional system for quantum information processing. The conventional realization of a general  $U(2)$  transformation involves three interactions: two phase gates and one rotation  $\mathbf{R}(\vartheta)$  [4],

$$\mathbf{U} = \Phi(\alpha_1, \alpha_2)\mathbf{R}(\vartheta)\Phi(0, \alpha_3). \quad (32)$$

Already for a qubit, the QHR implementations (19) and (26) are superior to Eq. (32) because they require only one QHR and one phase gate,

$$\mathbf{U} = \mathbf{M}(v)\Phi(\phi_1, \phi_2), \quad (33a)$$

$$\mathbf{U} = \mathbf{M}(v; \varphi_1)\Phi(0, \varphi_2). \quad (33b)$$

### 2. Qutrit

As a second example we consider a qutrit—a three-state quantum system. The most general transformation of a qutrit belongs to the  $U(3)$  group, which can be parametrized by nine real parameters; the  $SU(3)$  group is described by eight real parameters. A  $SU(2)$  factorization of  $SU(3)$  reads [11]

$$\mathbf{U} = \mathbf{R}_{23}(\alpha_1, \beta_1, \gamma_1)\mathbf{R}_{12}(\alpha_2, \beta_2, \alpha_2)\mathbf{R}_{23}(\alpha_3, \beta_3, \gamma_3), \quad (34)$$

where  $R_{mn}$  are  $SU(2)$  subgroups of  $SU(3)$ , with the  $SU(2)$  submatrix occupying the  $m$ th and  $n$ th rows and columns of  $R_{mn}$ . Hence this implementation (34) of  $SU(3)$  requires three  $SU(2)$  gates, each involving three qubit gates (32), i.e., nine qubit gates in total (which can be reduced to seven by combining adjacent phase gates). With the present QHR implementation (33) of  $SU(2)$  the number of operations can be reduced to six.

Already for  $SU(3)$  or  $U(3)$ , the present QHR implementations (19) and (26) are considerably more efficient because they require only two QHRs and a phase gate,

$$\mathbf{U} = \mathbf{M}(v_1)\mathbf{M}(v_2)\Phi(\phi_1, \phi_2, \phi_3), \quad (35a)$$

$$\mathbf{U} = \mathbf{M}(v_1; \varphi_1)\mathbf{M}(v_2; \varphi_2)\Phi(0, 0, \varphi_3). \quad (35b)$$

As an example, the arbitrarily chosen  $SU(3)$  gate

$$\mathbf{U} = \begin{bmatrix} 0.864e^{-2\pi i/3} & 0.282e^{15\pi i/19} & 0.416e^{-7\pi i/8} \\ 0.382e^{0.140\pi i} & 0.902e^{7\pi i/11} & 0.203e^{0.808\pi i} \\ 0.327e^{-0.789\pi i} & 0.328e^{4\pi i/5} & 0.886e^{0.035\pi i} \end{bmatrix} \quad (36)$$

(keeping three significant digits) can be realized with two standard QHRs and a phase gate, with

$$|v_1\rangle = [0.260e^{\pi i/3}, 0.734e^{0.140\pi i}, 0.628e^{-0.789\pi i}]^T, \quad (37a)$$

$$|v_2\rangle = [0, 0.651e^{-0.134\pi i}, 0.759e^{0.710\pi i}]^T, \quad (37b)$$

$$\Phi = \text{diag}\{e^{-0.667\pi i}, e^{0.866\pi i}, e^{-0.199\pi i}\}. \quad (37c)$$

Alternatively, the same  $SU(3)$  gate (36) can be realized by two generalized QHRs and a phase gate (35b), with  $\varphi_1 = -0.693\pi$ ,  $\varphi_2 = 0.653\pi$ ,  $\varphi_3 = 0.04\pi$ , and

$$|v_1\rangle = [0.955e^{0.307\pi i}, 0.226e^{-0.707\pi i}, 0.193e^{0.364\pi i}]^T, \quad (38a)$$

$$|v_2\rangle = [0, 0.987e^{0.347\pi i}, 0.161e^{-0.383\pi i}]^T. \quad (38b)$$

#### IV. QUANTUM FOURIER TRANSFORM

The quantum Fourier transform (QFT) is a key ingredient in quantum factoring, quantum search, generalized phase estimation, the hidden subgroup problem, and many other quantum algorithms [4]. The QFT is defined as the unitary operator with the following action on an orthonormal set of states  $|n\rangle$  ( $n=1, 2, \dots, N$ ):

$$\mathbf{U}_N^F |n\rangle = \frac{1}{\sqrt{N}} \sum_{k=1}^N e^{2\pi i(n-1)(k-1)/N} |k\rangle. \quad (39)$$

