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Fidelity estimation and entanglement verification for experimentally produced four-qubit cluster states

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We propose methods of fidelity estimation and entanglement verification for experimentally produced four-qubit cluster states. We show that we can obtain a high lower bound of the fidelity using only four local projective measurement settings. The lower bound is close to the exact fidelity, which is determined only by at least nine local projective measurement settings. We also present witness operators for distinguishing entanglement around a four-qubit cluster state from specific classes of genuine four-qubit entanglement, e.g., a class including GHZ and W types of entanglement.

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I. INTRODUCTION

Entanglement is a resource of quantum computation and communication. A specific class of entangled states called *cluster states* can be used for quantum computation that consists of only one-qubit measurements [1]. Consider a set of qubits that are located on lattice sites C. A cluster state $|\Phi\rangle_C$ is defined as

$$X^{(a)} \bigotimes_{a' \in ngbh(a)} Z^{(a')}|\Phi\rangle_C = |\Phi\rangle_C, \quad \forall\ a \in C, \tag{1}$$

where $ngbh(a)$ specifies the sites of all qubits that are located at next neighbors of the qubit at site $a \in C$. In this paper we denote X, Y, and Z as Pauli matrices. A four-qubit cluster state on a one-dimensional lattice $|\phi_4\rangle$ defined by Eq. (1) is equivalent to the simple form

$$|C_4\rangle = \tfrac{1}{2}(|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle) \tag{2}$$

under local Hadamard transformation $H^{(1)}H^{(4)}$ [2]. A state

$$|\chi\rangle = \tfrac{1}{2}[(|00\rangle + |11\rangle)|00\rangle + (|01\rangle + |10\rangle)|11\rangle], \tag{3}$$

which is equivalent to $|C_4\rangle$ under $H^{(1)}H^{(2)}$, is known as a resource of teleportation-based controlled-NOT gate [3,4].

Recently, several schemes for experimentally producing a four-qubit cluster state have been proposed [4–8]. In such experiments we must verify whether the produced state is a desired entangled state or not. An important verification method is to measure the fidelity

$$F(\rho) := \langle C_4|\rho|C_4\rangle = \mathrm{Tr}(|C_4\rangle\langle C_4|\rho) \tag{4}$$

between the produced state ρ and $|C_4\rangle$. We can also verify the class of entanglement of ρ by evaluating $F(\rho)$. For example, an entanglement witness operator is given by

$$\mathcal{W}_{C_4} := \tfrac{1}{2}\mathbb{1} - |C_4\rangle\langle C_4|, \tag{5}$$

where 1/2 comes from the maximal fidelity between $|C_4\rangle$ and any biseparable state [9–11]. This guarantees that if $\mathrm{Tr}(\mathcal{W}_{C_4}\rho) = 1/2 - F(\rho) < 0$ then ρ is not biseparable and has genuine four-qubit entanglement.

In this paper we show that we need at least nine measurement settings to obtain the exact fidelity $F(\rho)$ using local projective measurements. Then we show that we can obtain a high lower bound of $F(\rho)$ using only four local projective measurement settings. We also show that the lower bound is optimal within any four local projective measurement settings for the produced states with white noise. Furthermore, we show that we can discriminate classes of genuine multipartite entanglement by evaluating the fidelity of the produced state. Specifically, we present witness operators for distinguishing entanglement around a four-qubit cluster state from classes of genuine four-qubit entanglement, e.g., a class including GHZ and W types of entanglement.

This paper is organized as follows: In Sec. II we show that we need nine measurement settings for obtaining $F(\rho)$ using local projective measurements. In Sec. III we present the four local projective measurement settings for obtaining a high lower bound of $F(\rho)$ and show its optimality. In Sec. IV we propose witness operators for distinguishing classes of genuine four-qubit entanglement. Finally, we describe our conclusions in Sec. V.

II. OBTAINING THE FIDELITY USING LOCAL PROJECTIVE MEASUREMENTS

The fidelity between ρ and $|C_4\rangle$, $F(\rho)$, is obtained by measuring the expectation value of projection operator $|C_4\rangle\langle C_4|$ as described in Eq. (4). If we know the decomposition of $|C_4\rangle\langle C_4|$ into a small number of local projection operators, we can easily obtain $F(\rho)$ by the local projective measurements in experiments. For three-qubit GHZ or W states, such optimal decompositions were shown in Ref. [12]. Here we show an optimal decomposition of $|C_4\rangle\langle C_4|$ that can be measured by nine local projective measurement settings. In other words, we need at least nine local projective measurement settings to obtain the exact fidelity $F(\rho)$.

