Role of the electromagnetic field in the formation of domains in the process of symmetry-breaking phase transitions

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In the framework of quantum field theory we discuss the emergence of a phase locking among the electromagnetic modes and the matter components on an extended space-time region. We discuss the formation of extended domains exhibiting in their fundamental states nonvanishing order parameters, whose existence is not included in the Lagrangian. Our discussion is motivated by the interest in the study of the general problem of the stability of mesoscopic and macroscopic complex systems arising from fluctuating quantum components in connection with the problem of defect formation during the process of non-equilibrium symmetry breaking phase transitions characterized by an order parameter.

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I. INTRODUCTION

Complex systems made up of quantum components are usually remarkably stable at mesoscopic and macroscopic space-time scales. On the other hand, quantum fluctuations are the dominant feature at the microscopic scale of the quantum components. The necessity of taking into account such a double feature is reflected in the usual quantum field theory (QFT) prescription that the Lagrangian of the complex system built upon the quantum fields should be invariant under the local phase transformation of the quantum component field $\psi(\mathbf{x},t)\to\psi'(\mathbf{x},t)=\exp[ig\theta(\mathbf{x},t)]\psi(\mathbf{x},t)$. Local phase invariance is the QFT solution to the problem of building a stable system out of fluctuating components. The requirement of local phase invariance demands the introduction of gauge fields, e.g., the electromagnetic (e.m.) field $A_\mu(\mathbf{x},t)$, such that the Lagrangian be also invariant under the local gauge transformation $A_\mu(\mathbf{x},t)\to A'_\mu(\mathbf{x},t)-\partial_\mu\theta(\mathbf{x},t)$. Such a transformation is devised to compensate terms proportional to $\partial_\mu\theta(\mathbf{x},t)$ arising in the Lagrangian from the kinetic term for the matter field $\psi(\mathbf{x},t)$. This is a well known story. In the present paper, given the above connection between the matter field and the e.m. field, we wish to discuss, in the frame of QFT, the role played by the e.m. field in the locking of the phases of the e.m. modes and of the matter components on an extended space-time region. Furthermore, we will discuss extended domains exhibiting in their fundamental states nonvanishing order parameters, whose existence is not included in the Lagrangian.

The interest in the general problem of the stability of mesoscopic and macroscopic complex systems arising from fluctuating quantum components also finds one strong motivation in the study of the physically relevant problem of defect formation during the process of non-equilibrium symmetry breaking phase transitions characterized by an order parameter [1]. A topological defect may indeed appear in such a process whenever a region, surrounded by ordered domains, remains trapped in the "normal" or symmetric state. Examples of topological defects are vortices in superconductors and superfluids, magnetic domain walls in ferromagnets, and many other extended objects in condensed matter physics. On the other hand, topological defects, such as cosmic strings in cosmology, may have been also playing a role in the phase transition processes in the early Universe [2]. The phenomenological understanding of the defect formation in phase transitions is provided by the Kibble-Zurek scenario [3,4]. By considering the surprising analogy between defect formation in solid state physics and in high energy physics and cosmology [5], it has been also stressed that the analysis of the formation of defects in phase transitions becomes a "diagnostic tool" [6] in the study of non-equilibrium symmetry breaking processes in a wide range of energy scales. Questions such as why extended objects with topological singularities are observed only in systems showing some sort of ordered patterns, why defect formation is observed during the processes of phase transitions, why the features of the defect formation are shared by quite different systems, from condensed matter to cosmology, etc., have been specifically addressed in Refs. [7,8] and the dynamics of defect formation has been extensively studied in a large body of literature in QFT; see, as general Refs. [1,9].

In these studies, in dealing with the presence of a gauge field in the process of spontaneous symmetry breakdown a crucial role is played by the well known Anderson-Higgs-Kibble (AHK) mechanism [10,11], where the gauge field is expelled out of the ordered domains and confined, through self-focusing propagation, into "normal" regions, such as the vortex core, having a vanishing order parameter, i.e., where the long range correlation modes (the Nambu-Goldstone modes) responsible for the ordering are damped away. In the present paper, going beyond the well established AHK mechanism, our attention is focused on the dynamics governing the radiative gauge field and, as said above, its role in the onset of phase locking among the e.m. modes and the matter components. In the AHK mechanism the gauge field removes the order in the regions where it penetrates, thus describing the self-focusing gauge field propagation in ordered condensed matter as well as in asymmetric vacuum in elementary particle physics. Here we study the role of radiative gauge field in sustaining the phase locking in the coherent regime.

We choose as our model system an ensemble of a given number N of two-level atoms, which may represent rigid rotators endowed with an electric dipole. The interaction of these atoms with the e.m. quantum radiative modes will be considered. Moreover, we will examine the effects on the system of an electric field generated by an external source or else by an impurity introduced in the system, thus making contact with the family of the so called Jaynes-Cummings models [12], extensively studied in the literature in connection with quantum optics problems (see, e.g., Ref. [13] for detailed analysis). Our discussion in the present paper may be indeed of some relevance to quantum optics, as well. As a matter of fact, a system of N two-level atoms interacting with quantized e.m. modes has been known for a long time to provide from a formal point of view, under some convenient approximations and restrictions, a strong analogy between the laser phase transition and the conventional phase transitions in spontaneously broken symmetry theories [14,15], although the meaning of the constants in the potential function for the order parameter is different. The key point in such a connection between coherent (laser) light, the N atom system and phase transition is in the observation that, under convenient conditions, the behavior of the e.m. mode is described by the potential

$$V(u,u^*) = -\alpha|u|^2 + \beta|u|^4 + \frac{\alpha^2}{4\beta} = \beta\left(|u|^2 - \frac{\alpha}{2\beta}\right)^2, \quad (1.1)$$

where, in the Haken notation [see Eq. (2.19) of Ref. [14] or Eq. (VI.4.24) of Ref. [15]] α and β, with $\beta>0$, are convenient coefficients and u denotes the classical e.m. amplitude corresponding to the quantum e.m. field amplitude [in the interaction representation the electric field strength is decomposed as $E=u\exp(-i\omega_0 t)+u^*\exp(i\omega_0 t)$, where u is the slowly varying amplitude and ω_0 the atomic resonant frequency]. The essential point is that the (mean value of the) "order parameter" u minimizing the potential $V(u,u^*)$ is zero (disordered or symmetric state) for $\alpha<0$ and nonzero for $\alpha>0$, with $|u|^2=\frac{\alpha}{2\beta}\neq 0$ (ordered or asymmetric state). In this latter case the system is said to be above threshold (the threshold is set at $\alpha=0$), i.e., it is lasering. Of course, in the Lagrangian formalism the coefficient $(-\alpha)$ denotes the "squared mass" of the field, whose sign, as well known, controls the occurrence or not of spontaneous symmetry breakdown. In the Haken interpretation α is the pump parameter whose tuning may carry the system far from the equilibrium, i.e., in the lasering region. Thus in the phase transition between the disordered and the ordered state the order parameter u changes in time from zero to a value proportional to $\sqrt{\alpha}$. In his analysis, Haken also considers the Hamiltonian in the interaction representation

