Quantum dynamics in fluctuating traps: Master equation, decoherence, and heating

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We present a remarkably simple derivation of an exact time-local master equation describing the dynamics of quantum states in harmonic traps subject to arbitrary fluctuating forces. The relation between our master equation and known master equations of irreversible harmonic oscillator dynamics are established. Motivated by recent experiments, we focus on decoherence and in particular on the precise decoherence dynamics of a superposition of wave packets. We determine the decaying purity resulting from the time evolution based on our master equation and study its connection to experimentally accessible observables. Finally, we discuss the heating of the system subjected to arbitrary Gaussian noise.

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I. INTRODUCTION

Impressive advances in experimental studies of quantum dynamics on more and more macroscopic scales revives the old question about the quantum-classical transition when approaching the macroscopic level [1,2]. Quantum interference effects become more and more difficult to observe the more macroscopic the bodies and thus, the shorter the wave lengths involved. Indeed, interference fringes for more and more massive molecules have been seen [3] and a detailed discussion about the origins of decoherence in these experiments and the limits of seeing an interference pattern are available. In the very active field of quantum information processing questions about the stability and coherence of qubit registers is of paramount importance, and a wealth of theoretical results and proposals that hope to overcome the limitations set by decoherence are available [4].

A very remarkable feature of decoherence in systems with large Hilbert space dimension is the fact that certain superpositions of states can lose their coherence on a quantum time scale that may be significantly shorter than any dissipation or relaxation time scale [1,2,5–7]. Beautiful experiments on decoherence confirm these expectations [8–10]—they focused on the dependence of the decoherence time on a "distance" of the initially superposed states. In the first of these experiments [8], decoherence of a superposition of coherent states of a light mode in a microwave cavity has been investigated. The second experiment was based on the motional state of a single ion in a harmonic trap and the decoherence of various superpositions of states subjected to various environmental influences was studied [9,10]. The results presented in this paper are motivated by these latter experiments in ion traps.

In most theoretical investigations of decoherence, it is the coupling to unobserved quantum degrees of freedom that is put forward as the fundamental cause for decoherence—in other words, growing entanglement with an environment leads to decoherence. It appears sometimes that these investigations have led to the belief that entanglement is the primary cause of decoherence—see, however [11,12]. In such studies, standard methods are employed that allow one to derive effective master equations for the reduced dynamics of such quantum subsystems. More often than not, however, experimentally observed decoherence is due to rather "classical," i.e., technical and—from a fundamental point of view—rather simple causes. Experimental fluctuations are very often the dominating cause for decoherence. In one series of the experiments with the ion trap [9,10], an "environment" was in fact simulated by simply "shaking" the trap and thus "engineering" the effects of an environment.

Motivated by these experiments involving single ions in traps, in this paper we want to investigate quantum dynamics under the influence of simple *classical* fluctuations with an arbitrary correlation function. In other words, we focus on unitary, yet random dynamics. No entanglement is involved. Being conceptually much simpler than problems involving genuine dynamical quantum environments, the derivation of a time-local master equation for arbitrary Gaussian fluctuations is surprisingly simple and one of the main results of this paper. It can be seen as the classical version of the well-known time-local exact master equation for the damped harmonic oscillator coupled to a bath of harmonic oscillators [13]. The resulting master equation, nevertheless, is rich in that it contains all the relevant physics usually attributed to the dynamics of quantum subsystems. We are therefore able to investigate various aspects of decoherence in such systems using well-known and simple concepts. Earlier work in this spirit is the paper by Schneider and Milburn [14] who restrict themselves to white noise. Related considerations may also be found in the work of O'Connell and Zuo [15,16], who determine an "attenuation factor" as an indicator for (the loss of) coherence of an ion in a fluctuating trap shaken by white noise. Here we derive an underlying master equation for the ion dynamics for arbitrary Gaussian fluctuations and use purity as a measure for decoherence. Furthermore, we devote a whole section on the question of how master equation dynamics and the corresponding purity decay are related to the actually measured quantity. This relation turns out to hold in the weak coupling limit only. Remarkably, while experimentally decoherence due to a certain dephasing is measured, the master equation captures both dephasing and an additional random state diffusion. We show that both these causes for

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decoherence have the very same effect on purity decay (see Sec. VI for details). As a result, in the weak coupling limit, there is indeed a one-to-one correspondence between purity decay as predicted from master equation dynamics and the experimentally measured decay of certain probabilities reflecting the decaying coherences.

Our results are valid for arbitrary fluctuations and may result in nonexponential decay of coherences for noise with a finite correlation time. Moreover, since there is no mechanism for damping in shaken traps, the ion will heat up and we determine the precise dynamics of heating under the influence of arbitrary noise.

It should go without saying that there is a wealth of possibilities to describe quantum dynamics under the influence of classical fluctuations. In the white-noise case, matters are simple as one may employ a standard Ito calculus [17] (see also Sec. II). For a finite correlation time of the noise, a path integral approach may appear appropriate (see, for instance Feynman and Vernon [18]). The simplicity of harmonic oscillators, however, allows us to derive a *time-local* master equation for the dynamics with which we obtain all the following results in a rather straightforward way. It is the first aim of this paper to present this simple derivation of the master equation and to show applications next.

We focus on the analytical solution of our master equation and in particular investigate decoherence. We show that any initial state, after a finite time t_D will be a classical mixture of coherent states of the oscillator. In other words, no quantumness will prevail. Motivated by the experiments with single ions in traps [9,10], we further focus on the precise decoherence dynamics of a superposition of coherent states with a phase space distance Δz, a class of states that—as usual—we will refer to as "Schrödinger cat" states.

We calculate purity decay as resulting from our master equation and show the connection to the actually measured probability. It is of interest to note that the way the experiment is designed, it does in fact not really probe master equation dynamics but rather a specific type of dephasing.

Finally, we discuss the heating of the system due to arbitrary Gaussian fluctuations, and close with final conclusions and a brief outlook in Sec. VIII.

II. MASTER EQUATION FOR DYNAMICS IN A FLUCTUATING TRAP

We describe quantum dynamics in a harmonic oscillator trap with Hamiltonian $H_0=\frac{1}{2m}p^2+\frac{1}{2}m\omega^2q^2$ and commutation relation $[q,p]=i\hbar$, subject to classical noise. We restrict ourselves to one dimension. The generalization to three dimensions is straightforward and will yield no additional insight into what we are going to elaborate on. We assume that there is a classical random external force $F(t)$ acting on the particle, such that the total Hamiltonian reads

$$H(t)=H_0+qF(t)=\frac{p^2}{2m}+\frac{1}{2}m\omega^2q^2+qF(t). \tag{1}$$

The linear force term $qF(t)$ amounts to shaking the trap as it simply shifts randomly the origin of the otherwise unchanged harmonic potential.

We assume that the fluctuating force $F(t)$ is a classical Gaussian stochastic process determined by its first and second order moments

$$\langle\langle F(t)\rangle\rangle \equiv 0 \text{ and } \langle\langle F(t)F(s)\rangle\rangle \equiv C(t-s) \tag{2}$$

with an arbitrary correlation function $C(t-s)$. Assuming stationarity, the correlation function depends on the time difference $(t-s)$ only. This assumption is made purely for convenience—and could easily be dropped. It should be kept in mind throughout this paper that we use the notation $\langle\langle\cdots\rangle\rangle$ to denote the *classical ensemble average* over the classical noise $F(t)$. The simple brackets $\langle\cdots\rangle=\mathrm{Tr}[\cdots\rho]$ are reserved for quantum expectation values. For some concrete examples that will follow, we choose an Ornstein-Uhlenbeck process such that $C(t-s)=D\Gamma e^{-\Gamma|t-s|}/2$ with a constant $D=2\int_0^\infty dsC(s)$, related to diffusion. The frequency Γ is the inverse noise correlation (or memory) time. Keep in mind, however, that all main results of this paper are valid for any choice of $C(t-s)$, and in fact, even for a nonstationary $C(t,s)$.

As the fundamental Hamiltonian (1) depends on the noise $F(t)$, the solution $\rho(t)$ of von Neumann's equation

$$\frac{\partial}{\partial t}\rho(t)=\frac{1}{i\hbar}[H(t),\rho(t)] \tag{3}$$

is a stochastic process in the space of states. Of experimental relevance is the average state

$$\bar{\rho}(t)\equiv\langle\langle\rho(t)\rangle\rangle,$$

and it is the first aim of this paper to derive its evolution equation for an arbitrary correlation function $\langle\langle F(t)F(s)\rangle\rangle=C(t-s)$ of the fluctuating force.

As the relevant stochastic Hamiltonian (1) is quadratic, the formal solution of von Neumann's equation (3) is readily available. First, we introduce time dependent classical functions

$$P(t)=\int_0^t dsF(s)\cos\omega s \text{ and } Q(t)=\frac{1}{m\omega}\int_0^t dsF(s)\sin\omega s$$

which have dimension of momentum and position, respectively. Straightforward inspection then shows that the solution of Eq. (3) may be written in the form

$$\rho(t)=e^{-(i/\hbar)H_0t}e^{-(i/\hbar)P(t)q}e^{-(i/\hbar)Q(t)p}\rho(0)e^{(i/\hbar)Q(t)p}\times e^{(i/\hbar)P(t)q}e^{(i/\hbar)H_0t}. \tag{4}$$

As the operators in the exponents do not commute, taking the ensemble average $\langle\langle\cdots\rangle\rangle$ over the noise is a bit tedious. It is very easy, however, to derive the evolution equation for the average state $\bar{\rho}(t)$. Taking the ensemble average of the von Neumann equation (3) we find

$$\frac{\partial}{\partial t}\bar{\rho}=\frac{1}{i\hbar}[H_0,\bar{\rho}]+\frac{1}{i\hbar}[q,\langle\langle F(t)\rho\rangle\rangle].$$

At first sight, due to the appearance of the average $\langle\langle F(t)\rho\rangle\rangle$ this does not appear to lead to a closed evolution equation for $\bar{\rho}=\langle\langle\rho\rangle\rangle$. However, using the Novikov-formula [19]

$$\langle\langle F(t)\rho(t)\rangle\rangle = \int_0^t ds\langle\langle F(t)F(s)\rangle\rangle\left\langle\left\langle \frac{\delta\rho(t)}{\delta F(s)}\right\rangle\right\rangle, \tag{5}$$

it can be transformed to a time-local master equation for $\bar{\rho}$. Formula (5) is valid for any Gaussian process and simply reflects a partial integration over the functional distribution of the noise $F(t)$.