First, we describe a local decomposition of projector $|C_4\rangle\langle C_4|$ and show that it can be measured using nine local

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measurement settings. From Eq. (1) and $|C_4\rangle = H^{(1)}H^{(4)}|\phi_4\rangle$, $|C_4\rangle$ is defined by the stabilizers, $S_1 := Z^{(1)}Z^{(2)}$, $S_2 := X^{(1)}X^{(2)}Z^{(3)}$, $S_3 := Z^{(2)}X^{(3)}X^{(4)}$, and $S_4 := Z^{(3)}Z^{(4)}$. Here $S_1, \ldots, S_4$ and their products form a stabilizer group $\langle S_1, \ldots, S_4\rangle$. Therefore, $|C_4\rangle\langle C_4|$ is written as

$$|C_4\rangle\langle C_4| = \frac{1}{16}\prod_{k=1}^{4}(S_k + \mathbb{1})$$
$$= \frac{1}{16}(IIII + ZZII + XXZI + IZXX + IIZZ - YYZI + ZIXX + ZZZZ + XYYX + XXIZ - IZYY + YXYX - YYIZ - ZIYY + XYXY + YXXY), \qquad (6)$$

where the sites of operations are omitted [18]. Here we can use the same setting $XXZZ$ for measuring operator $XXZI$ and for $XXIZ$. Similarly we can use $ZZXX$ for $\{IZXX, ZIXX\}$, $YYZZ$ for $\{YYZI, YYIZ\}$, $ZZYY$ for $\{IZYY, ZIYY\}$, and $ZZZZ$ for $\{ZZII, IIZZ, ZZZZ\}$. Therefore, the expectation value of $|C_4\rangle\langle C_4|$ can be measured using the following nine local measurement settings:

$$XXZZ,\ ZZXX,\ YYZZ,\ ZZYY,\ ZZZZ,$$
$$XYXY,\ XYYX,\ YXXY,\ YXYX. \qquad (7)$$

Next we show that the decomposition is optimal, i.e., there exists no decomposition of $|C_4\rangle\langle C_4|$ into a sum of local operators that can be measured using less than nine local projective measurement settings. First, consider operators $\Lambda_{i,j} := \mathrm{Tr}_{1,4}[|C_4\rangle\langle C_4|(\sigma_i^{(1)}\sigma_j^{(4)})]$, for $i,j = 0, \ldots, 3$, where $\sigma_0 = I$, $\sigma_1 = X$, $\sigma_2 = Y$, and $\sigma_3 = Z$. Then it turns out that the sixteen $\Lambda_{i,j}$'s are linearly independent. Next, we generally describe a decomposition of $|C_4\rangle\langle C_4|$ into the sum of local projectors with N measurement settings: $|C_4\rangle\langle C_4| = \sum_{m=1}^{N}\sum_{r,s,t,u=0}^{1} c_{rstu}^{m} P_r^{1,m}\otimes P_s^{2,m}\otimes P_t^{3,m}\otimes P_u^{4,m}$, where $P_0^{k,m}$ and $P_1^{k,m} = \mathbb{1} - P_0^{k,m}$ are orthogonal local projectors for sites k, and c_{rstu}^{m}'s are constants. Each local projector is represented as a vector in a Bloch sphere; $P_r^{k,m} = \frac{1}{2}I + \frac{1}{2}(-1)^r\sum_{l=1}^{3} s_l^{k,m}\times\sigma_l = \frac{1}{2}I + \frac{1}{2}(-1)^r R^{k,m}$. Then, $\Lambda_{i,j}$ can be expanded as

$$\Lambda_{i,j} = \sum_{m=1}^{N}\sum_{s,t=0}^{1}\alpha_{i,j,s,t}^{m}\Big[I\otimes I + (-1)^s\sum_{l=1}^{3} s_l^{2,m}\sigma_l\otimes I + (-1)^t\sum_{l=1}^{3} s_l^{3,m} I\otimes\sigma_l + (-1)^{s+t}R^{2,m}R^{3,m}\Big],$$

where $\alpha_{i,j,s,t}^{m}$'s are constant. Thus, the sixteen linearly independent operators $\Lambda_{i,j}$ can be written as a linear combination of $7+N$ operators and hence $7+N \geq 16$. Therefore, the settings of nine local measurements (7) are optimal.