$$H = \hbar\,\gamma(b^\dagger S^- + bS^+), \quad (1.2)$$

which is a Jaynes-Cummings–like Hamiltonian, indeed. In Eq. (1.2) γ is a coupling constant which is proportional to the atomic dipole moment matrix element and to the inverse of the volume square root $V^{-1/2}$, b is the e.m. quantum field operator (associated to the c-number amplitude u), $S^\pm$ are the atomic polarization operators. In the Haken discussion the atomic variables are integrated out at some point of the computation since his interest is mostly focused on the e.m. lasering effect. In our following analysis, instead, we keep them and show that the phase locking between them and the e.m. mode can be reached under convenient boundary conditions.

II. THE MODEL

Let us start by assuming that transitions between the atomic levels are radiative dipole transitions. We thus disregard the static dipole-dipole interaction. Moreover, the system is assumed to be in a thermal bath kept at a nonvanishing temperature T. Under such conditions the system is invariant under dipole rotations. We use natural units $\hbar=1=c$. We assume the system be spatially homogeneous and denote by N the number of atoms per unit volume. The N atom system may be collectively described by the complex dipole wave field $\phi(\mathbf{x},t)$. In Sec. IV we will also use the known formal equivalence (see, e.g., Sec. III.6 of Ref. [15]) of the system of two-level atoms with a system of $\frac{1}{2}$ spins. The dipole wave field $\phi(\mathbf{x},t)$ integrated over the sphere of unit radius $\mathbf{r}$ gives

$$\int d\Omega|\phi(\mathbf{x},t)|^2 = N, \quad (2.1)$$

where $d\Omega=\sin\theta d\theta d\phi$ is the element of solid angle and (r,θ,ϕ) are the polar coordinates of $\mathbf{r}$. By introducing the rescaled field $\chi(\mathbf{x},t)=\frac{1}{\sqrt{N}}\phi(\mathbf{x},t)$ Eq. (2.1) becomes

$$\int d\Omega|\chi(\mathbf{x},t)|^2 = 1. \quad (2.2)$$

Since the atom density is assumed to be spatially uniform, the only relevant variables are the angular ones. Thus, in full generality, we may expand the field $\chi(\mathbf{x},t)$ in the unit sphere in terms of spherical harmonics

$$\chi(\mathbf{x},t) = \sum_{l,m} \alpha_{l,m}(t) Y_l^m(\theta,\phi), \quad (2.3)$$

which, by setting $\alpha_{l,m}(t)=0$ for $l\neq 0,1$, reduces to the expansion in the four levels $(l,m)=(0,0)$ and $(1,m), m=0,\pm 1$. The populations of these levels are given by $N|\alpha_{l,m}(t)|^2$ and at thermal equilibrium, in the absence of interaction, they follow the Boltzmann distribution. Moreover, the dipole rotational invariance implies that there is no preferred direction in the dipole orientation, which means that the amplitude of $\alpha_{1,m}(t)$ does not depend on m, and that no permanent polarization may develop for such a system in such conditions, i.e., the time average of the polarization $P_{\mathbf{n}}$ along any direction $\mathbf{n}$ must vanish. We thus write

$$\alpha_{0,0}(t) \equiv a_0(t) \equiv A_0(t)e^{i\delta_0(t)},$$

$$\alpha_{1,m}(t) \equiv A_1(t)e^{i\delta_{1,m}(t)}e^{-i\omega_0 t} \equiv a_{1,m}(t)e^{-i\omega_0 t}, \quad (2.4)$$

where $a_{1,m}(t)\equiv A_1(t)e^{i\delta_{1,m}(t)}$. $A_0(t)$, $A_1(t)$, $\delta_0(t)$, and $\delta_{1,m}(t)$ are real quantities. In Eqs. (2.4) we have also used $\omega_0\equiv\frac{1}{I}$, where I denotes the moment of inertia of the atom, which gives a relevant scale for the system: $\omega_0\equiv k=\frac{2\pi}{\lambda}$ (note that the eigenvalue of $\frac{\mathbf{L}^2}{2I}$ on the state $(1,m)$, $\mathbf{L}^2$ being the squared angular momentum operator, is $\frac{l(l+1)}{2I}=\frac{1}{I}=\omega_0$). By setting the $\mathbf{z}$ axis parallel to $\mathbf{n}$ and using the explicit expressions for the spherical harmonics

$$Y_0^0=\frac{1}{\sqrt{4\pi}},\quad Y_1^0=\sqrt{\frac{3}{4\pi}}\cos\theta,$$

$$Y_1^1=-\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}=-[Y_1^{-1}]^*, \tag{2.5}$$

we find

$$P_{\mathbf{n}}=\int d\Omega\chi^*(\mathbf{x},t)(\mathbf{x}\cdot\mathbf{n})\chi(\mathbf{x},t)=\frac{2}{\sqrt{3}}A_0(t)A_1(t)\cos(\omega-\omega_0)t, \tag{2.6}$$

where $\omega t\equiv\delta_{1,0}(t)-\delta_0(t)$ and whose time average is zero, as it should be. This confirms that the three levels $(1,m)$, $m=0,\pm1$ are in the average equally populated under normal conditions and that, as said above, we can safely write $\Sigma_m|\alpha_{1,m}(t)|^2=3|a_1(t)|^2$. On the other hand, the normalization condition (2.2) gives, at any time t,

$$|\alpha_{0,0}(t)|^2+\sum_m|\alpha_{1,m}(t)|^2=|a_0(t)|^2+3|a_1(t)|^2=1. \tag{2.7}$$

By defining Q as

$$Q\equiv|a_0(t)|^2+3|a_1(t)|^2, \tag{2.8}$$

we thus see from Eq. (2.7) that

$$\frac{\partial}{\partial t}Q=0, \tag{2.9}$$

i.e.,

$$\frac{\partial}{\partial t}|a_1(t)|^2=-\frac{1}{3}\frac{\partial}{\partial t}|a_0(t)|^2. \tag{2.10}$$