In our case, evaluating the required functional derivative of $\rho(t)$ with respect to earlier noise $F(s)$ is easy due to the explicit expression (4) for $\rho(t)$. We find the commutator

$$\frac{\delta\rho(t)}{\delta F(s)} = \frac{1}{i\hbar}\left[q\cos\omega(t-s) - \frac{p}{m\omega}\sin\omega(t-s), \rho(t)\right],$$

which luckily enough no longer explicitly depends on the noise $F(t)$. Taking the ensemble average is now trivial and leads to a time-local master equation for the averaged density operator which we write in the form

$$\frac{\partial}{\partial t}\bar{\rho} = \frac{1}{i\hbar}[H_0,\bar{\rho}] - \frac{\Theta(t)}{\hbar^2}[q,[q,\bar{\rho}]] + \frac{\Sigma(t)}{m\omega\hbar^2}[q,[p,\bar{\rho}]] \tag{6}$$

with time dependent functions

$$\Theta(t) \equiv \int_0^t dsC(s)\cos\omega s,$$

$$\Sigma(t) \equiv \int_0^t dsC(s)\sin\omega s. \tag{7}$$

We note that the "memory integrals" for the functions $\Theta(t)$, $\Sigma(t)$ extend over the correlation function so that typically, these time dependent functions approach constant asymptotic values for times larger than the noise correlation time. To be specific, very often we will study the case of an Ornstein-Uhlenbeck correlation function

$$C(s) = D\Gamma e^{-\Gamma|s|}/2 \tag{8}$$

for which we find the asymptotic values $\Theta(t)\to\Theta(\infty) = \frac{D}{2}\Gamma^2/(\Gamma^2+\omega^2)$ and $\Sigma(t)\to\Sigma(\infty)=\frac{D}{2}\Gamma\omega/(\Gamma^2+\omega^2)$.

The exact master equation (6) is one of the main results of this paper. As we have shown, its derivation is remarkably simple. It is valid for any Gaussian driving noise $F(t)$ with zero mean and correlation function $C(t-s)$ [in fact, any $C(t,s)$].

Applications and further investigations related to Eq. (6) will be presented in this paper: A phase space formulation of our master equation in terms of a Focker-Planck-type equation for the Wigner function will be given in Sec. IV. We study decoherence in shaken traps, and, motivated by decoherence experiments [8–10] we investigate in detail the decoherence of an initial superposition of coherent states of the oscillator in Sec. V.

We emphasize that the master equation (6) does not include any damping of the particle. The irreversible nature of the master equation arises solely from the classical noise, without any compensating damping, as it would arise from the coupling of the particle to a genuine (quantum) dynamical environment (see Sec. III). Therefore, an asymptotic stationary solution of (6) does not exist. On average, the particle will continuously increase its energy due to the driving fluctuations $F(t)$. We will determine the precise time dependence of the heating of the particle due to arbitrary Gaussian fluctuations $F(t)$ in Sec. VII.

We add a few notes for the more mathematically inclined reader: by construction, the exact master equation (6) defines a completely positive map [20] from the initial density operator $\rho(0)$ to $\rho(t)$. In fact, as it is obtained from an average of *unitary* propagators representing the Krauss operators, Eq. (6) defines a nice example of a *bistochastic* map [21] in the infinite dimensional Hilbert space of the trapped particle (or even more precise: it represents a *random external field* channel [22]). However, due to the time dependence of the coefficients $\Theta(t)$, $\Sigma(t)$ the evolution does not possess the semigroup property and therefore, the exact master equation (6) is *not* of the usual Lindblad type [23], not even in its asymptotic form, when the coefficients approach their constant asymptotic values.

In Sec. III we compare and contrast our master equation (6) to well-known master equations for irreversible harmonic oscillator dynamics. Furthermore, we determine the general solution of master equation (6) in terms of the Wigner function of $\bar{\rho}$ (or rather its Fourier transform) in Sec. IV. Before doing so, however, we discuss two important limiting cases of our general master equation (6), the white-noise limit and the rotating wave limit.

A. Standard white-noise limit

It is worth studying the limit of white noise, when formally

$$\langle\langle F(t)F(s)\rangle\rangle = C(t-s) \equiv D\delta(t-s). \tag{9}$$

This limit may be seen as arising from the more physical Ornstein-Uhlenbeck correlation function mentioned earlier in the limit of zero correlation time or $\Gamma\to\infty$. It is clear that in this case we find for any $t>0$ the constant $\Theta(t)=\Theta(\infty) = D/2$ and $\Sigma(t)=\Sigma(\infty)=0$, so that the exact master equation (6) reduces to

$$\frac{\partial}{\partial t}\bar{\rho} = \frac{1}{i\hbar}[H_0,\bar{\rho}] - \frac{D}{2\hbar^2}[q,[q,\bar{\rho}]]. \tag{10}$$

This master equation, which *is* of Lindblad type, has been used extensively in the past to study decoherence both in the framework of "quantum Brownian motion" (neglecting damping) and also in "quantum measurement" models [1,2,24–27]. Remarkably, for any finite noise correlation time, the true master equation (6) displays an additional mixed diffusion term $[q,[p,\bar{\rho}]]$. We repeat that equations of type (6) and (10) do not contain any damping terms and lead to continuous heating (see also Secs. III and VII).

For completeness we mention that quite generally, the white-noise limit (10) is obtained most conveniently using

stochastic Ito calculus [17]. We may introduce normalized Wiener increments through $dW(t)=\frac{1}{\sqrt{D}}\int_t^{t+dt}F(s)ds$ such that $dW^2=dt$. The stochastic Schrödinger equation corresponding to the Hamiltonian (1) reads

$$d|\psi\rangle = \frac{1}{i\hbar}H_0|\psi\rangle dt + \frac{\sqrt{D}}{i\hbar}q|\psi\rangle \circ dW,$$

which has to be interpreted as a Stratonovich stochastic differential equation [17]. Using $f\circ dg=fdg+\frac{1}{2}dfdg$ it can be transformed to its equivalent Ito form

$$d|\psi\rangle = \frac{1}{i\hbar}H_0|\psi\rangle dt - \frac{D}{2\hbar^2}q^2|\psi\rangle dt + \frac{\sqrt{D}}{i\hbar}q|\psi\rangle dW.$$

With the Ito rule $d(|\psi\rangle\langle\psi|)=(d|\psi\rangle)\langle\psi|+|\psi\rangle(d\langle\psi|)+(d|\psi\rangle)\times(d\langle\psi|)$ and $dW^2=dt$, it is now straightforward to show that the ensemble mean $\bar{\rho}=\overline{|\psi\rangle\langle\psi|}$ satisfies (10). Note that in this white-noise case, the Lindblad master equation (10) is formally valid for *any* H_0, i.e., for any potential governing the dynamics of the particle. Our exact master equation (6) holds for any correlation function $C(t,s)$ of the noise, but is restricted to the harmonic Hamiltonian H_0 as in (1).

B. Rotating wave approximation

Master equations for irreversible harmonic oscillator motion that include (thermal) fluctuations and possibly damping have been used in quantum optics and other fields for many decades [29]. In quantum optics, due to the large photon energy of atomic transitions, a rotating wave approximation (RWA) is a formidable simplification. In order to perform a rotating wave approximation in our case, we rewrite the master equation (6) in terms of the usual annihilation (creation) operator defined through $a=\sqrt{\frac{m\omega}{2\hbar}}\left(q+\frac{i}{m\omega}p\right)$ (and $a^\dagger$ accordingly). Moreover, we switch to an interaction representation with respect to the isolated harmonic oscillator Hamiltonian $H_0=\hbar\omega\left(a^\dagger a+\frac{1}{2}\right)$ such that $\bar{\rho}(t)\to\tilde{\rho}(t)\equiv e^{(i/\hbar)H_0t}\bar{\rho}(t)e^{-(i/\hbar)H_0t}$ and $a\to ae^{-i\omega t}$. Neglecting terms that rotate with phases $e^{\pm 2i\omega t}$ with respect to terms that do not rotate at all, we find that the time evolution of the transformed density operator may be written in the form

$$\frac{\partial}{\partial t}\tilde{\rho} = \frac{\Theta(t)}{2m\omega\hbar}(([a,\tilde{\rho}a^\dagger]+[a\tilde{\rho},a^\dagger])+([a^\dagger,\tilde{\rho}a]+[a^\dagger\tilde{\rho},a])). \tag{11}$$

The contribution of the term involving $\Sigma(t)$ disappears entirely. The rotating wave equation (11) is valid whenever the time scale induced by the prefactor $\frac{\Theta(t)}{2m\omega\hbar}$ (which has the dimension of inverse time) is slow compared to the time scale of the oscillations, i.e., as long as

$$\frac{\Theta(t)}{2m\omega} \ll \hbar\omega. \tag{12}$$

Recall that in the white-noise limit $\Theta=D/2$ with D a measure of the strength of the fluctuations according to the force correlation function (9). The left-hand side of condition (12) can be seen as consisting entirely of classical quantities, while the right-hand side is a quantized energy. We conclude that the rotating wave equation (11) can only be valid for rather weak fluctuations (i.e., of quantum scale), their strength being limited by condition (12).

III. RELATION TO ESTABLISHED MASTER EQUATIONS

Before we give the general solution of our master equation (6) of a particle in a fluctuating trap in Sec. IV, we want to compare and contrast it to two well-established master equations for irreversible harmonic oscillator dynamics in the next two subsections.

A. Soluble quantum Brownian motion model

A widely studied model in irreversible quantum dynamics is a harmonic oscillator, coupled bilinearly to a bath of environmental quantum oscillators [30,13]. The total Hamiltonian of this model reads $H_{\rm tot}=H_0+H_{\rm res}+H_{\rm int}$. The three contributions to the total energy represent the energy H_0 of the oscillator of interest as before, the energy $H_{\rm res}$ of all reservoir oscillators, and the interaction between the two, $H_{\rm int}$, involving coupling constants g_i. In detail, we have

$$H_0 = \frac{p^2}{2m}+\frac{1}{2}m\omega^2q^2,$$

$$H_{\rm res} = \sum_{i=1}^{N}\left(\frac{P_i^2}{2M_i}+\frac{1}{2}M_i\Omega_i^2Q_i^2\right),$$

$$H_{\rm int} = q\sum_{i=1}^{N}g_iQ_i \equiv q\hat{F}, \tag{13}$$

where the very last equality serves to define the force operator $\hat{F}=\Sigma_{i=1}^{N}g_iQ_i$.

In interaction representation with respect to the bath we can transform the relevant total energy to the form

$$\tilde{H}_{\rm tot}(t) = H_0+\tilde{H}_{\rm int}(t) = \frac{p^2}{2m}+\frac{1}{2}m\omega^2q^2+q\hat{F}(t) \tag{14}$$

with a time dependent quantum force operator

$$\hat{F}(t) = e^{iH_{\rm res}t/\hbar}\hat{F}e^{-iH_{\rm res}t/\hbar} = \sum_{i=1}^{N}g_i\left(Q_i\cos\Omega_it+\frac{P_i}{M_i\Omega_i}\sin\Omega_it\right).$$

The formal similarity of (14) with the "classically driven" Hamiltonian (1) is obvious. However, note that the environmental degrees of freedom (Q_i,P_i) that constitute the quantum force $\hat{F}(t)$ are noncommuting, and, moreover are dynamical variables whose time evolution is governed by Heisenberg's equations of motion involving $H_{\rm tot}$. Thus, the environmental degrees of freedom are able to absorb energy from the system: this model includes energy dissipation (damping). Correspondingly, the resulting exact master equation turns out to be more complex compared to our classically driven equation (6).

