Optimal partial estimation of quantum states from several copies

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We derive an analytical formula for the optimal trade-off between the mean estimation fidelity and the mean fidelity of the qubit state after a partial measurement on N identically prepared qubits. We also conjecture an analytical expression for the optimal fidelity trade-off in case of a partial measurement on N identical copies of a d-level system.

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I. INTRODUCTION

Certain operations permitted in classical physics cannot be done perfectly in quantum mechanics. This is best exemplified by the celebrated no-cloning theorem [1] which forbids the creation of two perfect copies of an unknown quantum state. The no-cloning theorem is closely related to another no-go theorem stating that one cannot gain some information on an unknown quantum state without disturbing it. That is, if this were possible one would be able to prepare two approximate replicas of this state, which would be better than the best ones allowed by quantum mechanics [2–5]. Therefore, in quantum mechanics any operation on an unknown quantum state giving some information on the state inevitably disturbs the state and, in addition, the more information it extracts the larger is the disturbance. This fundamental property of quantum operations is reflected in the plane of values of quantities quantifying the information gain and the state disturbance by a certain optimal trade-off curve that cannot be overcome by any quantum operation. Among all quantum operations particularly interesting are those that lie on this curve since they to the best possible extent as quantum mechanics allows approximate ideal disturbance-free measurement device. These operations, conventionally denoted as minimal disturbance measurements (MDMs), in general depend on the set of input states and their *a priori* distribution and also on the quantities quantifying the information gain and the state disturbance [6]. The most successful approach to finding the optimal trade-offs and corresponding MDMs proved to be that based on the quantification of the information gain by the mean estimation fidelity G and the state disturbance by the mean output fidelity F [7]. Using this approach it was possible to derive analytically the optimal trade-offs between G and F and to find the MDMs for a single copy of a completely unknown pure state of a d-level system [7], a completely unknown pure state of a d-level system produced by d independent phase shifts of some reference state [8], and a completely unknown maximally entangled state of two d-level systems [9]. In addition, the first two MDMs mentioned above were also demonstrated experimentally for $d=2$ (a qubit) [10]. The studies on MDMs were not restricted to finite-dimensional quantum systems; the MDM for a completely unknown coherent state was also found and realized experimentally in [11]. Multicopy MDMs were introduced by Banaszek and Devetak who considered a partial measurement on N identical copies of a pure qubit state. They assumed MDMs that output N disturbed quantum copies of the state and a classical estimate and they numerically found the optimal fidelity trade-off for this scenario [12]. The MDMs are not only of fundamental importance but they can be also applied to increase transmission fidelity of certain lossy and noisy channels [11,13].

In this paper we further investigate the minimal disturbance measurement on several copies of the state. In contrast to Ref. [12] we assume operations that output only a single quantum copy of the input state. We derive analytically the optimal trade-off between the mean estimation fidelity G and the mean output fidelity F for an ensemble of N identical pure qubits which is given by the formula

$$\sqrt{F-\frac{1}{N+2}}=\sqrt{\frac{N+1}{N+2}-G}+\sqrt{N\left(G-\frac{N}{N+2}\right)}. \quad (1)$$

Moreover, we also conjecture that the optimal fidelity trade-off for an ensemble of N identical pure states of a d-level system has the form

$$\sqrt{F-\frac{1}{N+d}}=\sqrt{(d-1)\left(\frac{N+1}{N+d}-G\right)}+\sqrt{N\left(G-\frac{N}{N+d}\right)}. \quad (2)$$

The paper is organized as follows. The general formalism allowing one to determine the MDM is presented in Sec. II. In Sec. III we find the optimal fidelity trade-off and the corresponding MDM for N identical qubits. In Sec. IV we present a conjecture of the optimal fidelity trade-off for N identical d-level systems. Finally, Sec. V contains conclusions.