Due to Eq. (2.1) [and the rescaling adopted for $\chi(\mathbf{x},t)$], Eq. (2.9) expresses nothing but the conservation of the total number N of atoms; it also means that, as shown in Eq. (2.10), due to the rotational invariance, the rate of change of the population in each of the levels $(1,m)$, $m=0,\pm1$, equally contributes, in the average, to the rate of change in the population of the level $(0,0)$, at each time t. Consistently with Eq. (2.7), in full generality we can set the initial conditions at $t=0$ as

$$|a_0(0)|^2=\cos^2\theta_0,\quad |a_1(0)|^2=\frac{1}{3}\sin^2\theta_0,\quad 0<\theta_0<\frac{\pi}{2}. \tag{2.11}$$

We exclude the values zero and $\frac{\pi}{2}$ since they correspond to the physically unrealistic conditions for the state (0,0) of being completely filled or completely empty, respectively. By properly tuning the parameter θ_0 in its range of definition one can adequately describe the physical initial conditions. For example, $\theta_0=\frac{\pi}{3}$ describes the equipartition of the field modes of energy $E(k)$ among the four levels $(0,0)$ and $(1,m)$, $|a_0(0)|^2\simeq|a_{1,m}(0)|^2$, $m=0,\pm1$, as typically given by the Boltzmann distribution when the temperature T is high enough, $k_BT\gg E(k)$. As we will see, however, the lower bound for the parameter θ_0 is imposed by the dynamics in a self-consistent way.

The possibility of obtaining a nonzero permanent polarization, and thus the dipole ordering in the system, is crucially conditioned by the ratio between the populations in the atomic levels. Namely, suppose that the atom system is under the influence of an electric field $\mathbf{E}$ due, e.g., to an impurity, or to any other external agent. Assume $\mathbf{E}$ to be parallel to the $\mathbf{z}$ axis. Then the term

$$\mathcal{H}=-\mathbf{d}\cdot\mathbf{E}, \tag{2.12}$$

where $\mathbf{d}$ is the electric dipole moment of the atom, will be added to the system energy and will break the dipole rotational symmetry. It will produce the mixing between the states Y_0^0 and Y_1^0: $Y_0^0\to Y_0^0\cos\tau+Y_1^0\sin\tau$ and $Y_1^0\to-Y_0^0\sin\tau+Y_1^0\cos\tau$, with

$$\tan\tau=\frac{\omega_0-\sqrt{\omega_0^2+4\mathcal{H}^2}}{2\mathcal{H}}. \tag{2.13}$$

Due to the mixing thus induced, the polarization $P_{\mathbf{n}}$ is now given by

$$P_{\mathbf{n}}=\frac{1}{\sqrt{3}}(A_0^2-A_1^2)\sin 2\tau+\frac{2}{\sqrt{3}}A_0(t)A_1(t)\cos 2\tau \times\cos([\omega-\sqrt{\omega_0^2+4\mathcal{H}^2}]t), \tag{2.14}$$

to be compared with Eq. (2.6) and whose time average is nonzero: $\overline{P_{\mathbf{n}}}=\frac{1}{\sqrt{3}}(A_0^2-A_1^2)\sin 2\tau$. The nonzero difference in the level populations $(A_0^2-A_1^2)$ is therefore crucial in obtaining the nonzero polarization. We will study under which conditions such an occurrence can be realized.

In the following, we restrict ourselves to the resonant radiative e.m. modes, i.e., those for which $k=\frac{2\pi}{\lambda}=\omega_0$, and we use the dipole approximation, i.e., we put $\exp(i\mathbf{k}\cdot\mathbf{x})\approx1$ in our formulas, since we are interested in the macroscopic behavior of the system. This means that the wavelengths of the e.m. modes we consider, of the order of $\frac{2\pi}{\omega_0}$, are larger (or comparable) than the system linear size. Let $c_r(\mathbf{k},t)$ denote the radiative e.m. field operator with polarization r and $u_r(\mathbf{k},t)=\frac{1}{\sqrt{N}}c_r(\mathbf{k},t)$ the rescaled one. The field equations for our system are [13,16]

$$i\frac{\partial\chi(\mathbf{x},t)}{\partial t}=\frac{\mathbf{L}^2}{2I}\chi(\mathbf{x},t)-i\sum_{\mathbf{k},r}d\sqrt{\rho}\sqrt{\frac{k}{2}}(\epsilon_r\cdot\mathbf{x})\times[u_r(\mathbf{k},t)e^{-ikt}-u_r^\dagger(\mathbf{k},t)e^{ikt}]\chi(\mathbf{x},t),$$

$$i\frac{\partial u_r(\mathbf{k},t)}{\partial t}=id\sqrt{\rho}\sqrt{\frac{k}{2}}e^{ikt}\int d\Omega(\epsilon_r\cdot\mathbf{x})|\chi(\mathbf{x},t)|^2, \tag{2.15}$$

where d is the magnitude of the electric dipole moment, $\rho\equiv\frac{N}{V}$ and ϵ_r is the polarization vector of the e.m. mode (for which the transversality condition $\mathbf{k}\cdot\epsilon_r=0$ is assumed to hold). We remark that the enhancement by the factor $\sqrt{N}$

appearing in the coupling $d\sqrt{\rho}$ in Eqs. (2.15) is due to the rescaling of the fields. We will comment more on this point later on.

III. THE FIELD EQUATIONS AND THE PHASE LOCKING

By resorting to the discussion of Eqs. (2.15) presented in Ref. [17], our task is now to analyze the implications of these equations with reference to the role played by the e.m. modes in the onset of the phase locking between them and the dipole field. First, let us observe that use of Eqs. (2.3) [and (2.4)] in Eq. (2.15) gives the set of coupled equations

$$\dot{a}_0(t) = \Omega \sum_m u_m^*(t) a_{1,m}(t), \tag{3.1}$$

$$\dot{a}_{1,m}(t) = -\Omega u_m(t) a_0(t), \tag{3.2}$$

$$\dot{u}_m(t) = 2\Omega a_0^*(t) a_{1,m}(t), \tag{3.3}$$

where $a_{1,m}(t) \equiv \alpha_{1,m}(t) e^{i\omega_0 t}$ [see Eq. (2.4)], the dot over the symbols denotes the time derivative, u_m is the amplitude of the e.m. mode coupled to the transition $(1,m) \leftrightarrow (0,0)$ and $\Omega \equiv \frac{2d}{\sqrt{3}} \sqrt{\frac{\rho}{2\omega_0}} \omega_0 \equiv G\omega_0$.