III. OBTAINING A HIGH LOWER BOUND OF THE FIDELITY USING FEWER LOCAL MEASUREMENT SETTINGS

In Sec. II we have shown that we need at least nine measurement settings to obtain the exact fidelity $F(\rho) = \mathrm{Tr}[|C_4\rangle\langle C_4|\rho]$ using local projective measurements. Here, we provide a method for obtaining a high lower bound of $F(\rho)$ using only four local projective measurement settings and show its optimality for the states that arise when white noise is mixed into $|C_4\rangle$.

Let us write $A \geq B$ when $A - B$ is a positive operator. When an operator B satisfies $|C_4\rangle\langle C_4| \geq B$, we can obtain a lower bound of the fidelity as $\mathrm{Tr}[|C_4\rangle\langle C_4|\rho] \geq \mathrm{Tr}[B\rho]$. Here, if we need fewer local measurement settings for measuring B than that of $|C_4\rangle\langle C_4|$, we can obtain the lower bound of $F(\rho)$ with smaller experimental effort than the exact fidelity. We can also obtain an entanglement witness operator $\mathcal{W}' := \frac{1}{2}\mathbb{1} - B$, since if a state is detected by $\mathcal{W}'$ then $\mathcal{W}_{C_4}$ must detect the state, i.e., for any state ρ detected by $\mathcal{W}'$, $0 > \mathrm{Tr}[\mathcal{W}'\rho] \geq \mathrm{Tr}[\mathcal{W}_{C_4}\rho]$ must hold [9,13].

An operator B_2 satisfying $|C_4\rangle\langle C_4| \geq B_2$ was presented in Refs. [9,13] for constructing a witness operator that can be measured by only two local measurement settings. The operator is described as $B_2 := \frac{1}{4}(ZZII + IZXX + ZIXX + XXZI + IIZZ + XXIZ) - \frac{1}{2}\mathbb{1}$ and can be measured by two measurement settings, $XXZZ$ and $ZZXX$. To evaluate how good the lower bound of the fidelity is for a family of experimentally produced states, we compare the exact fidelity and the lower bound of the fidelity of states with white noise

$$\rho(p_N) := \frac{p_N}{16}\mathbb{1} + (1 - p_N)|C_4\rangle\langle C_4|. \qquad (8)$$

Then, we obtain $F(\rho(p_N)) = 1 - \frac{15}{16}p_N$ and $\mathrm{Tr}[B_2\rho(p_N)] = 1 - \frac{3}{2}p_N$, and thus the lower bound of $F(\rho(p_N))$ obtained by B_2 is $\frac{9}{16}p_N$ smaller than $F(\rho(p_N))$. We also evaluate the detection capability of entanglement witnesses. The noise tolerance of the projector-based witness $\mathcal{W}_{C_4}$ is given by $\mathrm{Tr}\big[\big(\frac{1}{2}\mathbb{1} - |C_4\rangle\langle C_4|\big)\rho(p_N)\big] < 0$, which leads to $p_N < 8/15$, and thus it tolerates up to 53.3% noise. The noise tolerance of witness $\mathcal{W}_2 := \frac{1}{2}\mathbb{1} - B_2$ is similarly calculated and we obtain that it tolerates up to 33% noise.

Here we present an operator

$$B_4 := \frac{1}{8}(S_1 + \mathbb{1})(S_2 + S_3)(S_4 + \mathbb{1}) = \frac{1}{8}(XXZI + IZXX + ZIXX + XXIZ - YYZI - IZYY - ZIYY - YYIZ) \qquad (9)$$

that can be measured by four local measurement settings, $XXZZ$, $ZZXX$, $YYZZ$, and $ZZYY$. It satisfies $|C_4\rangle\langle C_4| \geq B_4$. This is simply shown from $\langle C_{ijkl}|(|C_4\rangle\langle C_4| - B_4)|C_{ijkl}\rangle \geq 0$ for any $i,j,k,l = 1,-1$, where $|C_{ijkl}\rangle$ is the simultaneous eigenstate of S_1, S_2, S_3, and S_4 with eigenvalues i, j, k, and l, respectively. Using B_4, we can obtain a high lower bound of $F(\rho)$. The lower bound of $F(\rho(p_N))$ is $\mathrm{Tr}[B_4\rho(p_N)] = 1 - p_N$. The noise tolerance of the entanglement witness $\mathcal{W}_4 := \frac{1}{2}\mathbb{1} - B_4$ is 50%.