II. MINIMAL DISTURBANCE MEASUREMENT

Let us investigate a general MDM for N identical pure states of a d-level system (qudit). Such states are represented by vectors in a d-dimensional Hilbert space $\mathcal{H}^{(d)}$ with an orthonormal basis $\{|0\rangle,|1\rangle,\ldots,|d-1\rangle\}$. The qudits form an orbit of the group SU(d) of $d\times d$ unitary matrices with determinant +1, $|\psi(g)\rangle=U_d(g)|0\rangle$, where $U_d(g)$, $g\in \mathrm{SU}(d)$, is a unitary representation of SU(d) on $\mathcal{H}^{(d)}$. We consider here quantum operations on N identical qudits

$$|\psi(g)\rangle^{\otimes N}=[U_d(g)]^{\otimes N}|0\rangle^{\otimes N}. \quad (3)$$

The operation outputs a single qudit—an approximate replica of $|\psi(g)\rangle$—and also yields a classical estimate of $|\psi(g)\rangle$.

Without loss of generality, these estimates can be labeled by the elements of the group SU(d). Note that the input Hilbert space of the operation is the symmetric subspace of the Hilbert space of N qudits, $\mathcal{H}_{\rm in}=\mathcal{H}_{+,N}^{(d)}$, and the output Hilbert space is the space of a single qudit, $\mathcal{H}_{\rm out}=\mathcal{H}^{(d)}$.

Our task is to find an operation that exhibits the best possible performance in the following protocol [7]. In each run of the protocol, the operation is applied on the quantum state (3). We assume that $|\psi(g)\rangle^{\otimes N}$ is chosen randomly with uniform *a priori* distribution from the set of states $\{|\psi(g)\rangle^{\otimes N}\}_{g\in \mathrm{SU}(d)}$. If the outcome $h\in\mathrm{SU}(d)$ is detected the operation produces a single-qudit output state $\rho(h|g)$. This state is not normalized and its trace $P(h|g)\equiv\mathrm{Tr}[\rho(h|g)]$ is the probability density of obtaining the outcome h on the state (3). The information on the state $|\psi(g)\rangle$ contained in the measurement outcome h is converted into a guess about the state which is in our case a single-qudit state $|\psi(h)\rangle$. The performance of this procedure can be quantified by two mean fidelities: the mean output fidelity F defined as

$$F=\int_{\mathrm{SU}(d)}\int_{\mathrm{SU}(d)}\langle\psi(g)|\rho(h|g)|\psi(g)\rangle dh\,dg, \tag{4}$$

which quantifies the average state disturbance, and the mean estimation fidelity G defined by the formula

$$G=\int_{\mathrm{SU}(d)}\int_{\mathrm{SU}(d)}P(h|g)|\langle\psi(g)|\psi(h)\rangle|^2 dh\,dg, \tag{5}$$

which quantifies the average information gain. Here, the integrals are taken over the whole group SU(d) and dg is the normalized invariant Haar measure on the group. Quantum mechanics sets a fundamental bound on the maximum value of the fidelity F that can be attained for a given value of the fidelity G for any considered quantum operation. The bound can be expressed in the form of a nontrivial optimal trade-off relation between F and G and the MDM is defined as a quantum operation for which the fidelities G and F satisfy the trade-off.

Two extreme cases of the trade-off are well known. First, if G is the optimal estimation fidelity of the qudit state from N identical copies, i.e., $G=(N+1)/(N+d)$ [14], then F can be at most equal to $F=(N+1)/(N+d)$. Second, if $F=1$ then G cannot be larger than the optimal estimation fidelity of a qudit from $N-1$ identical copies, i.e., $G=N/(N-1+d)$. To find the whole optimal trade-off we can use the method developed in [8]. With the help of Jamiolkowski-Choi isomorphism [15,16] we can represent the completely positive map corresponding to each particular outcome h by a positive-semidefinite operator $\chi_N^{(d)}(h)$ acting on the tensor product of the input and output Hilbert spaces $\mathcal{H}_{+,N}^{(d)}\otimes\mathcal{H}^{(d)}$. It holds that $\rho(h|g)=\mathrm{Tr}_{\rm in}\{\chi_N^{(d)}(h)[\psi^{\rm T}(g)]^{\otimes N}\otimes\mathbb{1}_{\rm out}\}$ where $\psi(g)\equiv|\psi(g)\rangle\langle\psi(g)|$. As shown in [8] the optimal partial measurement can be assumed to be covariant which means that $\chi_N^{(d)}(h)$ are generated from a single properly normalized operator $\chi_N^{(d)}$,