##### A. Qubit

For a qubit,  $\mathbf{U}^F$  is the Hadamard gate [4],

$$\mathbf{U}_2^F = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad (40)$$

which can be written as a single QHR,  $\mathbf{U}_2^F = \mathbf{M}(v)$ , with

$$|v\rangle = \frac{1}{2} [-\sqrt{2-\sqrt{2}}, \sqrt{2+\sqrt{2}}]^T. \quad (41)$$

Here the standard and generalized QHRs coincide.

##### B. Qutrit

For a qutrit the QFT matrix reads

$$\mathbf{U}_3^F = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{-2\pi i/3} \\ 1 & e^{-2\pi i/3} & e^{2\pi i/3} \end{bmatrix}. \quad (42)$$

The standard QHR decomposition reads

$$\mathbf{U}_3^F = \mathbf{M}(v_1)\mathbf{M}(v_2)\Phi(0, \pi/4, -3\pi/4), \quad (43a)$$

$$|v_1\rangle = \frac{1}{2} \sqrt{1 + \frac{1}{\sqrt{3}}} [1 - \sqrt{3}, 1, 1]^T, \quad (43b)$$

$$|v_2\rangle = \sqrt{\frac{1+\sqrt{2}}{2\sqrt{2}}} [0, 1 - \sqrt{2}, -i]^T. \quad (43c)$$

The generalized QHR decomposition reads

$$\mathbf{U}_3^F = \mathbf{M}(v_1; \pi)\mathbf{M}(v_2; \pi/2), \quad (44a)$$

$$|v_1\rangle = \frac{1}{2} \sqrt{1 + \frac{1}{\sqrt{3}}} [1 - \sqrt{3}, 1, 1]^T, \quad (44b)$$

$$|v_2\rangle = \frac{1}{\sqrt{2}} [0, 1, -1]^T. \quad (44c)$$

Here the first QHR  $\mathbf{M}(v_1; \pi) = \mathbf{M}(v_1)$  is the same for the standard and generalized QHR implementations.

##### C. Quartit

For a quartit ( $N=4$ ) the QFT matrix reads

$$\mathbf{U}_4^F = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}. \quad (45)$$

The standard QHR decomposition reads

$$\mathbf{U}_4^F = \mathbf{M}(v_1)\mathbf{M}(v_2)\Phi(0, \pi/4, 0, -3\pi/4), \quad (46a)$$

$$|v_1\rangle = \frac{1}{2} [-1, 1, 1, 1]^T, \quad (46b)$$

$$|v_2\rangle = \sqrt{\frac{1+\sqrt{2}}{2\sqrt{2}}} [0, 1 - \sqrt{2}, 0, -i]^T. \quad (46c)$$

The generalized QHR decomposition reads

$$\mathbf{U}_4^F = \mathbf{M}(v_1; \pi)\mathbf{M}(v_2; \pi/2), \quad (47a)$$

$$|v_1\rangle = \frac{1}{2} [-1, 1, 1, 1]^T, \quad (47b)$$

$$|v_2\rangle = \frac{1}{\sqrt{2}} [0, 1, 0, -1]^T. \quad (47c)$$

Again, the first QHR  $\mathbf{M}(v_1; \pi) = \mathbf{M}(v_1)$  is the same for the standard and generalized QHR implementations. Interestingly, the QFT for  $N=4$  is decomposed with only two QHRs, rather than three, without phase gates.