For the model given by (13) it is most often assumed that the total initial state is a product

$$\rho_{\text{tot}}(0) = \rho(0) \otimes \rho_{\text{res}}(0) \tag{15}$$

of an arbitrary system state $\rho(0)$ and a thermal environmental state $\rho_{\text{res}}(0)=e^{-H_{\text{res}}/kT}/Z$, with the partition function Z acting as the usual normalization factor. For the quantum force correlation function one finds the standard expression [31]

$$\langle \hat{F}(t)\hat{F}(s)\rangle = \sum_i \frac{\hbar g_i^2}{2M_i\Omega_i}[(2n_{\text{th}}(\Omega_i)+1)\cos\,\Omega_i(t-s) - i\,\sin\,\Omega_i(t-s)], \tag{16}$$

where $n_{\text{th}}(\Omega)=(e^{\hbar\Omega/kT}-1)^{-1}$ is the thermal number of quanta in the oscillator mode with angular frequency Ω and the expectation value $\langle\cdots\rangle=\text{Tr}[\rho_{\text{res}}(0)\cdots]$ refers to the initial environmental state.

We separate the real and imaginary part of the quantum correlation function (16) in the usual fashion and write

$$\langle \hat{F}(t)\hat{F}(s)\rangle = \kappa(t-s) + i\hbar\lambda(t-s). \tag{17}$$

Assuming the number N of bath oscillators to be large, it is customary [31] to introduce a spectral density

$$J(\Omega) \equiv \sum_i \frac{g_i^2}{2M_i\Omega_i}\delta(\Omega-\Omega_i)$$

such that the real and imaginary part of the correlation may be expressed

$$\kappa(t-s) = \left\langle \frac{1}{2}\{\hat{F}(t),\hat{F}(s)\}\right\rangle = 2\hbar\int_0^\infty d\Omega J(\Omega)\left[n_{\text{th}}(\Omega)+\frac{1}{2}\right]\times\cos\,\Omega(t-s),$$

$$\lambda(t-s) = \left\langle \frac{1}{2i\hbar}[\hat{F}(t),\hat{F}(s)]\right\rangle = -\int_0^\infty d\Omega J(\Omega)\sin\,\Omega(t-s). \tag{18}$$

While the real part $\kappa(t-s)$ describes equilibrium fluctuations (diffusion kernel) both of thermal $[n_{\text{th}}(\Omega)]$ and vacuum $(+\frac{1}{2})$ origin, the imaginary part $\lambda(t-s)$ reflects damping. The latter is related to the classical damping kernel $\gamma(t)=2\int_0^\infty d\Omega\frac{J(\Omega)}{\Omega}\cos\,\Omega(t)$ of this model through $\lambda(t-s)=\frac{1}{2}\partial_t\gamma(t-s)$—for details, see for instance [31].

The model based on the three energies (13) with initial state (15) allows for the derivation of an exact, time-local evolution equation for the reduced density operator [13], which can be written in the form

$$\frac{\partial}{\partial t}\rho = \frac{1}{i\hbar}[H_0,\rho] + \frac{a(t)}{i\hbar}[q^2,\rho] + \frac{b(t)}{i\hbar m\omega}[q,\{p,\rho\}] + \frac{c(t)}{m\omega\hbar^2}[q,[p,\rho]] - \frac{d(t)}{\hbar^2}[q,[q,\rho]], \tag{19}$$

with time dependent coefficients $a(t),\ldots,d(t)$ whose explicit expressions are well known yet some of them are somewhat complicated and lengthy [13,32,33]. Their time dependence may be expressed in terms of a special solution $q_0(t)$ of the damped classical equation

$$\ddot{q}_0 + \omega^2 q_0 + \frac{2}{m}\int_0^t ds\lambda(t-s)q_0(s) = 0 \tag{20}$$

with initial values $q_0(0)=0$ and $\dot{q}_0(0)=\omega$. To be more specific, one finds expressions of the form [33]

$$a(t) = \int_0^t ds\lambda(t-s)w(t,s),$$

$$b(t) = \int_0^t ds\lambda(t-s)x(t,s),$$

$$c(t) = \int_0^t ds\kappa(s)q_0(s) - \hbar\int_0^t ds\lambda(t-s)z_1(t,s),$$

$$d(t) = \frac{1}{\omega}\int_0^t ds\kappa(s)\dot{q}_0(s) - \hbar\int_0^t ds\lambda(t-s)y_1(t,s). \tag{21}$$

Here, $w(t,s),x(t,s),z_1(t,s)$, and $y_1(t,s)$ are real-valued functions that may be expressed in terms of the fundamental solution $q_0(t)$ from (20), for instance $w(x,t)=[\dot{q}_0(s)\dot{q}_0(t)-q_0(s)\ddot{q}_0(t)]/[\dot{q}_0^2(t)-q_0(t)\ddot{q}_0(t)]$. Explicit formulae for the remaining functions may be found in the Appendix of [33] [Eqs. (B18) and (B19)], but are of no relevance for our following discussion.

The relation between our master equation (6) for a classical fluctuating force and the quantum master equation (19) for a harmonic oscillator coupled to a bath of oscillator is now easily established. Obviously, the diffusion-type terms $[q,[q,\rho]]$ and $[q,[p,\rho]]$ appear in both master equations, while the two terms involving $[q^2,\rho]$ and $[q,\{p,\rho\}]$ are missing in our Eq. (6). The latter terms describe a dynamical potential renormalization due to the coupling (sometimes absorbed in a so-called "counterterm" [31]) and, more importantly, *damping*.

Let us for a moment formally neglect damping entirely, i.e., let us set the imaginary part of the correlation function equal to zero, $\lambda(t-s)\equiv 0$. Then the two coefficients $a(t)$ and $b(t)$ are zero and the quantum master equation (19) takes the very same form as our master equation (6). Moreover, without damping the special solution of the classical equation (20) is simply given by $q_0(t)=\sin\,\omega t$ and from Eq. (21) we see that the two remaining coefficients take the simple form $c(t)=\int_0^t ds\kappa(s)\sin(\omega t)$ and $d(t)=\int_0^t ds\kappa(s)\cos(\omega t)$. A glance at expressions (7) shows that we find complete formal equivalence of the two master equations once we identify—not surprisingly—the real part $\kappa(t-s)$ of the quantum bath correlation function with the correlation function (16) $C(t-s)$ of the classical noise (2).

We see that our master equation (6) arises as a special case of the quantum master equation (19) for $\kappa(t-s)=C(t-s)$ and $\lambda(t-s)=0$. However, the real and imaginary part of the quantum correlation function cannot be chosen independently. They are connected through (17) with (16), manifest-

ing a fluctuation-dissipation relation. As should be clear from (18), the no damping limit in this model can only be achieved through vanishing coupling strength $J(\Omega)\to 0$, or, microscopically, $g_i\to 0$ for all i in the interaction energy in (13). In order to retain a nonvanishing fluctuation kernel $\kappa(t-s)$ as $J(\Omega)\to 0$, we have to let the temperature go to infinity, such that the product $J(\Omega)n_{\rm th}(\Omega)$ remains finite.

We conclude that our master equation for a classically driven harmonic oscillator (6) may be obtained formally from the fully quantum model (13) in the limit of vanishing spectral density (i.e., vanishing interaction strength) and infinite temperature, such that the fluctuation kernel $\kappa(t-s)$ in (18) which involves their product, remains finite. Needless to say, our simple direct derivation of (6) presented in Sec. II is notably simpler and infinitely more straightforward.

As a final note we want to emphasize that the simple "classical limit" in the sense $\hbar\to 0$ does not lead us from the quantum master equation (19) to the classically driven equation (6), since the damping term has a proper classical limit as $\hbar\to 0$ and remains. Only if the coupling tends to zero (and the temperature goes to infinity), we may go from (19) to (6).

B. Quantum optical master equation for a damped harmonic oscillator

One of the most-employed master equations in quantum optics describes the (Markovian) dynamics of the state of a field mode of frequency ω inside a cavity. The mode is coupled to the field modes outside through a leaking mirror, leading to (energy) damping with a rate γ. At temperature T, such that the mean thermal occupation number is $\bar{n}=(e^{\hbar\omega/kT}-1)^{-1}$, one finds the master equation

$$\dot{\rho}=-i\omega[a^\dagger a,\rho]+\frac{\gamma}{2}(\bar{n}+1)([a,\rho a^\dagger]+[a\rho,a^\dagger])+\frac{\gamma}{2}\bar{n}([a^\dagger,\rho a]+[a^\dagger\rho,a]). \tag{22}$$

The derivation of this master equation rests on the Born-Markov- and rotating wave approximation and can be found in most quantum optics text books, for instance [34].

The relation between this dissipative quantum optical master equation (22) and our non-dissipative master equation for a fluctuating classical force (6) may be established similarly as in the previous subsection for the exact quantum Brownian motion master equation. Clearly, dissipation in (22) can only be suppressed for vanishing damping rate γ. In order to retain a nontrivial master equation as $\gamma\to 0$, we have to let the temperature go to infinity $T\to\infty$, such that the product $\bar{n}\gamma$ remains finite. More precisely, in the high-temperature limit we have $\bar{n}\approx kT/(\hbar\omega)$ and set $\tilde{\Theta}\equiv mkT\gamma$. In this nondissipative $(\gamma\to 0)$, infinite-temperature $(T\to\infty)$ limit with constant $\tilde{\Theta}$ it is clear that the quantum optical master equation (22) reduces to

$$\frac{\partial}{\partial t}\rho=-i\omega[a^\dagger a,\rho]+\frac{\tilde{\Theta}}{2m\hbar\omega}[([a,\rho a^\dagger]+[a\rho,a^\dagger])+([a^\dagger,\rho a]+[a^\dagger\rho,a])].$$

As could be expected, we find complete formal agreement with our master equation for classical fluctuations in *rotating wave approximation* (11) through the identification $\tilde{\Theta}\equiv\Theta(\infty)$, employing the long-time (or Markovian) limit.

The damped quantum optical harmonic oscillator described by Eq. (22) has very often been the basis for investigating decoherence phenomena both theoretically [5] and, more recently, experimentally [8–10]. In the mainly experimental paper [10] (part of) the theoretical analysis was carried out using (22) when in fact (part of) the experiment was performed with classical fluctuations so that Eqs. (6) and (11) appear more appropriate. Clearly, the experimental parameters were chosen such that the identification of (22) and (6) was possible through the limit discussed in this section. In Sec. V we discuss decoherence due to general classical fluctuations, in line with actual experiments [9,10]. Importantly, we may relieve ourselves from the restrictions given by the aimed correspondence of an experiment involving classical fluctuations and the quantum optical master equation (22). Moreover, we may investigate the influence of a finite correlation time of the classical fluctuations and the influence of nonrotating wave terms present in the exact master equation (6).

IV. SOLUTION OF THE MASTER EQUATION AND GENERAL DECOHERENCE

Here we write the exact master equation for a fluctuating trap (6) in terms of the Wigner function of $\bar{\rho}(t)$,

$$W(q,p)=\frac{1}{2\pi\hbar}\int dx\langle q-x/2|\bar{\rho}|q+x/2\rangle e^{(i/\hbar)xp}.$$

An analytical expression for the time evolved $W(q,p,t)$ for an arbitrary initial state can be given. Our presentation follows closely—and is in fact simpler than—similar investigations [7,28,32,33] based on the exact master equation (19) of quantum Brownian motion.

Transforming the master equation (6) for the averaged density operator to a differential equation for the Wigner function leads to a Focker-Planck-type differential equation

$$\frac{\partial}{\partial t}W=\left(-\frac{p}{m}\frac{\partial}{\partial q}+m\omega^2 q\frac{\partial}{\partial p}+\Theta(t)\frac{\partial^2}{\partial p^2}+\frac{\Sigma(t)}{m\omega}\frac{\partial^2}{\partial q\partial p}\right)W, \tag{23}$$

which highlights the meaning of the irreversible terms in our master equation (6). They describe momentum diffusion and an additional mixed position-momentum diffusion. As mentioned before, a damping term is missing.