Equations (3.1)–(3.3), as well as Eqs. (2.15), from which they have been derived, appear to be not invariant under time-dependent phase transformations of the field amplitudes. We want to investigate how gauge invariance can be recovered.

Use of the conjugate of Eq. (3.2) in Eq. (3.1) gives, consistently with Eq. (2.9), the conservation law $\dot{Q}=0$. Moreover, use of the conjugate of Eq. (3.2) in Eq. (3.3) leads to

$$\frac{\partial}{\partial t} |u_m(t)|^2 = -2 \frac{\partial}{\partial t} |a_{1,m}(t)|^2. \tag{3.4}$$

Since the amplitude $|\alpha_{1,m}(t)| = |a_{1,m}(t)|$ does not depend on m, Eq. (3.4) shows that also the amplitude $|u_m(t)|$ does not depend on m. Equation (3.4) shows the existence of another constant of motion; namely, putting $|u(t)| \equiv |u_m(t)|$ and using $|a_1(t)| \equiv |a_{1,m}(t)|$, we can write

$$|u(t)|^2 + 2|a_1(t)|^2 = \frac{2}{3} \sin^2 \theta_0, \tag{3.5}$$

for any t, where we have also used the initial condition (2.11) and set

$$|u(0)|^2 = 0. \tag{3.6}$$

We observe that since $|u(t)|^2 > 0$, Eq. (3.5) imposes $|a_1(t)|^2 \leq \frac{1}{3} \sin^2 \theta_0$ and therefore $|a_0(t)|^2 \geq \cos^2 \theta_0$ due to Eq. (2.7). Note that Eq. (3.5) gives [cf. Eq. (2.11)]

$$|u(t)|^2 = 2[|a_1(0)|^2 - |a_1(t)|^2], \tag{3.7}$$

for any t, which, by exploiting Eq. (2.11), reads

$$|u(t)|^2 = \frac{2}{3}[|a_0(t)|^2 - \cos^2 \theta_0]. \tag{3.8}$$

It is also useful to write

$$u_m(t) = U(t) e^{i\varphi_m(t)}, \tag{3.9}$$

with $U(t)$ and $\varphi_m(t)$ real quantities.

By combining Eqs. (2.4) and (3.9) with Eqs. (3.1)–(3.3) and equating real and imaginary parts, we get

$$\dot{A}_0(t) = \Omega U(t) A_1(t) \cos \alpha_m(t), \tag{3.10}$$

$$\dot{A}_1(t) = -\Omega U(t) A_0(t) \cos \alpha_m(t), \tag{3.11}$$

$$\dot{U}(t) = 2\Omega A_0(t) A_1(t) \cos \alpha_m(t), \tag{3.12}$$

$$\dot{\varphi}_m(t) = 2\Omega \frac{A_0(t) A_1(t)}{U(t)} \sin \alpha_m(t), \tag{3.13}$$

where we have put

$$\alpha_m \equiv \delta_{1,m}(t) - \delta_0(t) - \varphi_m(t). \tag{3.14}$$

Similarly, we can derive equations for $\dot{\delta}_{1,m}$ and $\dot{\delta}_0$.

From Eqs. (3.10)–(3.12) we see that since their left hand sides are independent of m, so the right hand sides have to be, i.e., either $\cos \alpha_m(t)=0$ for any m at any t, or α_m is independent of m at any t. In both cases, Eq. (3.13) shows that φ_m is then independent of m, which in turn implies, together with Eq. (3.14), that $\delta_{1,m}(t)$ is independent of m. Phases thus turn out to be independent of m. We will therefore put $\varphi \equiv \varphi_m$, $\delta_1(t) \equiv \delta_{1,m}(t)$ and $\alpha \equiv \alpha_m$. We observe that in general the phases can be always changed by arbitrary constants. The independence of m of the phases is dictated by the requirement to not violate the gauge invariance. Should exist a difference between the phases having different m, this difference could be changed by a rotation of the axes and would spoil the gauge invariance. In the present case, the independence of m of the phases is of dynamical origin and we will find that the phase locking among $\delta_0(t)$, $\delta_1(t)$, and $\varphi(t)$, has indeed the meaning of recovering the gauge invariance. We will discuss this point in the Sec. II.B.

Summarizing, we can now write $u(t) \equiv u_m(t)$ and $a_1(t) \equiv a_{1,m}(t)$ and from Eqs. (3.1)–(3.3) we get the known [17] set of equations:

$$\dot{a}_0(t) = 3\Omega u^*(t) a_1(t), \tag{3.15}$$

$$\dot{a}_1(t) = -\Omega u(t) a_0(t), \tag{3.16}$$

$$\dot{u}(t) = 2\Omega a_0^*(t) a_1(t). \tag{3.17}$$

Equations (3.15)–(3.17) are fully consistent with the original normalization condition (2.2) [or Eq. (2.7)], with the conservation (2.9) and with the dipole rotational invariance expressed by the zero average polarization [cf. Eq. (2.6)]. In Eq. (3.15) the rate of change of the amplitude of the level (0,0) is shown to depend on the coupling between the levels $(1,m)$, $m=0,\pm 1$ and the radiative e.m. mode of corresponding polarization. Each of these couplings contribute in equal measure, due to rotational invariance, to the transitions to (0,0). Similarly, in Eq. (3.16) the rate of change of the amplitude of each level $(1,m)$ is shown to depend on the coupling between the level (0,0) and the corresponding radiative e.m. mode. Finally, in Eq. (3.17) the transitions $(0,0) \leftrightarrow (1,m)$, $m=0,\pm 1$ control the rate of change of the amplitude of the radiative e.m. mode of corresponding polarization. These equations thus reflect the correct selection rules in radiative and absorption processes [18–20]. Equation (3.15) describes the fact that each of the levels $(1,m)$ may find in the e.m. field the proper mode to couple with, in full respect of the selection rules. In this sense, the field concept, as a full collection of e.m. modes with all possible polarizations, is crucial here. As already said, Eqs. (3.15)–(3.17) are fully consistent with the physical boundary conditions and the motion equation (2.15) from which they are derived.