$$\chi_N^{(d)}(h)=[U_d^{*\otimes N}(h)\otimes U_d(h)]\chi_N^{(d)}[U_d^{T\otimes N}(h)\otimes U_d^{\dagger}(h)]. \tag{6}$$

The overall operation must be trace preserving, which imposes the constraint,

$$\int_{\mathrm{SU}(d)}\mathrm{Tr}_{\rm out}[\chi_N^{(d)}(h)]dh=\mathbb{1}_{\rm in}, \tag{7}$$

where $\mathrm{Tr}_{\rm out}$ stands for the partial trace over the output single-qudit Hilbert space and $\mathbb{1}_{\rm in}$ denotes the identity operator on the input space $\mathcal{H}_{+,N}^{(d)}$. The formula (7) expresses the completeness of the measurement carried on the input state. The unitary representation $U_d^{\otimes N}$ of SU(d) acts irreducibly on $\mathcal{H}_{+,N}^{(d)}$. For the covariant map (6) the integral in Eq. (7) can thus be evaluated with the help of Schur's lemma and we get $D(N,d)^{-1}\mathrm{Tr}[\chi_N^{(d)}]\mathbb{1}_{\rm in}$ where $D(N,d)=\binom{N-1+d}{d-1}$ is the dimension of the symmetric Hilbert space $\mathcal{H}_{+,N}^{(d)}$. The trace-preservation condition (7) thus boils down to the proper normalization of the map which should read $\mathrm{Tr}[\chi_N^{(d)}]=D(N,d)$.

The operator $\chi_N^{(d)}$ generating the optimal partial measurement is proportional to a rank-1 projector and can be written as $|\chi_N^{(d)}\rangle\langle\chi_N^{(d)}|$ where $|\chi_N^{(d)}\rangle$ is the eigenvector of a positive-semidefinite operator

$$R_p^{(d)}=pR_F^{(d)}+(1-p)R_G^{(d)},\quad p\in[0,1], \tag{8}$$

corresponding to its maximum eigenvalue [8]. Here

$$R_F^{(d)}=\int_{\mathrm{SU}(d)}[\psi(g)^{\otimes N}]^{\rm T}\otimes\psi(g)dg, \tag{9}$$

$$R_G^{(d)}=\mathrm{Tr}_{\rm out}[R_F^{(d)}\mathbb{1}_{\rm in}\otimes\psi(0)]\otimes\mathbb{1}_{\rm out}. \tag{10}$$

Using the map $\chi_N^{(d)}$ the fidelities F and G can be expressed as

$$F=\mathrm{Tr}[\chi_N^{(d)}R_F^{(d)}],\quad G=\mathrm{Tr}[\chi_N^{(d)}R_G^{(d)}]. \tag{11}$$

The operator $R_F^{(d)}$ can be easily evaluated using Schur's lemma and after some algebra we arrive at [17]

$$R_F^{(d)}=\frac{1}{D(N+1,d)}(\Pi_{+,N+1}^{(d)})^{T_N}, \tag{12}$$

where $(\ )^{T_N}$ stands for the partial transposition with respect to the first N qudits and $\Pi_{+,N+1}^{(d)}$ is the projector onto the subspace $\mathcal{H}_{+,N+1}^{(d)}$. In what follows it is convenient to work with the occupation number basis