We may solve Eq. (23) with the help of the Fourier representation of the Wigner function

$$W(q,p)=\frac{1}{2\pi\hbar}\int d\mu\int d\nu\chi_{\rm W}(\mu,\nu)e^{-i/\hbar(\mu q-\nu p)} \tag{24}$$

involving the (Wigner) characteristic function

$$\chi_{\rm W}(\mu,\nu)=\frac{1}{2\pi\hbar}{\rm Tr}(e^{i/\hbar(\mu\hat{q}-\nu\hat{p})}\bar{\rho}). \tag{25}$$

In terms of $\chi_{\rm W}$, the master equation (6) reads

$$\frac{\partial}{\partial t}\chi_{\mathrm{W}} = \left(-\frac{\mu}{m}\frac{\partial}{\partial \nu} + m\omega^2\nu\frac{\partial}{\partial \mu} - \frac{\Theta(t)}{\hbar^2}\nu^2 + \frac{\Sigma(t)}{m\omega\hbar^2}\mu\nu\right)\chi_{\mathrm{W}}.$$

The nonderivative terms may be eliminated through the Gaussian ansatz

$$\chi_{\mathrm{W}}(\mu,\nu) = e^{-(1/2\hbar^2)[\alpha(t)\mu^2+2\beta(t)\mu\nu+\gamma(t)\nu^2]}\zeta(\mu,\nu).$$

For the factor $\zeta(\mu,\nu)$ we find

$$\frac{\partial}{\partial t}\zeta = \left(-\frac{\mu}{m}\frac{\partial}{\partial \nu} + m\omega^2\nu\frac{\partial}{\partial \mu}\right)\zeta, \tag{26}$$

provided the coefficients α,β and γ satisfy the following set of inhomogeneous linear differential equations:

$$\frac{\partial}{\partial t}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = \begin{pmatrix} 0 & -\frac{2}{m} & 0 \\ m\omega^2 & 0 & -\frac{1}{m} \\ 0 & 2m\omega^2 & 0 \end{pmatrix}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{1}{m\omega}\Sigma(t) \\ 2\Theta(t)\end{pmatrix}. \tag{27}$$

The solution of (27) for the required initial conditions $\alpha(0)=\beta(0)=\gamma(0)=0$ may be written in closed form involving an integral over the noise correlation function. We find

$$\alpha(t) = \frac{1}{2m^2\omega^2}\int_0^t dsC(s)[2\omega(t-s)\cos\omega s + \sin\omega s - \sin\omega(2t-s)],$$

$$\beta(t) = \frac{1}{2m\omega^2}\int_0^t dsC(s)[\cos\omega(2t-s) - \cos\omega s],$$

$$\gamma(t) = \frac{1}{2\omega}\int_0^t dsC(s)[2\omega(t-s)\cos\omega s - \sin\omega s + \sin\omega(2t-s)]. \tag{28}$$

The remaining Eq. (26) may be solved through its characteristic equations which yields $\mu(t)=\mu_0\cos\omega t - m\omega\nu_0\sin\omega t$ and $\nu(t)=\mu_0/(m\omega)\sin\omega t+\nu_0\cos\omega t$.

The required time evolved function $\zeta(\mu,\nu,t)$ may thus be written in the form

$$\zeta(\mu,\nu,t) = \int d\mu_0\int d\nu_0\zeta(\mu_0,\nu_0,0)\delta[\mu-\mu(t)]\delta[\nu-\nu(t)],$$

or equivalently,

$$\zeta(\mu,\nu,t) = \zeta\left(\mu\cos(\omega t) + m\omega\nu\sin(\omega t), \nu\cos(\omega t) - \frac{\mu}{m\omega}\sin(\omega t), 0\right).$$

We conclude that the required time evolved characteristic function $\chi_{\mathrm{W}}(t)$ is given by the product of a Gaussian and the shifted initial function $\chi_{\mathrm{W}}(0)$,

$$\chi_{\mathrm{W}}(\mu,\nu,t) = \exp\left[-\frac{1}{2\hbar^2}(\alpha(t)\mu^2 + 2\beta(t)\mu\nu + \gamma(t)\nu^2)\right] \times\chi_{\mathrm{W}}\left(\mu\cos(\omega t) + m\omega\nu\sin(\omega t), \nu\cos(\omega t) - \frac{\mu}{m\omega}\sin(\omega t), 0\right). \tag{29}$$

Finally, the corresponding time evolved Wigner function can be expressed as a convolution of the initial Wigner function with a (shifted) Gaussian:

$$W(q,p,t) = \frac{1}{2\pi\sqrt{\det M}}\int dQ\int dP e^{-1/2(Q-q(t),P-p(t))M^{-1}\binom{Q-q(t)}{P-p(t)}} \times W(Q,P,0), \tag{30}$$

where $q(t)=q\cos\omega t - p/(m\omega)\sin\omega t$ and $p(t)=p\cos\omega t + qm\omega\sin\omega t$ are the classical trajectories that link the given phase space point (q,p) with its location at $t=0$ through the action of the harmonic H_0. The 2×2 variance matrix M involves the coefficients (28) and is given by $M=\begin{pmatrix}\alpha(t) & -\beta(t)\\ -\beta(t) & \gamma(t)\end{pmatrix}$. With expression (30), we have a closed expression for the time evolved Wigner function $W(q,p,t)$ for an arbitrary initial state $W(Q,P,0)$.

A. Long-time limit

It is of interest to consider the long time limit. First note that for times much longer than the force correlation time we can replace

$$\int_0^t dsC(s) \approx \int_0^\infty dsC(s) \equiv D/2$$

where the last equality serves to define the constant D. With the additional assumption $\omega t \gg 1$, and in leading order in t, the variance matrix M in (30) takes the simple diagonal form $M=\frac{Dt}{2}\begin{pmatrix}(m\omega)^{-2} & 0\\ 0 & 1\end{pmatrix}$ so that the Wigner function (30), for large times t, may be written in the form

$$W(q,p,t) = \frac{m\omega}{\pi Dt}\int dQ \times\int dP e^{-((2m/Dt)([P-p(t)]^2/2m)+(m\omega^2/2)[Q-q(t)]^2)} \times W(Q,P,0).$$

It is obvious that the convoluting Gaussian becomes excessively broader in phase space as time t grows and thus, any fine details of the initial Wigner function will be smeared out after some time. This effect may be seen as a manifestation of decoherence and we will devote a good part of the remainder of this paper to study decoherence in fluctuating traps in more detail. In particular, in Sec. V we discuss the decoherence of an initial superposition of coherent states. Before we do so, however, we give a few remarks about decoherence of general initial states in the next subsection.

B. Coherent state representation and general decoherence

It has been investigated in many circumstances that a quantum system subjected to environmental noise will eventually become a simple classical mixture of well-localized, classical states [1,2]—for some more recent work see [27,28]. Any "quantumness" of the initial state will eventually disappear due to the effects of the environment. Unlike most investigations in this context, here we do not deal with a dynamically interacting environment but rather with simple classical fluctuations driving the quantum system unitarily, yet randomly. No entanglement with other quantum degrees of freedom is involved in our case. We will see that under the influence of the considered classical Gaussian noise, *any* initial state, after a finite time t_D, will be a classical mixture of coherent states $|z\rangle$,

$$\bar{\rho}(t) = \int \frac{d^2z}{\pi} P(z,t)|z\rangle\langle z| \text{ for } t > t_D \tag{31}$$

with a *positive* P-function $P(z,t) \geqslant 0$. In other words, there is a finite time t_D such that for all $t > t_D$ and for any initial state, the averaged density operator can be represented as a classical mixture of minimum-uncertainty wave packets (coherent states) of the harmonic oscillator. No quantum coherences will prevail that extend over larger regions than the elementary phase space cell of volume $\hbar$, as covered by a coherent state $|z\rangle$. Here we denote as usual coherent states as eigenstates of the annihilation operator $a|z\rangle = z|z\rangle$ with the dimensionless complex number z, related to the mean values of position and momentum of the harmonic oscillator degrees of freedom through the standard relation $z = \sqrt{\frac{m\omega}{2\hbar}}\left(\langle q\rangle + \frac{i}{m\omega}\langle p\rangle\right)$.

In order to prove positivity (and in fact existence) of the P function for $t > t_D$ in (31), we argue similar to [27,28]. Crucially, we use the simple relationship [34] between the characteristic functions of P–, Wigner- and Q-function $Q(z) \equiv \langle z|\bar{\rho}|z\rangle$ given by

$$\chi_P(\eta) = e^{(1/2)|\eta|^2}\chi_W(\eta) = e^{|\eta|^2}\chi_Q(\eta). \tag{32}$$

It should be noted that in our dimensionless variable η relation (24) between Wigner function and its corresponding characteristic function $\chi_W(\eta) \equiv \mathrm{Tr}[e^{\eta a^+ - \eta^* a}\bar{\rho}]$ reads

$$W(z) = \int \frac{d^2\eta}{\pi}\chi_W(\eta)e^{\eta^* z - \eta z^*} \tag{33}$$

by virtue of the identification $\eta = \sqrt{\frac{m\omega}{2\hbar}}\left(\nu + \frac{i}{m\omega}\mu\right)$ [compare with (25)]. The very same relation (33) holds for the P and Q function and their corresponding characteristic functions χ_P, χ_Q. Importantly, we note that the Q-function $Q(z) \equiv \langle z|\bar{\rho}|z\rangle = \int \frac{d^2\eta}{\pi}\chi_Q(\eta)e^{\eta^* z - \eta z^*}$ is always positive by definition.

A little algebra shows that in terms of the dimensionless variable η the time evolved characteristic function (30) may be expressed in the form

$$\chi_W(\eta,t) = e^{-(1/2)(2A|\eta|^2 + B^*\eta^2 e^{2i\omega t} + B\eta^{*2}e^{-2i\omega t})}\chi_W(\eta e^{i\omega t},0) \tag{34}$$

with time dependent functions linearly related to the coefficients (28) that may be written in the form

$$A = A(t) = \frac{1}{m\omega\hbar}\int_0^t ds C(s)[(t-s)\cos\omega s],$$

$$B = B(t) = \frac{e^{i\omega t}}{m\omega^2\hbar}\int_0^t ds C(s)\sin\omega(t-s)$$

$$= \frac{e^{i\omega t}}{m\omega^2\hbar}[\Theta(t)\sin\omega t - \Sigma(t)\cos\omega t]. \tag{35}$$

These two time dependent functions turn out to be most relevant for the remaining part of this paper. Note that for large times $A(t) \to \frac{\Theta(t)}{m\hbar\omega}t \approx \frac{\Theta(\infty)}{m\hbar\omega}t$ so that $A(t)$ not only grows over all bounds but is larger than $B(t)$ by a large factor ωt.

The simple relations (32) between characteristic functions allow us to express the time evolved P function in terms of the characteristic function of the Q function,

$$P(z,t) = \int \frac{d^2\eta}{\pi}e^{-1/2(2(A-1)|\eta|^2 + B^*\eta^2 + B\eta^{*2})}\chi_Q(\eta,0)e^{\eta^* z e^{i\omega t} - \eta z^* e^{-i\omega t}}. \tag{36}$$

It should be noted that due to the subtraction of unity, the factor $(A-1)$ in the exponent will be negative for small enough times so that it is obvious from (36) that the P function may not even exist, initially.