A. The ground state

We want to study now the vacuum or ground state of the system for each of the modes $a_0(t)$, $a_1(t)$, and $u(t)$ described by Eqs. (3.15)–(3.17) [i.e., by Eq. (2.15)]. It is convenient to differentiate once more with respect to time both sides of Eq. (3.15). By using Eqs. (3.16) and (3.17) and the constants of motion (2.7) and (3.5) we eliminate the variables $a_1(t)$ and $u(t)$. We thus find

$$\ddot{a}_0(t) = 4\Omega^2\gamma_0^2(\theta_0)a_0(t) - 4\Omega^2|a_0(t)|^2 a_0(t), \qquad (3.18)$$

where $\gamma_0^2(\theta_0) \equiv \frac{1}{2}(1+\cos^2\theta_0)$. Equation (3.18) can be written in the form

$$\ddot{a}_0(t) = -\frac{\delta}{\delta a_0^*}V[a_0(t),a_0^*(t)], \qquad (3.19)$$

where the potential $V[a_0(t),a_0^*(t)]$ is

$$V[a_0(t),a_0^*(t)] = 2\Omega^2[|a_0(t)|^2 - \gamma_0^2(\theta_0)]^2. \qquad (3.20)$$

In a standard fashion (see, e.g., Ref. [11]) we may adopt the semiclassical ("mean field") approximation in order to study the ground state of the theory. We thus search for the minima of the potential V. Let $a_{0,R}(t)$ and $a_{0,I}(t)$ denote the real and the imaginary component, respectively, of the $a_0(t)$ field $|a_0(t)|^2 = A_0^2(t) = a_{0,R}^2(t) + a_{0,I}^2(t)$. The potential has a relative maximum at $a_0=0$ and a (continuum) set of minima given by

$$|a_0(t)|^2 = \frac{1}{2}(1+\cos^2\theta_0) = \gamma_0^2(\theta_0). \qquad (3.21)$$

These minima correspond to the points on the circle of squared radius $\gamma_0^2(\theta_0)$ in the $(a_{0,R}(t),a_{0,I}(t))$ plane. We thus recognize that we are in the familiar case of a theory where the cylindrical SO(2) symmetry (the phase symmetry) around an axis orthogonal to the plane $(a_{0,R}(t),a_{0,I}(t))$ is spontaneously broken. The order parameter is given by $\gamma_0(\theta_0)$. Note that Eq. (3.21) does not fix the (ground state expectation) value of the phase field $\delta_0(t)$. The points on the circle represent (infinitely many) possible vacua for the system and they transform into each other under shifts of the field δ_0: $\delta_0 \to \delta_0 + \alpha$ [SO(2) rotations in the $(a_{0,R}(t),a_{0,I}(t))$ plane]. The phase symmetry is broken when one specific ground state is singled out by fixing the value of the δ_0 field.

By proceeding as usual in these circumstances [11], we transform to new fields: $A_0(t) \to A_0'(t) \equiv A_0(t) - \gamma_0(\theta_0)$ and $\delta_0'(t) \to \delta_0(t)$, so that $A_0'(t)=0$ in the ground state for which $A_0(t)=\gamma_0(\theta_0)$. Use of these new variables in V leads to recognize that the amplitude $A_0'(t)$ describes a quasiperiodic mode with pulsation $m_0 = 2\Omega\sqrt{(1+\cos^2\theta_0)}$ [a "massive" mode with real mass $2\Omega\sqrt{(1+\cos^2\theta_0)}$] and that the field $\delta_0'(t)$ corresponds to a zero-frequency mode (a massless mode) playing the role of the so-called Nambu-Goldstone (NG) field or collective mode implied by the spontaneous breakdown of symmetry.

We note that when Eq. (3.21) holds, use of Eqs. (2.7) and (3.5) gives $A_1^2 = \frac{1}{6}\sin^2\theta_0$ and $\bar{U}^2 = \frac{1}{3}\sin^2\theta_0$, moving away from the initial condition values (2.11) and (3.6), respectively. In this respect, it is remarkable that the value $a_0=0$, which we have excluded in our initial conditions, cf. Eq. (2.11), on the basis of physical considerations, consistently appears to be the relative maximum for the potential, and therefore an instability point out of which the system (spontaneously) runs away. Moreover, as already observed, use of the constant of motion laws (2.7) and (3.5) shows that $|a_0|^2 = 0$ implies $U^2 = -\frac{2}{3}\cos^2\theta_0$ which is not possible since U is real. Finally, we remark that the bound $|a_0(t)|^2 \ge \cos^2\theta$ discussed above [see the comment after Eq. (3.6)] is consistently satisfied by $|a_0(t)|^2 = \gamma_0^2(\theta_0)$ [see Eq. (3.21)].

We consider now the time derivative of both sides of Eq. (3.16) and by simple manipulations we find the following equation for the amplitude $a_1(t)$:

$$\ddot{a}_1(t) = -\sigma^2 a_1(t) + 12\Omega^2|a_1(t)|^2 a_1(t), \qquad (3.22)$$

where $\sigma^2 = 2\Omega^2(1+\sin^2\theta_0)$. The potential from which the right-hand side of Eq. (3.22) is derivable is

$$V[a_1(t),a_1^*(t)] = \sigma^2|a_1(t)|^2 - 6\Omega^2[|a_1(t)|^2]^2. \qquad (3.23)$$

In this case there is a relative minimum at $a_1=0$ and a (continuum) set of relative maxima on the circle of squared radius

$$|a_1(t)|^2 = \frac{1}{6}(1+\sin^2\theta_0) \equiv \gamma_1^2(\theta_0). \qquad (3.24)$$

Note that, for $|a_1(t)|^2 = \gamma_1^2(\theta_0)$, $U^2 = -\frac{1}{3}\cos^2\theta_0 < 0$, which is not acceptable since U is real. The values on the circle of radius $\gamma_1(\theta_0)$ are thus forbidden for the amplitude A_1. This is consistent with the intrinsic instability of the excited levels $(1,m)$. We have also seen that the conservation law (3.5) and the reality condition for U require that $|a_1(t)|^2 \le \frac{1}{3}\sin^2\theta_0$ which lies indeed below $\gamma_1^2(\theta_0)$, and we note that the value $\frac{1}{6}\sin^2\theta_0$ taken by A_1^2 when $|a_0(t)|^2 = \gamma_0^2(\theta_0)$ also lies below the bound. In conclusion, the potential $V[a_1(t),a_1^*(t)]$ involved in the dynamics must be lower than $\frac{1}{3}\sin^2\theta$.

This is enough about the consistency between Eqs. (3.18) and (3.22). As mentioned above, we exclude that the amplitude A_1 be constantly zero (at the minimum of $V[a_1(t),a_1^*(t)]$), since this would correspond to the physically unrealistic situation of the $(0,0)$ level completely filled. In conclusion, within these dynamical bounds, the field $a_1(t)$ described by Eq. (3.22) is a massive field with (real) mass (pulsation) $\sigma^2 = 2\Omega^2(1+\sin^2\theta_0)$.