$$|\{N_i\};N\rangle=|N_0,N_1,\ldots,N_{d-1};N\rangle$$
$$=\sqrt{\frac{N!}{\prod_{i=0}^{d-1}N_i!}}S_N|\underbrace{00\cdots0}_{N_0}\underbrace{11\cdots1}_{N_1}\cdots\underbrace{d-1\cdots d-1}_{N_{d-1}}\rangle, \tag{13}$$

which forms an orthonormal basis in the subspace $\mathcal{H}_{+,N}^{(d)}$. Here $S_N=(1/N!)\sum_{\{\pi\}}P_\pi^{(N)}$ is the symmetrization operator for N qudits, the symbol $\{\pi\}$ stands for summation over all $N!$ permutations of N qudits and $P_\pi^{(N)}$ denotes the permutation operator of N qudits; the integers N_i, $N\geqslant N_i\geqslant 0$, $i=0,\cdots,d-1$ are the numbers of qudits in the states $|i\rangle$, $i=0,\cdots,d-1$ that satisfy the constraint $\sum_{i=0}^{d-1}N_i=N$. Making use of the occupation number basis the operator $\Pi^{(d)}_{+,N+1}$ can be expressed as

$$\Pi^{(d)}_{+,N+1} = \sum_{\sum_{i=0}^{d-1} N_i = N+1} |\{N_i\};N+1\rangle\langle\{N_i\};N+1|. \quad (14)$$

To find the desired MDM for N identical qudits we have to diagonalize a large matrix $R^{(d)}_p$. For a general d this is a complex task which can be solved numerically. However, if we resort to the qubit case ($d=2$) we can find the optimal fidelity trade-off and the MDM analytically. The obtained result then can be used to make at least a conjecture about the optimal fidelity trade-off for N qudits.

III. *N* IDENTICAL QUBITS

For qubits the operator (14) reads as

$$\Pi^{(2)}_{+,N+1} = \sum_{k=0}^{N+1} |N+1,k\rangle\langle N+1,k|, \quad (15)$$

where $|N,k\rangle \equiv |N_0=N-k, N_1=k;N\rangle$ is a completely symmetric state of N qubits in which k qubits are in the basis state $|1\rangle$ and the remaining $N-k$ qubits are in the basis state $|0\rangle$. Hence, making use of the formula

$$|N+1,k\rangle = \sqrt{\frac{N-k+1}{N+1}}|N,k\rangle|0\rangle + \sqrt{\frac{k}{N+1}}|N,k-1\rangle|1\rangle \quad (16)$$

and Eq. (12) one finds that

$$R^{(2)}_F = \frac{1}{(N+1)(N+2)} \times \sum_{k=0}^{N+1} [(N-k+1)|N,k\rangle|0\rangle\langle N,k|\langle 0| + k|N,k-1\rangle|1\rangle\langle N,k-1|\langle 1| + \sqrt{k(N-k+1)}(|N,k\rangle|1\rangle\langle N,k-1|\langle 0| + |N,k-1\rangle|0\rangle\langle N,k|\langle 1|)]. \quad (17)$$

Further, substitution of the obtained expression into Eq. (10) gives the operator $R^{(2)}_G$ in the form

$$R^{(2)}_G = \sum_{k=0}^{N+1} \frac{(N-k+1)}{(N+1)(N+2)}|N,k\rangle\langle N,k| \otimes \mathbb{1}_{\text{out}}. \quad (18)$$

In order to determine the optimal $\chi^{(2)}_N$ we have to find the maximum eigenvalue and the corresponding eigenvector of the matrix $R^{(2)}_p$. The matrix $R^{(2)}_p$ has a block-diagonal structure with two one-dimensional blocks and N two-dimensional blocks. The elements of the one-dimensional blocks are the eigenvalues $\lambda^{(0)} = \frac{(1-p)N+1}{(N+1)(N+2)}$ and $\lambda^{(N+1)} = 1/(N+1)(N+2)$ with the characteristic subspaces spanned by the basis vectors $|N,0\rangle|1\rangle$ and $|N,N\rangle|0\rangle$, respectively. The two-dimensional blocks correspond to the invariant subspaces spanned by the basis vectors $\{|N,k-1\rangle|0\rangle, |N,k\rangle|1\rangle\}$, $k=1,2,\ldots,N$ and have the form $\mathbf{M}_k/(N+1)(N+2)$, where