However, due to the increasing nature of $A(t)$ as described after Eq. (35), there will be a time t_D such that $A(t) > 1$ and moreover

$$[A(t)-1]^2 - |B(t)|^2 > 0 \text{ for all } t > t_D. \tag{37}$$

For these times we can use the Gaussian integral representation

$$e^{-1/2[2(A-1)|\eta|^2 + B^*\eta^2 + B\eta^{*2}]}$$

$$= (\tilde{a}^2 - |\tilde{b}|^2)^{1/2}\int \frac{d^2\xi}{\pi}e^{-1/2(2\tilde{a}|\xi|^2 + \tilde{b}^*\xi^2 + \tilde{b}\xi^{*2})}e^{\xi^*\eta - \xi\eta^*}$$

where $\tilde{a} = \tilde{a}(t) = \frac{(A-1)}{(A-1)^2 - |B|^2} > 0$ and $\tilde{b} = \tilde{b}(t) = \frac{B}{(A-1)^2 - |B|^2}$. It allows us to express the P function in (36) as a convolution of a real Gaussian with the initial Q function,

$$P(z,t) = (\tilde{a}^2 - |\tilde{b}|^2)^{1/2}\int \frac{d^2\xi}{\pi}e^{-1/2(2\tilde{a}|\xi|^2 + \tilde{b}^*\xi^2 + \tilde{b}\xi^{*2})}Q(ze^{i\omega t} - \xi, 0).$$

As the Q function is manifestly positive, we have thus explicitly proven that for times $t > t_D$, i.e., when condition (37) is satisfied, *any* initial state will allow a P representation (31) with *positive* $P(z,t) > 0$ for all $t > t_D$. In other words, after the time t_D has elapsed, any initial quantum state will have turned into a mere classical mixture of coherent states.

The time t_D defines an overall decoherence time and will in general depend in a complicated way on the details of the

correlation function $C(t-s)$. In the often-employed Markov approximation $C(s)\to D\delta(s)$, when in addition we assume weak coupling so that the "rotating-wave" condition (12) is satisfied (here $D/2m\omega \ll \hbar\omega$), we can give a simple analytical expression for t_D. Note first that in this limit $B(t)$ may be neglected with respect to unity in (37). Therefore, we find a long overall decoherence time given by $\omega t_D = 2m\omega^2\hbar/D \gg 1$. We see a dependence of t_D on the ratio (m/D) for the trapped particle subjected to classical fluctuations, while for a free particle it was shown in previous work [27] on quantum-Brownian motion type models that $t_D \sim \sqrt{m/D}$. We stress that this t_D is long compared to the rapid decay times of coherences for some special initial states as discussed in much more detail in the next two sections, where we also highlight the connection of the underlying master equation (6) to actual experiments.

V. DECOHERENCE OF A SUPERPOSITION OF COHERENT STATES

In this section we determine the time dependence of purity $\mathcal{P}=\mathrm{Tr}[\bar{\rho}^2]$ as a measure for decoherence. We base our calculations on the nice equivalence of purity and integrals over squared Wigner- and corresponding characteristic function, given by

$$\mathcal{P} = \mathrm{Tr}[\bar{\rho}^2] = \int \frac{d^2z}{\pi} W(z)^2 = \int \frac{d^2\eta}{\pi} |\chi_W(\eta)|^2. \tag{38}$$

In combination with the simple analytical expression (34), the time dependent purity $\mathcal{P}(t)$ for an arbitrary initial state can be determined. Note that in particular, for (a sum of) Gaussian initial states, the integral in (38) can be evaluated analytically, leading to a closed expression for $\mathcal{P}(t)$.

Before we calculate the decay of purity for an initial superposition of coherent states, let us focus on a *single* coherent state $|\psi(0)\rangle = |z_1\rangle$ first. The characteristic function is simply $\chi_W(\eta,0) = e^{-1/2|\eta|^2 + \eta z_1^* - \eta^* z_1}$ so that with (34) we find

$$\mathcal{P}(t) = 1/\sqrt{d(t)} \text{ with } d(t) \equiv (2A+1)^2 - |2B|^2. \tag{39}$$

The decay of purity for a *single* coherent state is entirely due to diffusive broadening of the initial coherent state. The corresponding time scale, captured in the time dependence of $d(t)$, is essentially governed by the increase of $A(t)$ as discussed after Eq. (35). In Fig. 1 we display the decaying purity (black curves) for a single initial coherent state and an exponentially decaying force correlation function as in (8) with various choices of the strength parameter $R = D/(m\omega\hbar)$ [see also Eq. (49)] and a fairly long noise correlation time given by $\Gamma=\omega$. The grey curves show the increasing function $A(t)$ which essentially determines—through Eq. (39)—the decay of purity.

This classical diffusion time scale that determines purity decay for a single coherent state is typically slow compared to the rapid decay of $\mathcal{P}(t)$ experienced by a *superposition* of two coherent states, as will be investigated next. Note also that for a single coherent state $|z_1\rangle$, the purity $\mathcal{P}(t)$ is independent of the initial location z_1.

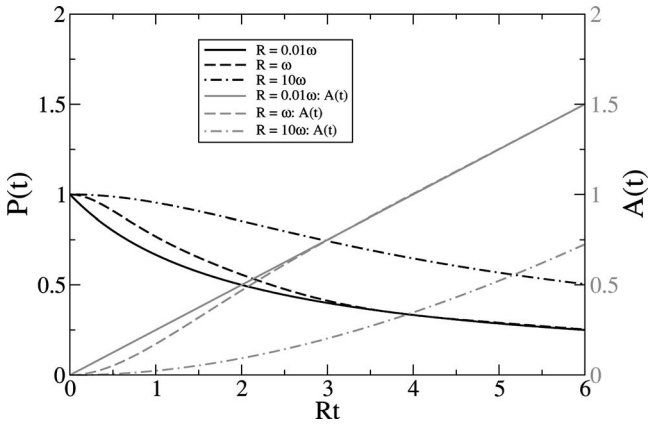

FIG. 1. Purity decay of an initial *single* coherent state of the ion (black curves) for an Ornstein-Uhlenbeck noise [see Eq. (8)]. Parameters chosen are $\Gamma=\omega$ and various fluctuation strengths $R=0.01\omega, \omega, 10\omega$. The gray curves show the corresponding increase of the quantity $A(t)$ of Eq. (35) which displays obvious (anti-)correlation with purity.

Decoherence and its dependence on the "size" of the initial quantum state is most strikingly seen for an initial superposition of two distinct coherent states so that

$$|\psi(0)\rangle = \frac{1}{\sqrt{\mathcal{N}}}(|z_1\rangle + |z_2\rangle) \tag{40}$$

with a normalization constant $\mathcal{N}=2(1+e^{-1/2|z_1-z_2|^2}\cos\Phi)$ where $\Phi=(z_1^* z_2 - z_1 z_2^*)/2i$. Such an initial state was also envisaged in decoherence experiments [8–10] and the starting point of many theoretical investigations on decoherence, see for instance [5,7,32].

With initial state (40), the characteristic function χ_W consists of four terms

$$\chi_W(\eta,0) = \frac{e^{-1/2|\eta|^2}}{\mathcal{N}}\{e^{\eta z_1^* - \eta^* z_1} + e^{\eta z_2^* - \eta^* z_2} + e^{-1/2|z_1 - z_2|^2}[e^{\eta z_2^* - \eta^* z_1 - i\Phi} + e^{\eta z_1^* - \eta^* z_2 + i\Phi}]\}. \tag{41}$$

Obviously, the first two terms in the curly brackets in (41) arise from the diagonal contributions $|z_i\rangle\langle z_i|$ while the last two terms reflect the (initial) presence of coherences $|z_1\rangle\langle z_2|$ and $|z_2\rangle\langle z_1|$. All integrals being Gaussian, the purity may be determined analytically without any approximation. We find an exact result that we choose to write in the form

$$\mathcal{P}(t) = \frac{1}{2\sqrt{d(t)}}[1 + \mathcal{D}(t)] \tag{42}$$

with a *coherence term*

$$\mathcal{D}(t) = \{e^{-a|\Delta z|^2 - b^*\Delta z^2 - b\Delta z^{*2}} + 2\cos\Phi e^{-1/2|\Delta z|^2}[e^{-b^*\Delta z^2} + e^{-b\Delta z^{*2}} - 1] + e^{-|\Delta z|^2}[e^{a|\Delta z|^2 - b^*\Delta z^2 - b\Delta z^{*2}} + \cos^2\Phi - 1]\}/(1 + e^{-1/2|\Delta z|^2}\cos\Phi)^2. \tag{43}$$

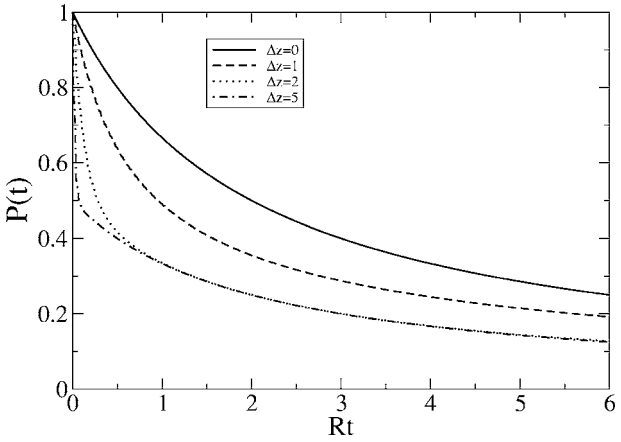

FIG. 2. Purity decay of an initial *superposition* of coherent states of distance Δz of the ion. We chose parameters with a weak strength $R=0.01\omega$ of the fluctuations and a decay rate of the noise correlation $\Gamma=\omega$. The initial separation of the two coherent states varies from $\Delta z=0$ over $\Delta z=1$, $\Delta z=2$, to $\Delta z=5$. Obviously, the large Δz, the faster the (initial) decay of purity.

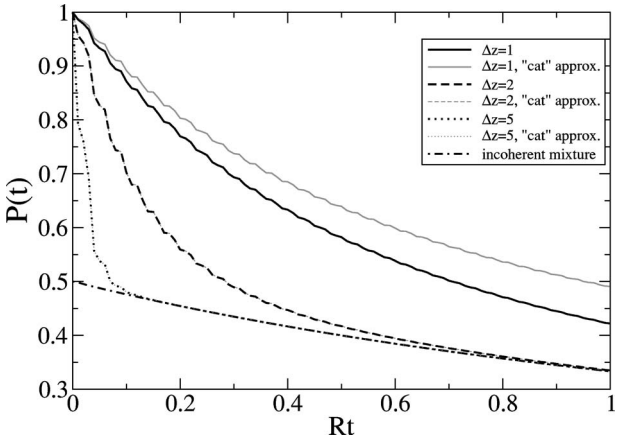

FIG. 3. Detail of Fig. 2. As before, we chose noise parameters $R=0.01\omega$ and a decay rate of the noise correlation $\Gamma=\omega$. The initial separation of the two coherent states varies from $\Delta z=1$ over $\Delta z=2$ to $\Delta z=5$. The gray curves are the results for the simple "cat approximation," valid for $\Delta z \gg 1$. We see good agreement already for $\Delta z=2$.