Finally, we focus on the e.m. mode $u(t)$ and consider Eq. (3.17). By proceeding as above by differentiating once more with respect to time we find

$$\ddot{u}(t) = -\mu^2 u(t) - 6\Omega^2 |u(t)|^2 u(t), \tag{3.25}$$

where $\mu^2 = 2\Omega^2 \cos 2\theta_0$. The right-hand side of Eq. (3.25) is derivable from the potential

$$\begin{aligned} V[u(t),u^*(t)] &= \mu^2 |u(t)|^2 + 3\Omega^2 |u(t)|^4 + \frac{1}{3}\Omega^2 \cos^2 2\theta_0 \\ &= 3\Omega^2 \left[|u(t)|^2 + \frac{1}{3}\cos 2\theta_0 \right]^2 \end{aligned} \tag{3.26}$$

and we note that $V[u(t),u^*(t)]$ is nothing but the potential for the e.m. mode given in Eq. (1.1) for $-\alpha = \mu^2$ and $\beta = 3\Omega^2$. We are in the case of a theory where the symmetry can be spontaneously broken or not, according to the negative or positive value of the squared mass μ^2 of the field (the pump in the Haken interpretation), respectively.

Again, in the semiclassical approximation we search for the minima of the potential $V[u(t),u^*(t)]$ and see that $\mu^2 \geq 0$ for $\theta_0 \leq \frac{\pi}{4}$ and the only minimum is at $u_0 = 0$. Equation (3.25) then describes quasiperiodic modes with pulsation $\mu = \Omega\sqrt{2\cos 2\theta_0}$, typically expected for a paraboloid potential $V[u(t),u^*(t)]$ with cylindrical SO(2) symmetry about an axis orthogonal to the plane $(u_R(t),u_I(t))$ and minimum at $u_0 = 0$. Here $u_R(t)$ and $u_I(t)$ denote the real and the imaginary component, respectively, of the $u(t)$ field. In such a case we have the symmetric solution with zero order parameter $u_0 = 0$. This solution describes the system when the initial condition (3.6) holds at any time. This occurrence is, however, not consistent with the dynamical evolution of the system moving away from the initial conditions exhibited by Eq. (3.18), as mentioned above. Luckily, consistency is dynamically recovered provided $\theta_0 > \frac{\pi}{4}$. In such a case, indeed, $\mu^2 = 2\Omega^2 \cos 2\theta_0 < 0$ and the potential has a relative maximum at $u_0 = 0$ and a (continuum) set of minima given by

$$|u(t)|^2 = -\frac{1}{3}\cos 2\theta_0 = -\frac{\mu^2}{6\Omega^2} \equiv v^2(\theta_0), \quad \theta_0 > \frac{\pi}{4}. \tag{3.27}$$

The fact that in the present case $u_0 = 0$ is a maximum for the potential means that the system dynamics evolves away from it, consistently with the similar situation noticed above for the a_0 mode where the system spontaneously evolves away from the initial conditions. The symmetric solution of the minimum at $u_0 = 0$ is thus excluded for internal consistency and the lower bound $\frac{\pi}{4}$ for θ_0 is thus dynamically imposed in a self-consistent way.

In Eq. (3.27) the minima are the points of the circle of squared radius $v^2(\theta_0)$ in the $(u_R(t),u_I(t))$ plane. As in the case of the amplitude a_0 analyzed above, the points on the circle represent (infinitely many) possible vacua for the system and they transform into each other under shifts of the field φ: $\varphi \to \varphi + \alpha$. For $\theta_0 > \frac{\pi}{4}$ the phase symmetry is broken, the order parameter is given by $v(\theta_0) \neq 0$ and one specific ground state is singled out by fixing the value of the φ field.

As usual [11], we transform to new fields $U(t) \to U'(t) \equiv U(t) - v(\theta_0)$ and $\varphi'(t) \to \varphi(t)$ so that in the ground state $U'(t) = 0$. Use of these new variables in $V[u(t),u^*(t)]$ shows that the amplitude $U'(t)$ describes a "massive" mode with real mass $\sqrt{2|\mu^2|} = 2\Omega\sqrt{|\cos 2\theta_0|}$ (a quasiperiodic mode) and that the field $\varphi'(t)$ is a zero-frequency mode (a massless mode). This field, also called the "phason" field [21], plays the role of the Nambu-Goldstone (NG) collective mode in the theories where symmetry is spontaneously broken. When Eq. (3.27) holds, it is $A_1^2 = \frac{1}{6}$ which lies below the upper bound $\frac{1}{3}\sin^2\theta$ provided $\theta > \frac{\pi}{4}$. Similarly, Eq. (3.27) implies $A_0^2 = \frac{1}{2}$ which satisfies the constraint of being greater than $\cos^2\theta$ for $\theta > \frac{\pi}{4}$.

In conclusion, the e.m. field, as an effect of the spontaneous breakdown of the phase symmetry [Eq. (3.27)] gets a massive component (the amplitude field), as indeed expected in the Anderson-Higgs-Kibble mechanism, and there is also a (surviving) massless component (the phase field) playing the role of the NG mode. In the following we show that such a massless component is crucially involved in the phase locking of the e.m. and matter fields.

The emerging picture is then the following. The system may be prepared with initial conditions dictated by the conservation of the particle number and given by Eqs. (2.11) and (3.6), where the value of the parameter θ_0 is in principle arbitrary within reasonable physical conditions. According to the field equations (2.15), the system then evolves towards the minimum energy state where $|a_0(t)|^2 \neq 0$ as in Eq. (3.21) and the amplitude $|u(t)|^2$ departs from its initial zero value. This implies a succession of (quantum) phase transitions [22] from the initial $u_0 = 0$ symmetric vacuum to the asymmetric vacuum $|u(t)|^2 \neq 0$, which means that in Eq. (3.26) θ_0 has to be greater than $\frac{\pi}{4}$. In this way the lower bound for θ_0 is dynamically fixed and the phase symmetry is dynamically broken in the process of phase transition to the coherent regime. The role of the phason mode φ is to recover such a symmetry, thus reestablishing the gauge invariance of the theory. This is done through the emergence of the coherence implied by the phase locking between the matter field and the e.m. field. Let us see how this happens.