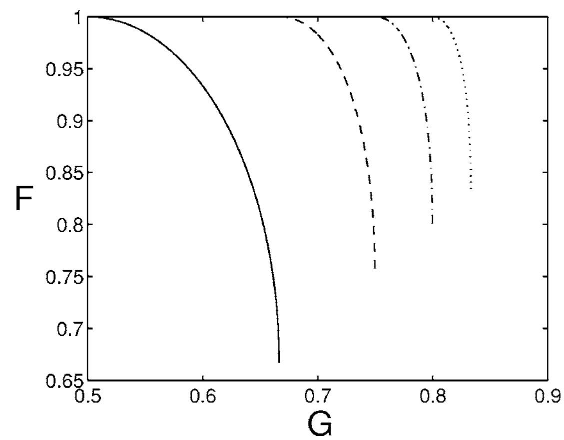

FIG. 1. Optimal trade-off between the fidelities F and G for $N=1$ (solid curve), 2 (dashed curve), 3 (dot-dashed curve), and 4 (dotted curve) identical pure qubits.

$$\mathbf{M}_k = \begin{pmatrix} N-k+2 & p\sqrt{k(N-k+1)} \\ p\sqrt{k(N-k+1)} & N-k+1+p(2k-N) \end{pmatrix}. \quad (19)$$

The matrix $\mathbf{M}_k$ possesses two eigenvalues

$$\mu^{(k)}_{1,2} = \frac{2N+3-pN}{2} - k(1-p) \pm \frac{\sqrt{(1+pN)^2 - 4pk(1-p)}}{2}, \quad (20)$$

from which one obtains the remaining eigenvalues of the matrix $R^{(2)}_p$ as $\lambda^{(k)}_{1,2} = \mu^{(k)}_{1,2}/(N+1)(N+2)$. The larger eigenvalue $\lambda^{(k)}_1$ is a decreasing function of k attaining maximum for $k=1$. Obviously, $\lambda^{(1)}_1 \geq \lambda^{(N+1)}$. Moreover, using in Eq. (20) the inequality $\sqrt{(1+Np)^2 - 4p(1-p)} \geq 1-p$ following from the inequality $N \geq 1$ one can show that also $\lambda^{(1)}_1 \geq \lambda^{(0)}$ holds and therefore $\lambda^{(1)}_1$ is the maximum eigenvalue of the matrix $R^{(2)}_p$. The eigenvalue is nondegenerate and its eigenvector determining the optimal map $\chi^{(2)}_N$ reads as

$$|\chi^{(2)}_N\rangle = \sqrt{N+1}(\alpha|N,0\rangle|0\rangle + \beta|N,1\rangle|1\rangle), \quad (21)$$

where α and β are nonnegative real numbers satisfying the condition $\alpha^2+\beta^2=1$. On inserting $\chi^{(2)}_N = |\chi^{(2)}_N\rangle\langle\chi^{(2)}_N|$ into Eqs. (11) one arrives after some algebra at the optimal fidelities

$$F = \frac{1}{N+2}[(\sqrt{N}\alpha + \beta)^2 + 1], \quad (22)$$

$$G = \frac{1}{N+2}(N+\alpha^2). \quad (23)$$

Expressing now the parameters α, β using Eq. (23) and the normalization condition $\alpha^2+\beta^2=1$ and substituting the obtained formulas into Eq. (22) we finally obtain the optimal fidelity trade-off for N identical qubits (1). The trade-off is depicted for several numbers of copies N in Fig. 1.