While $\Phi(t)$ and $d(t)$ were introduced above, two new time dependent functions

$$a = a(t) = \frac{2A(t)+1}{(2A+1)^2 - |2B|^2} \text{ and } b = b(t) = \frac{B(t)}{(2A+1)^2 - |2B|^2} \tag{44}$$

appear whose complicated time dependence is determined from the two functions (35). Note that initially $a(0)=1$ and $b(0)=0$ so that $\mathcal{D}(0)=1$ and $\mathcal{P}(0)=1$, as expected. The coherence term $\mathcal{D}(t)$ will decay, however, and the extreme case $\mathcal{D}=0$ reflects the purity of an *incoherent mixture* of two distinct coherent states, as it amounts to half the purity of a single coherent state according to (39). Thus the decay of $\mathcal{D}(t)$ marks the transition from a fully coherent superposition ($\mathcal{D}=1$) to an incoherent mixture ($\mathcal{D}\to 0$) of the two coherent states. This transition may occur on a time scale that is notably shorter than the overall decoherence (or diffusive) time scale t_D introduced earlier in Eq. (37). Note that the decay of purity as given by (42) with (43) depends on the initial state through the *distance* $\Delta z = z_1 - z_2$ between the two superposed states and the angle $\Phi = (z_1^* z_2 - z_1 z_2^*)/2i$.

In complete generality, the time dependence of the purity (42) will be complicated. We give some numerical results at the end of this section in Figs. 2–7, where we interpret these results in more detail.

Of particular interest is the case of an initial superposition of two very distinct wave packets, when $|\Delta z| \gg 1$ (large "Schrödinger cat"). In this limit there is a very nice alternative representation of the quantity $\mathcal{D}(t)$ that was used in [7] to quantify the decay of coherence. With obvious notations, the time dependent Wigner function can be written in the form $W(t)=W_{11}(t)+W_{22}(t)+W_{12}(t)+W_{21}(t)$. While the diagonal contributions represent Gaussian distributions near the wave packets $|z_1\rangle$, $|z_2\rangle$, the two off-diagonal contributions reflect the coherence between these wave packets. They are located near the phase space region given by $z \approx (z_1+z_2)/2$. It is this "amount of Wigner function" residing in the middle that indicates coherence between the two superposed states, and it is indeed easy to show that for $|\Delta z| \gg 1$ and within the relevant time span, we can write $\mathcal{D}(t)/\sqrt{d(t)} = 2\int \frac{d^2z}{\pi} |W_{12}(z,t) + W_{21}(z,t)|^2$, highlighting once again the meaning of $\mathcal{D}(t)$ as a measure for the coherence between the two wave packets. In the limit $|\Delta z| \gg 1$ expression (43) simplifies considerably and we find the much more appealing "Schrödinger cat" approximation

$$\mathcal{D}(t) = e^{-(1-a)|\Delta z|^2 - b^* \Delta z^2 - b \Delta z^{*2}} \text{ for } |\Delta z| \gg 1, \tag{45}$$

which no longer depends on the angle Φ.

As $a(t)$ starts off at the value unity and decays initially, it is clear that for $|\Delta z| \gg 1$ only a tiny decrease of $a(t)$ is re-

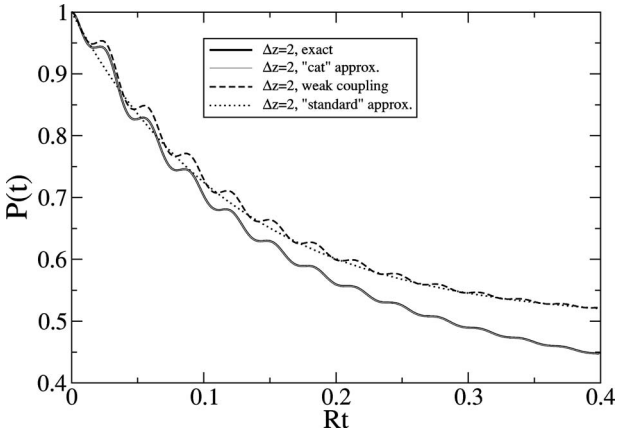

FIG. 4. Details of the initial purity decay of the initial *superposition* of coherent states of the ion with $\Delta z=2$. Again, parameters are $R=0.01\omega$ and $\Gamma=\omega$. The full black curve shows the exact decay, the full gray curve the "cat approximation," as before. The dashed curve is the weak coupling approximation, and the dash-dotted curve corresponds the "standard" exponential decay formula.

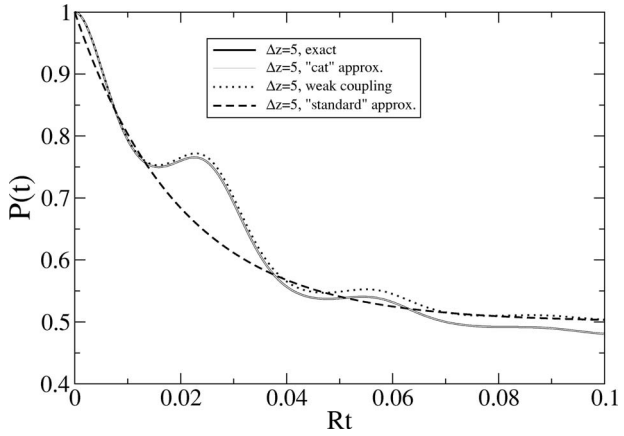

FIG. 5. Same as Fig. 4 but for a larger initial separation $\Delta z=5$ of the two superposed coherent states.

quired to bring $\mathcal{D}(t)$ essentially to zero. So even for a seemingly small influence of the fluctuating forces, the coherences will disappear quickly in the limit $|\Delta z| \gg 1$.

A. Weak fluctuating forces and large "Schrödinger cat" $|\Delta z| \gg 1$

A very important case deserving special attention is the weak coupling limit (or the early times limit), when $A \ll 1$ and $B \ll 1$ in Eqs. (44) [see expressions (35)]. A systematic expansion in lowest orders in A and B renders $a \approx 1-2A$, $b \approx B$, and $d \approx 1$ so that for $|\Delta z| \gg 1$ and the relevant early time spans, the decaying purity may be written in the form

$$\mathcal{P}(t) = \frac{1}{2}[1 + \mathcal{D}(t)], \quad |\Delta z| \gg 1 \tag{46}$$

with

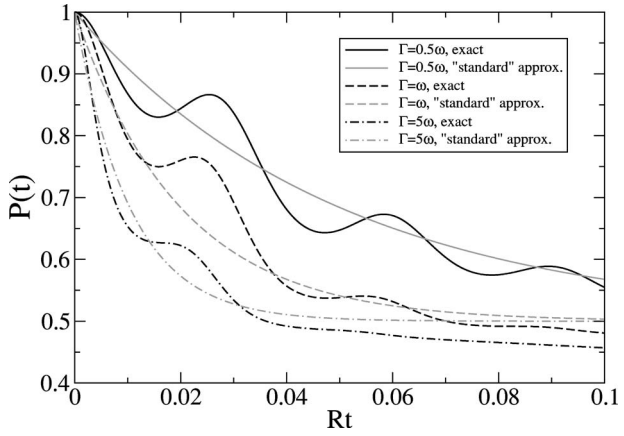

FIG. 6. Purity decay of an initial *superposition* of coherent states of the ion (black curves) with $\Delta z=5$. Parameters chosen are a strength $R=0.01\omega$ of the fluctuations and various $\Gamma=0.5\omega, \omega, 5\omega$. The gray curves show the corresponding exponential decay predicted by the "standard" approximation.

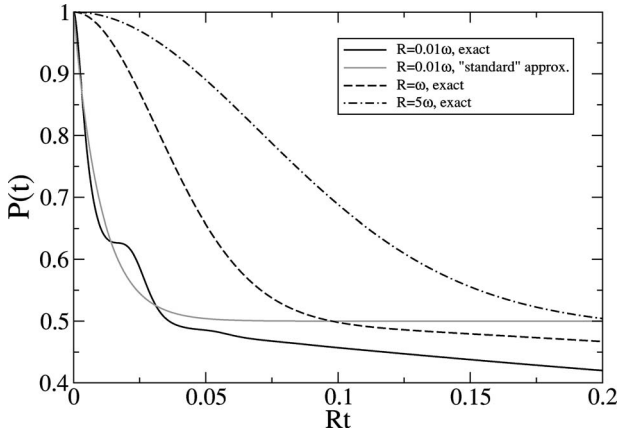

FIG. 7. Purity decay of an initial *superposition* of coherent states of the ion (black curves) with $\Delta z=5$. Parameters chosen are various strengths $R=0.01\omega, \omega, 5\omega$ of the fluctuations and $\Gamma=5\omega$. The gray curves show the "standard" approximation.

$$\mathcal{D}(t) = e^{-2A|\Delta z|^2 - B^*\Delta z^2 - B\Delta z^{*2}} \text{ for } |\Delta z| \gg 1 \text{ and } A, B \ll 1, \tag{47}$$

which is the "weak coupling" approximation.

This expression for weak fluctuations and large $|\Delta z|$ is of great significance. In contrast to the more theoretically motivated results (43) and (45) as measures for decoherence based on purity decay, expression (47) (or rather its square root) is directly measurable—and was in fact measured in the experiments [9,10]. This important point will be elaborated on in more detail in Sec. VI.

In the long time limit, as discussed before, we have $A(t) \approx \frac{\Theta(\infty)}{m\hbar\omega}t$ and we may neglect $B(t)$ altogether. Then the result (47) for weak fluctuations takes the very familiar form for coherence decay of two coherent states captured by the simple exponential formula

$$\mathcal{D}(t) = e^{-r|\Delta z|^2 t} \text{ for } |\Delta z| \gg 1, \quad A, B \ll 1, \quad \text{and } t \to \infty \tag{48}$$

with $r = \frac{2\Theta(\infty)}{m\hbar\omega}$. The rate r is proportional to the strength of the fluctuations and determined by the dependence of $\Theta(t)$ on the force correlation function as in (7). This final expression (48) scales with the squared distance $|\Delta z|^2$ and similar expressions have been around for a long time [5,9]—derived from Markovian master equations. We stress, however, that for the considered fluctuating forces, the general coherence decay for an initial state (40), as given by the expressions (43) and (45) or (47) for purity decay, is more complicated than the simple familiar expression (48).

B. Numerical results for purity decay

In this section we show the decay of purity $\mathcal{P}(t)$ for an initial superposition of coherent states for various separations $\Delta z = z_2 - z_1$. As before (8), we choose an Ornstein-Uhlenbeck exponentially decaying noise correlation function

$$C(s) = D\Gamma e^{-\Gamma|s|}/2.$$

A good measure for the strength of the fluctuations is the quantity

$$R \equiv \frac{D}{m\omega\hbar} \tag{49}$$

which is a rate. As before, $D=2\int_0^\infty C(s)ds$ denotes the integral over the force correlation function. The physical meaning of Eq. (49) becomes obvious once we introduce the fundamental oscillator momentum uncertainty scale $\Delta p_{\rm osc}=\sqrt{\hbar m\omega}$. As is apparent from (23), D is a momentum diffusion constant and so R^{-1} can be seen as the time the fluctuating momentum requires to become of the order of the fundamental oscillator quantum momentum unit $\Delta p_{\rm osc}$, i.e., $DR^{-1}=(\Delta p_{\rm osc})^2$.