The phase locking

As shown above, provided $\theta_0 > \frac{\pi}{4}$, a time-independent amplitude $U(t) \equiv \bar{U}$ is compatible with the system dynamics [e.g., the ground state value of A_0 in Eq. (3.21) implies $\bar{U} = \text{const}$, as noticed above]. Equations (3.12) and (3.13) with $\alpha_m \equiv \alpha = \delta_1(t) - \delta_0(t) - \varphi(t)$ are

$$\dot{U}(t) = 2\Omega A_0(t) A_1(t) \cos \alpha(t), \tag{3.28}$$

$$\dot{\varphi}(t) = 2\Omega \frac{A_0(t)A_1(t)}{U(t)} \sin \alpha(t). \tag{3.29}$$

We thus see that $\dot{U}(t) = 0$, i.e., a time-independent amplitude $\bar{U} = \text{const}$ exists, if and only if the phase locking relation

$$\alpha = \delta_1(t) - \delta_0(t) - \varphi(t) = \frac{\pi}{2} \tag{3.30}$$

holds. In such a case,

$$\dot{\varphi}(t) = \dot{\delta}_1(t) - \dot{\delta}_0(t) = \omega, \tag{3.31}$$

which shows that any change in time of the difference between the phases of the amplitudes $a_1(t)$ and $a_0(t)$ is compensated by the change of the phase of the e.m. field. When Eq. (3.30) holds we also have $\dot{A}_0=0=\dot{A}_1$ [see Eqs. (3.10) and (3.11)]. Provided $\theta_0 > \frac{\pi}{4}$, the phase relation (3.30) can be thus regarded as a further constant of motion implied by the dynamics: $\dot{\alpha}=0$. It expresses nothing but the gauge invariance of the theory. Since δ_0 and φ are the NG modes, Eqs. (3.30) and (3.31) also exhibit the coherent feature of the collective dynamical regime: the system of N dipoles and of the e.m. field is characterized by the "in phase" dynamics expressed by Eq. (3.30) (phase locking). In other words, the gauge invariance of the theory is preserved by the dynamical emergence of the coherence between the matter field and the e.m. field. In such a regime we have

$$\bar{A}_0^2 - \bar{A}_1^2 = \cos^2 \theta_0 - \frac{1}{3}\sin^2 \theta_0 + 2\bar{U}^2 \neq 0, \tag{3.32}$$

to be compared with $A_0^2(t) - A_1^2(t) \approx 0$ at the thermal equilibrium in the absence of the collective dynamical regime discussed here. Equation (3.32) shows the relevant role played by the occurrence of a time-independent e.m. amplitude $\bar{U}^2$; the collective dynamical regime, which sets in for $\theta_0 > \frac{\pi}{4}$, allows that a nonzero permanent polarization $P_{\mathbf{n}}$ appears when an electrical field is applied, as discussed in deriving Eq. (2.14). In the following we will come back to this point.

In conclusion we recognize that, starting at $t=0$ from the initial condition $|u(0)|^2=0$, and, correspondingly, from the zero order parameter $u_0=0$, a nonzero time–independent e.m. amplitude can develop (phase transition), provided $\theta_0 > \frac{\pi}{4}$, as an effect of the radiative dipole-dipole interaction. This results in turn in the phase locking (3.30) and in the subsequent coherence in the time behavior of the phase fields [see Eq. (3.31)]. Equations (3.30) and (3.31) show the role played by the phason field φ in recovering the gauge invariance in the process of phase transition to the collective dynamical regime.

In the collective dynamical regime considered above the values of the amplitudes A_0 and A_1 are related to the amplitude $\bar{U}$ through the relations (2.7) and (3.8). Moreover, we also obtain

$$A_0^2 = \frac{1}{3}\left[1 + \cos^2 \theta_0 + \left(1 - \frac{1}{4}\sin^2 2\theta_0\right)^{1/2}\right] \tag{3.33}$$

which used in Eq. (3.18) shows that the oscillations around the ground state for A_0 have pulsation $\nu = 2\sqrt{2}(1 - \frac{1}{4}\sin^2 2\theta_0)^{1/4}$.

The physical meaning of the phase locking can be stated as follows. The gauge arbitrariness of the field A_μ is meant to compensate exactly the arbitrariness of the phase of the matter field in the covariant derivative $D_\mu = \partial_\mu - igA_\mu$. Should one of the two arbitrarinesses be removed by the dynamics, the invariance of the theory requires the other arbitrariness, too, must be simultaneously removed, namely the appearance of a well defined phase of the matter field implies that a specific gauge function must be selected. The above link between the phase of the matter field and the gauge of A_μ is stated by the equation $A_\mu = \partial_\mu \varphi$ (A_μ is a pure gauge field). When $\varphi(\mathbf{x},t)$ is a regular (continuous differentiable) function then $\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} + \nabla A_0 = (-\frac{\partial}{\partial t}\nabla + \nabla \frac{\partial}{\partial t})\varphi = 0$, since in such a case time derivative and the gradient operator can be interchanged. Analogously, in the space of the regular functions $\varphi(\mathbf{x},t)$ it is $\mathbf{B} = \nabla \times \mathbf{A} = \nabla \times \nabla \varphi = 0$. Thus the existence of nonvanishing fields $\mathbf{E}$ and $\mathbf{B}$ in a coherent region implies that the time and space derivatives should act on a space larger than the space of regular functions, namely, $\varphi(\mathbf{x},t)$ should exhibit a (divergence or a topological) singularity within the region [8]. This is precisely what is observed, e.g., in type II superconductors when penetrated by the lines of a quantized flux in a vortex core.

IV. DISCUSSION AND CONCLUSIVE REMARKS

We have seen above that the rescaling of the field by the factor $\sqrt{N}$ [see Eq.(2.2)] induces the enhancement by the same factor of the coupling constants appearing in the field equations (2.15) [see also the coupling Ω introduced in Eqs. (3.1)–(3.3)]. This implies that for large N the collective interaction time scale is much shorter (by the factor $\frac{1}{\sqrt{N}}$) than the short range interactions among the atoms. Hence the mesoscopic/macroscopic stability of the system vs the quantum fluctuations in the short range dynamics of the microscopic components. For the same reason, for sufficiently large N the collective interaction is protected against thermal fluctuations. Indeed, thermal fluctuations could affect the collective process only when kT is comparable or larger than the energy gap, whose value thus determines the height of the protection; the larger the energy gap, the more robust the protection. We do not present in this paper an estimate of the energy gap. We will do that in a future work. The role of the factor $\sqrt{N}$ in setting the time scale of the system can be understood also in connection with the influence on the system of atoms of an electric field $\mathbf{E}$ due, e.g., to an impurity (or to any other external agent). By closely following Ref. [23], we will see indeed that for large N the system of atoms behaves as a collective whole. The interaction $\mathcal{H} = -\mathbf{d}\cdot\mathbf{E}$ [Eq. (2.12)] with the electrical field can be written [13] as