Figure 2 shows the time evolution of the propagator  $\mathbf{U}_N(t)$  toward the corresponding QFT matrix  $\mathbf{U}_N^F$ , for  $N=2, 3, 4$ , for realizations with generalized QHRs. As time progresses, the deviation of  $\mathbf{U}_N(t)$  from  $\mathbf{U}_N^F$  vanishes steadily in all cases. As predicted, QFT is realized with just a single QHR for  $N=2$  and with just two QHRs for  $N=3$  and 4.

#### V. DISCUSSION AND CONCLUSIONS

We have proposed a simple physical implementation of the quantum Householder reflection in a coherently driven  $N$ -pod system. We have shown that the most general  $U(N)$  transformation of a qunit can be constructed by at most  $N-1$  standard QHRs and an  $N$ -dimensional phase gate, or by  $N-1$  generalized QHRs (each having an extra phase parameter compared to the standard QHR) and a one-dimensional phase gate, i.e., by only  $N$  physical operations. This significant improvement over the existing setups [involving  $O(N^2)$  operations] can be crucial in making quantum-state engineering and operations with qunits experimentally feasible.

The Householder gate is superior already for a qubit because the general  $U(2)$  gate needs just two gates, a QHR and a phase gate, compared to three gates in existing implementations. For a qutrit, the QHR realization of  $U(3)$  requires only three gates, compared to at least seven hitherto. The

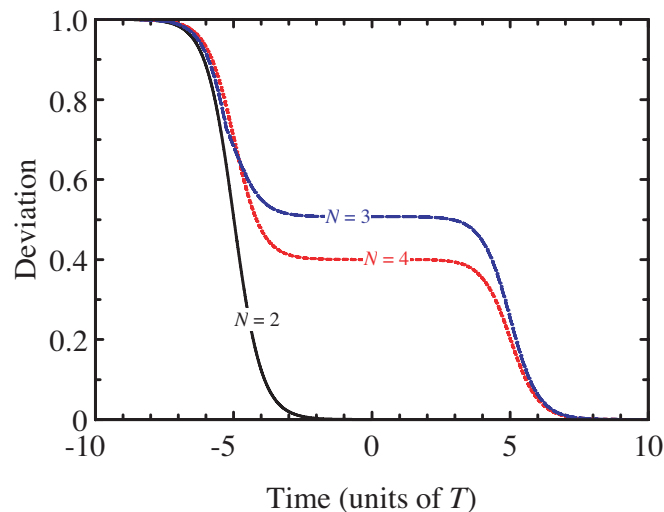


FIG. 2. (Color online) Deviation  $\sum_{j,k=1}^N |(U_N)_{jk} - (U_N^F)_{jk}|$  of the propagator  $U_N(t)$  from the QFT matrix  $U_N^F$  versus time for  $N=2, 3, 4$ , for generalized QHR implementations. The pulses for  $N=2$  are centered at time  $\tau=-5T$ , whereas for  $N=3$  and 4 at times  $\tau=-5T$  and  $5T$ . We have assumed sech pulse shapes [Eq. (14a)] and rms pulse area  $A=2\pi$  ( $\chi=2$ ). The individual couplings  $\chi_n$  are given by the components of the generalized QHR vectors (41) for  $N=2$ , (44b) and (44c) for  $N=3$ , (47b) and (47c) for  $N=4$ , each multiplied by  $\chi$ . The detunings are  $\Delta_0=0$  for the first steps and  $\Delta_0=1/T$  for the second.

QHR implementation of the  $U(N)$  gate is particularly important for qutrits because of the straightforward physical implementation in a  $J=1 \leftrightarrow J=0$  transition (Fig. 1); the results, of course, apply to any  $N$ , and can be accomplished, for instance, by using more  $J=1$  levels.

We have given examples for QHR implementations of quantum Fourier transforms. The QHR realization of QFT for a qubit requires a single interaction step, compared to two steps hitherto. The QHR realization is particularly efficient for a quartit ( $N=4$ ), where the QFT is synthesized with only two QHR gates [as for a qutrit ( $N=3$ )], much fewer than  $O(4^2)$  in the existing  $SU(2)$  proposals.

The components of the Householder vectors are the amplitudes of the respective couplings. It is important that all QHR phases are relative phases of the external control fields, e.g., relative laser phases, which are much easier to control than dynamic and geometric phases.