Weak fluctuations refers to the limit $R \ll \omega$. In this case the increase of the average momentum during an oscillator period is small compared to the intrinsic oscillator momentum uncertainty $p_{\rm osc}$.

The decaying purity can now be written as a function of five parameters: oscillator frequency ω, inverse force correlation time Γ, fluctuation strength measured by the rate R from (49), separation $\Delta z=z_2-z_1$, and initial angle $\Phi=(z_1^* z_2 - z_1 z_2^*)/2i$. The latter we set equal to zero, choosing for instance, as in experiments, $z_2=-z_1$. The well-known decoherence formula (48) is expected to be valid asymptotically (for $|\Delta z| \gg 1$) with a rate $r=R\frac{\Gamma^2}{\Gamma^2+\omega^2}$.

In Fig. 2 we show the purity $\mathcal{P}(t)$ using the exact expression (42) [with (43)] for rather weak fluctuations ($R=0.01\omega$) and a fairly long correlation time of the noise ($\Gamma=\omega$). Obviously, the larger the initial distance Δz of the superposed coherent states, the faster the (initial) decay of purity, as expected.

A detailed glance at the initial decay displayed in Fig. 3 (a detail of Fig. 2) shows two very different dynamical decay regimes. First, a fast decay of purity that is dependent on the initial separation Δz and that is in fact *very* fast for $\Delta z \gg 1$. Second, a relatively slow decay that is nothing but the loss of purity as given by a *mixture* of two coherent states, i.e., half the value of the curve displayed in Fig. 1. In Fig. 3, next to the exact curves for purity decay (42) (black) we display the "cat" approximation (45) (gray) valid for large Δz and see very good agreement already for $\Delta z=2$.

Yet more details of purity decay for the case $\Delta z=2$ are displayed in Fig. 4. Again, parameters chosen are $R=0.01\omega$ and $\Gamma=\omega$. The full black curve shows the exact decay according to Eq. (42), the full gray curve the "cat" approximation Eq. (45) valid for large Δz with very good agreement. The dashed curve is the weak coupling approximation Eq. (46) [with Eq. (47)] while the dash-dotted curve corresponds the "standard" exponential decay formula Eq. (48). Fig. 5 shows the same quantities as the previous figure Fig. 4, yet for a larger initial separation $\Delta z=5$.

Finally, we show some graphs that highlight the dependence of purity decay on the parameters of the force correlation function. In Fig. 6 we display the decaying purity for an initial superposition of coherent states with $\Delta z=5$ and fairly weak fluctuations, $R=0.01\omega$. The black curves show the exact decay for three different noise correlation times determined by $\Gamma=0.5\omega$ (full curve), $\Gamma=\omega$ (dashed curve), and $\Gamma=5\omega$ (dashed-dotted curve). The gray curves show the corresponding exponentially decaying result given by the standard approximation of Eq. (48) which appears to predict the average decay on a very course time scale reasonably well.

In Fig. 7 we show the dependence of purity decay on the strength of the fluctuations for a fixed (short) bath correlation time $\Gamma=5\omega$ and an initial separation $\Delta z=5$ of the superposed coherent states. In the weak coupling case $R=0.01\omega$ (full black curve: exact; full gray curve: "standard" approximation), we still do not see a pure exponential decay, even though the exponential formula gives at least the correct average decay. For stronger coupling, $R=\omega$ (dashed curve), or even $R=5\omega$ (dashed-dotted curve), purity decay is clearly nonexponential.

We stress again that we here display a decay of purity as a measure for decoherence. In the next section we will discuss a quantity that is actually being measured in the decoherence experiment. As we will explain in great detail, it is in the weak coupling case only that the measured quantity can be related to purity as determined in this section.

VI. RELATION TO THE ION TRAP DECOHERENCE EXPERIMENT

We will see that the quantity measured in the actual experiment involving fluctuating forces [9,10] is numerically related to expressions (46) and (47) for purity decay in the weak coupling limit. In general, however, the dynamics given by the master equation (6) are not probed directly by the way the experiment is performed. We will explain these differences toward the end of this section in some detail. For the following analysis see also [9].

For the experiment, an internal two-level structure (states $|\uparrow\rangle$, $|\downarrow\rangle$) of the ion is employed. A superposition of motional states of the ion, as used in the last Sec. V may be obtained starting from an initial product state $|\Psi_0\rangle=\frac{1}{\sqrt{2}}(|\uparrow\rangle+|\downarrow\rangle) \otimes |0\rangle$ of internal and motional states of the ion. Applying a Jaynes-Cummings-type interaction for a certain time t amounts to an initial-state-dependent unitary shift operation

$$D(z_i) \equiv e^{z_i a^\dagger - z_i^* a} \tag{50}$$

with certain displacements $z_1=-z_2$ whose actual values are of little significance here. Thus, after the interaction the initial state is transferred to an entangled "Schrödinger cat" state (see [10] for more details)

$$|\Psi_1\rangle = \frac{1}{\sqrt{2}}(|\downarrow\rangle \otimes |z_1\rangle + |\uparrow\rangle \otimes |z_2\rangle) \tag{51}$$

of the ion with motional coherent states $|z_i\rangle=D(z_i)|0\rangle$. With the help of a $\frac{\pi}{2}$ pulse on the two-level part and a subsequent measurement, one could indeed create a motional superposition of the form $|\psi\rangle \sim |z_1\rangle+|z_2\rangle$, the starting point of the earlier investigations in Sec. V. This was the route taken in the Paris experiment [8]. Allowing for the action of a fluctuating force $F(t)$, the time evolution would then be governed indeed by our master equation (6).

In the Boulder ion trap experiments [9,10] the fluctuations are switched on with an initial total state (51) (i.e., with a reduced motional mixed state $\rho \sim |z_1\rangle\langle z_1| + |z_2\rangle\langle z_2|$). While numerically closely related, we will see that nevertheless, one should distinguish between decoherence resulting from master equation dynamics with (6)—captured in the decaying purity $\mathcal{P}(t)$—and the experimentally measured decay of certain coherences. It is important to realize—and we will explain these issues at the end of this section in detail—that the loss of purity described by the master equation (6) contains both, relative dephasing and *random diffusion* of the states. In contrast, the way the experiment is done, the diffusion of states is not measured and relative dephasing is the only cause of decoherence. We hasten to add that still, a simple numerical relation between the decaying purity (42) obtained from the master equation (6) and the experimentally measured quantity exists, in the weak coupling regime.

The stochastic time evolution of state (51) under the influence of the random force is easily evaluated. In interaction representation with respect to the harmonic oscillator Hamiltonian H_0, the action of the fundamental fluctuating Hamiltonian (1) up to a time t amounts to a random unitary shift operation $D(\beta)$ (independent of the internal state) with

$$\beta(t) = \frac{-i}{\sqrt{2m\omega\hbar}} \int_0^t ds F(s) e^{i\omega s} \tag{52}$$

and an overall irrelevant phase. In other words, the time evolved stochastic state is

$$|\psi_\beta(t)\rangle = e^{-iH_0/\hbar t} D(\beta) |\psi(0)\rangle,$$

whose dependence on the stochastic force $F(t)$ is indicated through the subscript β.

In the experiment, after allowing the fluctuations to act for a period of length t, the initial shift operation (50) that created the "Schrödinger-cat" state is reversed through the inverse Jaynes-Cummings interaction, i.e., a state dependent shift operation $D(-z_i)$ is applied. This last step is crucial as the sequential application of the three shifts amounts to a single shift with

$$D(-z_i)D(\beta)D(z_i) = e^{\beta z_i^* - \beta^* z_i} D(\beta). \tag{53}$$

In other words, the final state of the ion factorizes and may be written in the form [10]

$$|\Psi_2\rangle = \frac{1}{\sqrt{2}}(|\downarrow\rangle + e^{\beta^* \Delta z - \beta \Delta z^*} |\uparrow\rangle) \otimes |\beta\rangle,$$

where, as before, $\Delta z = z_1 - z_2$. Thus, in the experiment the accumulated (different) fluctuating phases of the two coherent states ($|z_1\rangle$, $|z_2\rangle$) are transferred to the two-state system. This dephasing can be measured with the help of an additional $\frac{\pi}{2}$ pulse, to give the state

$$|\Psi_3\rangle = \frac{1}{2}[(1 + e^{\beta^* \Delta z - \beta \Delta z^*})|\downarrow\rangle + (1 - e^{\beta^* \Delta z - \beta \Delta z^*})|\uparrow\rangle] \otimes |\beta\rangle. \tag{54}$$

Thus, the average probability to find the ion in state $|\downarrow\rangle$ is given by

$$P_\downarrow = \frac{1}{4}\langle\langle |1 + e^{\beta^* \Delta z - \beta \Delta z^*}|^2 \rangle\rangle = \frac{1}{2}(1 + e^{-1/2(2A|\Delta z|^2 + B^* \Delta z^2 + B \Delta z^{*2})}), \tag{55}$$

where the last equality follows from the analytical evaluation of the Gaussian ensemble mean over the fluctuations. The time dependent functions $A(t)$ and $B(t)$ are the ones defined in (35).

A. Probability decay versus purity decay

A glance at expressions (46) with (47) shows that for the case $|\Delta z| \gg 1$ and weak coupling, the measurable probability $P_\downarrow$ from (55) is closely related to the decaying purity. Indeed, with the coherence term $\mathcal{D}(t)$ from (47) we might as well write

$$P_\downarrow = \frac{1}{2}(1 + \sqrt{\mathcal{D}}).$$

This shows that the somewhat abstract quantity $\mathcal{D}(t)$ obtained from purity-decay considerations given by master equation dynamics may directly be related to a measurable quantity.

We emphasize, however, that expression (55) holds for any strength of the fluctuating forces, while the similar purity decay law (46) [with (47)] holds in the weak coupling limit only. For stronger coupling, the time dependence of purity as obtained from the master equation (6) and the measured probability (55) are simply different quantities that do not show a simple relation. One should not be misled in believing that the experimental setup actually tests quantum dynamics as given by our master equation (6)—or, for that matter, as given by related master equations as discussed in Sec. III. More specifically, the actual experiment does not probe the diffusion experienced by the two coherent states [as encoded in the state $|\beta\rangle$ in (54)]. The latter, however, contributes to the decaying purity in (42) and is part of our master equation (6). While clearly well suited to measure decoherence in the sense explained above, the experiment does not directly probe master equation dynamics as such.

B. Dephasing and state diffusion

To make this point clearer, let us artificially separate two causes for decoherence. Suppose—as originally envisaged—the motional initial state to be a superposition of two coherent states as in (40). After allowing a fluctuating force to act for a certain time t, the resulting stochastic motional state may be written in the form

$$\rho_\beta(t) = \frac{e^{-(i/\hbar)H_0 t}}{\mathcal{N}}(|z_1 + \beta\rangle\langle z_1 + \beta| + |z_2 + \beta\rangle\langle z_2 + \beta| + e^{1/2(\beta^* \Delta z - \beta \Delta z^*)}|z_2 + \beta\rangle\langle z_1 + \beta| + e^{-1/2(\beta^* \Delta z - \beta \Delta z^*)}|z_1 + \beta\rangle\langle z_2 + \beta|)e^{(i/\hbar)H_0 t}. \tag{56}$$

with the stochastic displacement β from (52). Clearly, the state ρ_β depends on the fluctuations β in two ways: they contribute to a fluctuating phase in front of the coherences,

and they lead to a random displacement $z_i \to z_i + \beta$ of the coherent states themselves. Taking the ensemble over the fluctuations, we recover the average $\bar{\rho}(t) = \langle\langle \rho_\beta(t) \rangle\rangle$, the solution of our master equation (6) for initial state (40).