$$H = \hbar\,\gamma(b^\dagger \sigma^- + b\sigma^+), \tag{4.1}$$

which is a Jaynes-Cummings–like Hamiltonian, as already mentioned in Sec. I in connection with Haken analysis. In Eq. (4.1) γ is the coupling constant which is proportional to the matrix element of the atomic dipole moment and to the inverse of the volume square root $V^{-1/2}$, b is the electric field quantum operator, $\sigma^\pm$ are the atomic polarization operators. Let $|0\rangle_i$ and $|1\rangle_i$, $i=1,\ldots,N$, denote the ground state and the excited state of each of the N two-level atoms, respectively, associated to the eigenvalues $\mp\frac{1}{2}$ of the operator $\sigma_{3i} = \frac{1}{2}(|1\rangle_{ii}\langle 1| - |0\rangle_{ii}\langle 0|)$, no summation on i (see, e.g., Sec. III.6

of Ref. [15] for the formal equivalence of the system of two-level atoms with a system of $\frac{1}{2}$ spins). The operators $\sigma_i^+ = |1\rangle_{ii}\langle 0|$ and $\sigma_i^- = (\sigma_i^+)^\dagger$ generate the transitions between the two levels induced by the action of the electric field. The N-atom system is thus described by $\sigma^\pm = \Sigma_{i=1}^N \sigma_i^\pm, \sigma_3 = \Sigma_{i=1}^N \sigma_{3i}$ with the fermionlike su(2) algebra

$$[\sigma_3, \sigma^\pm] = \pm \sigma^\pm, \quad [\sigma^-, \sigma^+] = -2\sigma_3. \tag{4.2}$$

Suppose that the electric field action induces the transition $|0\rangle_i \to |1\rangle_i$ for a certain number of atoms, say l (as far as $N \gg l$ our conclusions will not be affected by the fact that initially some of the atoms are not in their ground state). The system state may be then represented as the normalized superposition $|l\rangle$ given by

$$|l\rangle \equiv (|0\rangle_1|0\rangle_2 \cdots |0\rangle_{N-l}|1\rangle_{N-l+1}|1\rangle_{N-l+2} \cdots |1\rangle_N + \cdots + |1\rangle_1|1\rangle_2 \cdots |1\rangle_l|0\rangle_{l+1}|0\rangle_{l+2} \cdots |0\rangle_N) \Big/ \sqrt{\binom{N}{l}}. \tag{4.3}$$

The difference between the number of atoms in the excited state and the ones in the ground state is measured by σ_3:

$$\langle l|\sigma_3|l\rangle = l - \frac{1}{2}N \tag{4.4}$$

and the nonzero value of this quantity (proportional to the system polarization) signals that the dipole rotational [SU(2)] symmetry is broken. Operating with $\sigma^\pm$ on $|l\rangle$ gives

$$\sigma^+|l\rangle = \sqrt{l+1}\sqrt{N-l}|l+1\rangle,$$

$$\sigma^-|l\rangle = \sqrt{N-(l-1)}\sqrt{l}|l-1\rangle. \tag{4.5}$$

Equations (4.4) and (4.5) show that σ_3 and $\frac{\sigma^\pm}{\sqrt{N}}$ are represented on $|l\rangle$ by

$$\sigma_3 = S^+S^- - \frac{1}{2}N,$$

$$\frac{\sigma^+}{\sqrt{N}} = S^+\sqrt{1 - \frac{S^+S^-}{N}}, \quad \frac{\sigma^-}{\sqrt{N}} = \sqrt{1 - \frac{S^+S^-}{N}}S^-, \tag{4.6}$$

where $S^- = (S^+)^\dagger$, $[S^-, S^+] = 1$, $S^+|l\rangle = \sqrt{l+1}|l+1\rangle$, and $S^-|l\rangle = \sqrt{l}|l-1\rangle$, for any l. Equations (4.6) are the Holstein-Primakoff nonlinear boson realization of SU(2) [24,25]. $\frac{\sigma^\pm}{\sqrt{N}}$ in Eqs. (4.6) still satisfy the su(2) algebra (4.2). However, since for $N \gg l$ Eqs. (4.5) give

$$\frac{\sigma^\pm}{\sqrt{N}}|l\rangle = S^\pm|l\rangle, \tag{4.7}$$

the su(2) algebra (4.2) contracts in the large N limit to the (projective) $e(2)$ algebra (or Weyl-Heisenberg algebra) [26,27,23]

$$[S_3, S^\pm] = \pm S^\pm, \quad [S^-, S^+] = 1, \tag{4.8}$$

where $S_3 \equiv \sigma_3$. From Eqs. (4.7) and (4.8) we see that, for large N, $S^\pm$ denote the creation and annihilation *boson* operators associated to the quanta of collective dipole waves excited by the electric field. The interaction (4.1) can now be written in terms of $S^\pm$ as

$$H = \hbar\sqrt{N}\gamma(b^\dagger S^- + bS^+). \tag{4.9}$$

We thus conclude that in the large N limit the collection of single two-level (fermionlike) atoms appears as a collective bosonic system. The original coupling of the individual atoms to the field gets enhanced by the factor $\sqrt{N}$ and appears as the coupling of the collective modes $S^\pm$ (the system as a whole) to the field. We observe that, as shown by Eq. (2.14), the polarization persists as far as τ is nonzero, namely, as far as the field $\mathbf{E}$ is active (i.e., $\mathcal{H} \neq 0$). The system finite size prevents indeed from having a persistent polarization surviving the $\mathcal{H} \to 0$ limit [8]. In such a limit the dipole rotational symmetry is thus restored.

Finally, we note that the collective dynamical features presented here are not substantially affected by energy losses from the system volume, which we have not considered in the discussion above. These losses are related with the different lifetimes of our different modes, according to the different time scales associated to the pulsations m_0, σ, and μ. An analysis of energy losses when the system is enclosed in a cavity has been presented elsewhere in connection with the problem of efficient cooling of an ensemble of N atoms [23]. Another problem which we have not considered in this paper is the one related to how much time the system demands to set up the collective regime. This problem, which is a central one in the domain formation in the Kibble-Zurek scenario, will be the object of our study in a future work. Here we remark only that, since the correlation among the elementary constituents is kept by a pure gauge field, the communication among them travels at the phase velocity of the gauge field.

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