The specific feature of the optimal map (21) is that it can be rewritten as the following coherent superposition of two maps:

$$|\chi_N^{(2)}\rangle = \sqrt{N+1}\left(\alpha'|N,0\rangle|0\rangle + \sqrt{\frac{2N}{N+1}}\beta' S_N|0\rangle^{\otimes N-1}|\Phi_+^{(2)}\rangle\right), \tag{24}$$

where $\alpha'=\alpha-\sqrt{N}\beta$, $\beta'=\sqrt{N+1}\beta$, and $|\Phi_+^{(2)}\rangle=(1/\sqrt{2})(|00\rangle+|11\rangle)$. The first map in the superposition is described by the vector $|N,0\rangle|0\rangle$ and corresponds to the choice $\alpha'=1$ ($\beta'=0$). Since in this case $F=G=\frac{N+1}{N+2}$ the map apparently realizes optimal estimation of a qubit from N identical copies [18]. The second map is obtained by choosing $\beta'=1$ ($\alpha'=0$) and it is represented by the second vector on the right-hand side of Eq. (24). It gives $F=1$ and $G=N/(N+1)$, which corresponds to optimal estimation of a qubit from $N-1$ copies while one copy is left intact by the map.

IV. OPTIMAL PARTIAL MEASUREMENT ON N QUDITS

Interestingly, the fact that one can create a MDM as a coherent superposition of the two extreme maps is a general property of the MDMs that is not only valid for the present case of N qubits, but holds also for a single phase-covariant qudit [8], two maximally entangled qudits [9], or a single completely unknown qudit when

$$|\chi_1^{(d)}\rangle = \sqrt{d}(\tilde{\alpha}|00\rangle + \tilde{\beta}|\Phi_+^{(d)}\rangle), \tag{25}$$

where $|\Phi_+^{(d)}\rangle=(1/\sqrt{d})\Sigma_{j=0}^{d-1}|jj\rangle$ is the maximally entangled state of two qudits and $\tilde{\alpha},\tilde{\beta}\geqslant 0$ satisfy the condition $\tilde{\alpha}^2+\tilde{\beta}^2+2\tilde{\alpha}\tilde{\beta}/\sqrt{d}=1$. Thus, although we are not able to solve analytically the above optimization task of finding the MDM for N qudits for a general d we can use the superposition principle together with Eqs. (24) and (25) to guess that the desired optimal map reads

$$|\chi_N^{(d)}\rangle = \sqrt{D(N,d)}(\bar{\alpha}|0\rangle^{\otimes N+1} + \bar{\beta}S_N|0\rangle^{\otimes N-1}|\Phi_+^{(d)}\rangle), \tag{26}$$

where $\bar{\alpha},\bar{\beta}\geqslant 0$ and

$$\bar{\alpha}^2 + \frac{2\bar{\alpha}\bar{\beta}}{\sqrt{d}} + \frac{N+d-1}{Nd}\bar{\beta}^2 = 1. \tag{27}$$

In order to facilitate the following calculations we rearrange the terms on the right-hand side of the map (26) and rewrite it in the form

$$|\chi_N^{(d)}\rangle = \sqrt{D(N,d)}\left(\alpha|0\rangle^{\otimes N+1} + \frac{\beta}{\sqrt{d-1}} \times \sum_{j=1}^{d-1}|N_0=N-1,N_j=1\rangle|j\rangle\right), \tag{28}$$

where $\alpha,\beta\geqslant 0$ satisfy the condition $\alpha^2+\beta^2=1$ and where we have used the shorthand notation $|N_0=N-1,N_j=1\rangle$ for a completely symmetric state of N qudits containing $N-1$ qudits in the basis state $|0\rangle$ and a single qudit in the basis state $|j\rangle$. The fidelities F and G for this map can be again calculated with the help of Eq. (11). Substituting Eq. (28) into Eq. (11) and taking into account the symmetry of the projector $\Pi_{+,N+1}^{(d)}$ which implies $(\Pi_{+,N+1}^{(d)})^{T_N}=(\Pi_{+,N+1}^{(d)})^{T_{\text{out}}}$ the problem of finding F and G reduces to the calculation of the following scalar products:

$$A_j = \langle N_0=N-1,N_j=1|\langle 0|N_0=N,N_j=1\rangle,$$

$$B_j = \langle N_0=N,N_j=1|0\rangle^{\otimes N}|j\rangle,$$

$$C_{kj} = \langle N_0=N-1,N_k=1|\langle j|N_0=N-1,N_k=1,N_j=1\rangle, \tag{29}$$

where we have used the shorthand notation $|N_0=N-1,N_k=1,N_j=1\rangle$ for a completely symmetric state of $N+1$ qudits containing $N-1$ qudits in the basis state $|0\rangle$, a single qudit in the state $|k\rangle$, and a single qudit in the state $|j\rangle$. The scalar products can be easily evaluated using Eq. (13) as $A_j=\sqrt{N/(N+1)}$, $B_j=1/\sqrt{N+1}$, and $C_{jk}=\sqrt{(1+\delta_{jk})/(N+1)}$. Hence, one obtains

$$F = \frac{1}{N+d}[(\sqrt{N}\alpha + \sqrt{d-1}\beta)^2 + 1],$$

$$G = \frac{1}{N+d}(N+\alpha^2). \tag{30}$$

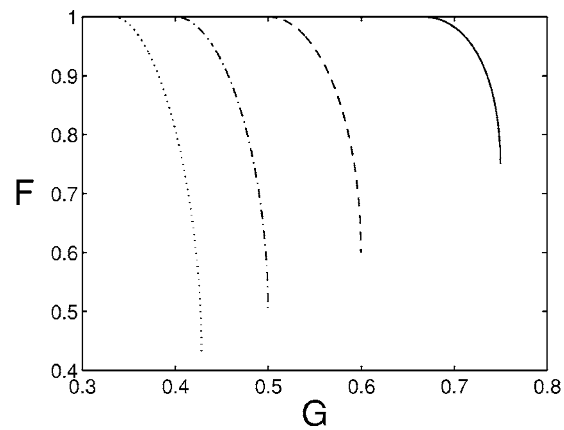

FIG. 2. Trade-off between the fidelities F and G for $N=2$ and $d=2$ (solid curve), 3 (dashed curve), 4 (dot-dashed curve), and 5 (dotted curve).

Eliminating now the parameters α and β from these equations using the same procedure as in the qubit case we arrive finally at the fidelity trade-off (2). Although the trade-off found was not shown to be optimal here, there are several indications supporting our conjecture that it is really optimal. First, for $\beta=0$ we obtain $F=G=\frac{N+1}{N+d}$ using Eqs. (30) and therefore these optimal fidelities satisfy our trade-off. Second, by putting $\alpha=\sqrt{N/(N+d-1)}$ and $\beta=\sqrt{(d-1)/(N+d-1)}$ one finds that $F=1$ and $G=N/(N+d-1)$, which means that also the second extreme case is satisfied. Finally, for $d=2$ the trade-off reduces to the optimal trade-off for N identical qubits (1) while for $N=1$ it boils down to the optimal trade-off for a single completely unknown qudit [7]. The trade-off (2) is depicted in Fig. 2 for $N=2$ and $d=2,3,4,5$.

V. CONCLUSIONS

In summary, in the present paper we have derived analytically the optimal trade-off between the mean estimation fidelity and the mean output fidelity for partial measurements on *N* identical pure qubits. Furthermore, based on the structure of the optimal map saturating the trade-off we have made a conjecture about the optimal fidelity trade-off for partial measurements on *N* identical pure qudits. The results obtained provide an insight into the generic structure and properties of MDMs. The optimal partial measurements saturate the fundamental bound on conversion of quantum information onto classical information and may thus find applications in quantum communication and information processing.

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