The ion trap experiment is sensitive to phase fluctuations. The diffusion of the states themselves is not observed due to the three successive shifts as given by Eq. (53). To clarify this point further, consider by contrast the following two artificial ensembles. On the one hand, we could envisage a purely dephasing ensemble given by

$$\rho_\beta^{\text{deph}}(t) = \frac{e^{-(i/\hbar)H_0 t}}{\mathcal{N}}(|z_1\rangle\langle z_1| + |z_2\rangle\langle z_2| + e^{(1/2)(\beta^* \Delta z - \beta \Delta z^*)}|z_2\rangle \times \langle z_1| + e^{-1/2(\beta^* \Delta z - \beta \Delta z^*)}|z_1\rangle\langle z_2|)e^{(i/\hbar)H_0 t}, \tag{57}$$

where we neglect state diffusion $z_i \to z_i + \beta$ when compared to the "true" ensemble (56), yet retain the fluctuating phases. It is clear that these phases are just the ones that also appear in the actual experiment [see state (54)]. In fact, the phases differ by a factor $\frac{1}{2}$, which is due to the two shifts that create and recombine the motional state according to (53). On the other hand, let us investigate a purely diffusing ensemble given by

$$\rho_\beta^{\text{diff}}(t) = \frac{e^{-(i/\hbar)H_0 t}}{\mathcal{N}}(|z_1 + \beta\rangle\langle z_1 + \beta| + |z_2 + \beta\rangle\langle z_2 + \beta| + |z_2 + \beta\rangle \times \langle z_1 + \beta| + |z_1 + \beta\rangle\langle z_2 + \beta|)e^{(i/\hbar)H_0 t}, \tag{58}$$

where we omit relative dephasing when compared to the "true" ensemble (56), yet keep state diffusion. We emphasize that none of the two ensembles $\langle\langle \rho_\beta^{\text{deph}} \rangle\rangle$ and $\langle\langle \rho_\beta^{\text{diff}} \rangle\rangle$ is a solution of the master equation (6).

The decaying purity for the realistic ensemble (56) is given in Sec. V for any choice of parameters [Eq. (42)]. Let us here concentrate on the case $|\Delta z| \gg 1$ and weak fluctuations $(A, B \ll 1)$ so that the purity decay of the true ensemble (56) is in fact well described by (46) with (45). In similar fashion we can compute purity decay of the two artificial ensembles (57) and (58). Remarkably, it turns out that in this limit of weak fluctuating forces they both suffer the same purity decay, namely

$$\mathcal{P}^{\text{deph}}(t) \approx \mathcal{P}^{\text{diff}}(t) \approx \frac{1}{2}(1 + e^{-1/4(2A|\Delta z|^2 + B^* \Delta z^2 + B \Delta z^{*2})}).$$

Apparently, we find a decay law with essentially the same functional dependence on Δz and the same time dependent functions $A(t)$ and $B(t)$ as both the experimentally observed probability (55) and the purity decay of the true ensemble, with, however, a smaller exponent $\frac{1}{4}$. We conclude that dephasing and state diffusion, both contained in the underlying master equation (6), contribute equally to decoherence in shaken traps. The experiment probes dephasing only.

We conclude this section by stating that the setup of the ion trap experiment allows one to measure the precise time dependence of decoherence of a superposition of coherent states in a shaken trap. However, we also see that there is no universal connection between the usually investigated purity decay as obtained from a master equation and the measured probability. The relation is indirect and may be established quantitatively for weak fluctuating forces only.

VII. HEATING

We stressed earlier that the fundamental Hamiltonian (1) contains fluctuations—yet there is no compensating damping mechanism. Thus, the average (here kinetic) energy of the shaken particle will increase with time—see also [35]. Let us determine the precise dynamics of heating due to arbitrary Gaussian fluctuations, by computing the average kinetic energy

$$\bar{E}_{\text{kin}}(t) = \text{Tr}\left(\frac{p^2}{2m}\bar{\rho}(t)\right).$$

For simplicity, we choose an initial vacuum state $\rho(0) = |0\rangle\langle 0|$.

While we could use the exact master equation (6) and its analytical solution provided in Sec. IV to determine $\bar{E}_{\text{kin}}(t)$, we here choose to calculate it directly. As already explained in the previous Sec. VI, under the influence of a fluctuating force, the initial state $|\psi(0)\rangle = |0\rangle$ evolves to $|\psi(t)\rangle = e^{-iH_0 t/\hbar} D(\beta)|0\rangle$. Thus, for a single realization

$$E_{\text{kin}}(t) = \langle\psi(t)|\frac{p^2}{2m}|\psi(t)\rangle = \frac{\hbar\omega}{4}(2|\beta|^2 - \beta^2 e^{-2i\omega t} - \beta^{*2} e^{2i\omega t} + 1),$$

where we used $p = i\sqrt{\frac{m\omega\hbar}{2}}(a^\dagger - a)$, the earlier expression (52) for β, and the averages

$$\langle\langle|\beta|^2\rangle\rangle = A(t) \text{ and } \langle\langle\beta^2\rangle\rangle = -B(t),$$

with the two time dependent functions $A(t)$ and $B(t)$ from (35) involving integrals over the force correlation function. We obtain an exact analytical expression

$$\bar{E}_{\text{kin}}(t) = \frac{\hbar\omega}{4}[2A(t) + B(t)e^{-2i\omega t} + B^*(t)e^{2i\omega t} + 1].$$

Forgetting about the influence of the (small) contribution $B(t)$, it becomes apparent that the time it takes for the kinetic energy to be much larger than the fundamental quantum $\hbar\omega$ is set by the condition $A(t) \gg 1$. This is the same time scale than the time t_D we defined through Eq. (37)—namely the time it takes for the density operator to become a classical mixture of coherent states.

Of importance is the long time limit where we find the linear increase $\bar{E}_{\text{kin}}(t) \approx \frac{\Theta(\infty)}{2m}t$ as predicted from Markovian master equations. For the case of an Ornstein-Uhlenbeck process we find asymptotically $\frac{\bar{E}_{\text{kin}}(t)}{\hbar\omega} \approx \frac{Rt}{4}\frac{\Gamma^2}{\Gamma^2 + \omega^2}$. Note that for strong enough fluctuations, however, the classical regime $\bar{E}_{\text{kin}} \gg \hbar\omega$ may well be reached long before the linear regime becomes relevant. We stress again that the time scale to reach this classical regime is essentially the time t_D found in the investigations for overall decoherence in Sec. IV.

In Fig. 8 we show the increase of the average kinetic energy for the previously employed Ornstein-Uhlenbeck correlation function (8) for various fluctuation strengths R

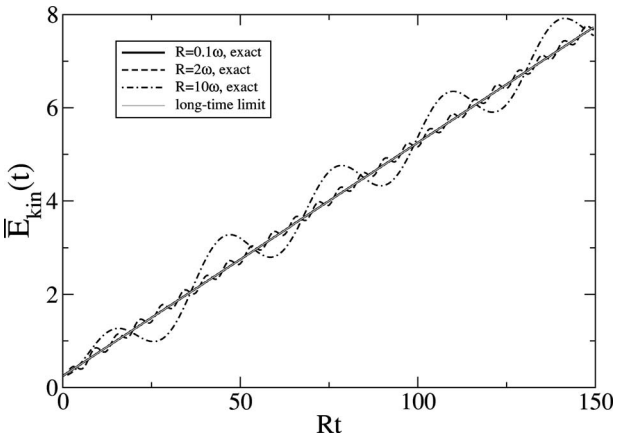

FIG. 8. Kinetic energy increase ("heating") for an initial vacuum coherent states of the ion. Parameters chosen are various strengths $R=0.1\omega, 2\omega, 10\omega$ of the fluctuations and a noise correlation determined through $\Gamma=0.5\omega$. The gray curve shows the asymptotic linear increase and lies perfectly on top of the exact result for weak coupling ($R=0.1\omega$). Clearly, for stronger fluctuations the increase in kinetic energy is far from linear, even after a time when the classical regime $\bar{E}_{\mathrm{kin}} \gg \hbar\omega$ has been reached.

$=0.1\omega$ (full curve), $R=2\omega$ (dashed curve), and $R=10\omega$ (dashed-dotted curve). We chose a correlation rate $\Gamma=0.5\omega$ of the driving noise. The gray curve shows the asymptotic linear increase and lies perfectly on top of the exact result for weak coupling ($R=0.1\omega$). Clearly, for stronger fluctuations, the increase in kinetic energy is far from linear, even though the classical regime $\bar{E}_{\mathrm{kin}} \gg \hbar\omega$ is reached quite early.

VIII. CONCLUSIONS

We investigate quantum dynamics in a shaken trap. We show that there is a very simple derivation of a time-local master equation governing the dynamics for an arbitrary correlation function $C(t,s)=\langle\langle F(t)F(s)\rangle\rangle$ of the fluctuating force. We compare and contrast our master equation for classically shaken traps with well-established master equations for irreversible harmonic-oscillator dynamics and see that it corresponds to the no-damping and infinite-temperature limit of the fully quantum model (13). We investigate decoherence of quantum states in some detail. First, we show that *any* quantum state will turn to a mere classical mixture of coherent states after a finite time t_D. Technically speaking, we prove that for $t > t_D$ any state has a P representation with positive $P(z,t) > 0$. No entanglement with other quantum degrees of freedom is involved here. Decoherence in this model is entirely due to unitary, yet random dynamics. More specifically, we investigate the loss of coherence of an initial superposition of coherent states with separation $|\Delta z|$. We find that it is in the weak coupling and long-time limit only that the familiar exponential decay of the form $e^{-r|\Delta z|^2 t}$ gives a reasonably good description. For not even large values of $|\Delta z|$ this asymptotic regime may be entirely irrelevant as the coherences have disappeared much earlier, following a nonexponential decay. The precise decay law is governed by a more complicated expression involving nonrotating wave terms and possibly effects from a finite memory time of the correlation function. Finally, heating of trapped particles due to general fluctuations are considered. We find that the average kinetic energy turns larger than the fundamental quantum $\hbar\omega$ for a time that corresponds to the decoherence time t_D for the evolution to a total classical mixture introduced above.

It would be enlightening to see ion trap decoherence beyond the limits of weak fluctuations and white noise experimentally. Then the correspondence to the simple exponential decay as predicted by the quantum optical master equation for a damped oscillator fails and the results of the general master equation (6) apply. From a theoretical side, we would like to combine our results for a shaken trap with quantum feedback control schemes as developed by Wiseman and Milburn [36] to prevent the heating of the ion. For such very recent experimental and theoretical developments see [37]. We believe that an application of feedback control theories to arbitrarily shaken traps as considered in this work are of interest.

ACKNOWLEDGMENTS

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