The generalized QHR requires off-resonant pulsed interactions, appropriately detuned from resonance. The standard QHR can be realized both on and off resonance. The off-resonance implementation has the advantage that only negligible transient population is placed into the (possibly decaying) ancillary excited state; however, it requires a specific value of the detuning.

In the existing  $SU(2)$  proposals, each interaction step involves a single  $SU(2)$  (or Givens) rotation. The difference between the Givens rotation and the Householder reflection is that, when applied to an arbitrary matrix, the Givens rotation nullifies a single matrix element; the Householder reflection nullifies an entire row (or column). When the matrix is unitary, a single Householder reflection nullifies one column and one row simultaneously. Hence the Householder reflection is  $N$  times faster than the Givens  $SU(2)$  transformation.

In atoms and ions qubits are encoded usually in degenerate ground sublevels, and the coupling between them is accomplished by off-resonant interactions, via an intermediate state, which is eliminated adiabatically to produce an effective Raman coupling. In doing so, the phase relation between the two Raman fields is lost. In our proposal we use resonant or nearly resonant fields; no adiabatic elimination is performed and the phase relation is preserved in the resulting QHR propagator. Therefore, already for  $N=2$ , the QHR contains an additional phase parameter compared to previous realizations, which reduces the number of steps for  $U(2)$  operations from 3 to 2. Hence, even for a qubit there is a clear improvement. It is also significant that resonant interactions, which we use, require less interaction energy than off-resonant interactions; this may be crucial in the case of weak couplings.

We conclude by emphasizing that the widespread use of the Householder reflection in classical data analysis promises that the proposed quantum implementation has the potential to become a powerful tool for quantum-state engineering and quantum-information processing.

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- [1] B. W. Shore, *The Theory of Coherent Atomic Excitation* (Wiley, New York, 1990).
- [2] N. V. Vitanov, T. Halfmann, B. W. Shore, and K. Bergmann, *Annu. Rev. Phys. Chem.* **52**, 763 (2001).
- [3] K. Bergmann, H. Theuer, and B. W. Shore, *Rev. Mod. Phys.* **70**, 1003 (1998); N. V. Vitanov, M. Fleischhauer, B. W. Shore, and K. Bergmann, *Adv. At., Mol., Opt. Phys.* **46**, 55 (2001).
- [4] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, U.K., 2000); D. Bouwmeester, A. Ekert, and A.

Zeilinger, *The Physics of Quantum Information: Quantum Cryptography, Quantum Teleportation, Quantum Computation* (Springer-Verlag, Berlin, 2000).

- [5] M. Reck, A. Zeilinger, H. J. Bernstein, and P. Bertani, *Phys. Rev. Lett.* **73**, 58 (1994); A. Muthukrishnan and C. R. Stroud, Jr., *Phys. Rev. A* **62**, 052309 (2000); G. K. Brennen, D. P. O'Leary, and S. S. Bullock, *ibid.* **71**, 052318 (2005); S. S. Bullock, D. P. O'Leary, and G. K. Brennen, *Phys. Rev. Lett.* **94**, 230502 (2005); A. B. Klimov, R. Guzmán, J. C. Retamal, and C. Saavedra, *Phys. Rev. A* **67**, 062313 (2003).

- [6] A. S. Householder, *J. ACM* **5**, 339 (1958).  
[7] J. H. Wilkinson, *Comput. J.* **3**, 23 (1960); *Numer. Math.* **4**, 354 (1962); J. M. Ortega, *ibid.* **5**, 211 (1963); D. J. Mueller, *ibid.* **8**, 72 (1966).  
[8] E. S. Kyoseva and N. V. Vitanov, *Phys. Rev. A* **73**, 023420 (2006).  
[9] J. R. Morris and B. W. Shore, *Phys. Rev. A* **27**, 906 (1983).  
[10] N. Rosen and C. Zener, *Phys. Rev.* **40**, 502 (1932).  
[11] D. J. Rowe, B. C. Sanders, and H. de Guise, *J. Math. Phys.* **40**, 7 (1999); J. B. Bronzan, *Phys. Rev. D* **38**, 1994 (1988).