We show that the lower bound is optimal within any four local projective measurement settings for the produced state with white noise $\rho(p_N)$. For a set of four local projective measurements, there exist density matrices that are indistinguishable from $\rho(p_N)$. A lower bound of fidelity between $\rho(p_N)$ and $|C_4\rangle$ obtained by the measurement settings must be less than or equal to the fidelity between such density matrices and $|C_4\rangle$. In the following we show that for a set of four local projective measurements, there always exists a density matrix $\rho'(p_N)$, that is indistinguishable from $\rho(p_N)$ and its fidelity is $F(\rho'(p_N))=1-p_N$, which corresponds to $\mathrm{Tr}[B_4\rho(p_N)]$. We generally describe local projective measurements as a collection of projectors $\{M^m_{rstu}\}$, where $M^m_{rstu} := P^{1,m}_r\otimes P^{2,m}_s\otimes P^{3,m}_t\otimes P^{4,m}_u$, for $r,s,t,u=0,1$ and measurement settings $m=1,\ldots,4$. There exists an operator W_1 on the first qubit such that $\mathrm{Tr}\,W_1=0$, $\mathrm{Tr}[(P^{1,1}_r)W_1]=0$, $\mathrm{Tr}[(P^{1,2}_r)W_1]=0$, and $\|W_1\|=1$. Similarly, there exists an operator W_4 on the fourth qubit such that $\mathrm{Tr}\,W_4=0$, $\mathrm{Tr}[(P^{4,3}_u)W_4]=0$, $\mathrm{Tr}[(P^{4,4}_u)W_4]=0$, and $\|W_4\|=1$. From the fact that $|C_4\rangle$ is a maximally entangled state between sites (1,4) and (2,3), there exists an operator $W_{2,3}$ on the second and third qubits such that $W^{(1)}_1W^{(4)}_4|C_4\rangle=W^{(2,3)}_{2,3}|C_4\rangle$, and the eigenvalues of $W_{2,3}$ are 1 and -1, which are the same as the eigenvalues of $W_1\otimes W_4$. For $W:=W_1\otimes(W_{2,3})^{-1}\otimes W_4$, we obtain $\mathrm{Tr}\,W=0$, $W|C_4\rangle=|C_4\rangle$, and $\|W\|=1$. Thus, there exists a density operator $\rho'(p_N)=\rho(p_N)-(p_N/16)W$. From the definition of W_1 and W_4, $\mathrm{Tr}[M^m_{rstu}W]=0$ for any M^m_{rstu} and hence $\mathrm{Tr}[M^m_{rstu}\rho(p_N)]=\mathrm{Tr}[M^m_{rstu}\rho'(p_N)]$. Therefore, we cannot distinguish between $\rho(p_N)$ and $\rho'(p_N)$ using four local projective measurement settings. The fidelity between $\rho'(p_N)$ and $|C_4\rangle$ is $\langle C_4|\rho'(p_N)|C_4\rangle=1-p_N$. Therefore, B_4 provides the optimal lower bound of the fidelity between $\rho(p_N)$ to $|C_4\rangle$ when we use four local projective measurement settings.

Although B_4 is optimized for obtaining the lower bound of the fidelity, an upper bound of the fidelity is also obtained using B_4. Actually, $\frac{1}{2}B_4+\frac{1}{2}\mathbb{1}\geq|C_4\rangle\langle C_4|$ is satisfied. Thus, the upper bound of the fidelity of $\rho(p_N)$ is calculated as $\mathrm{Tr}\left[\left(\frac{1}{2}B_4+\frac{1}{2}\mathbb{1}\right)\rho(p_N)\right]=1-\frac{1}{2}p_N$.

IV. DISTINGUISHING CLASSES OF GENUINE FOUR-QUBIT ENTANGLEMENT

The entanglement witness (5) confirms that the detected state is not biseparable but a genuine four-qubit entangled state. However, there are many types of genuine four-qubit entangled states that are not cluster types [14,15]. It would be better if we could distinguish other types of genuine four-qubit entangled states from $|C_4\rangle$ using witness operators. For genuine three-qubit entanglement, Acín *et al.* provided a witness that detects a GHZ\\W class of entanglement, namely if a state is detected, then the entanglement of the state is not in the W class [16].

Here we present witnesses that allow us to distinguish between genuine four-qubit entangled states around $|C_4\rangle$ and certain classes of genuine four-qubit entangled states, e.g., a class including GHZ and W states. They are given by calculating the maximal fidelity between $|C_4\rangle$ and classes of states with specific genuine four-qubit entanglement. The classification is obtained by applying the idea of Schmidt number witness [17,19] to multipartite systems.

First we review the definition of Schmidt number [19] and show a construction of projection-based Schmidt number witnesses. A bipartite pure state $|\psi\rangle\in\mathcal{H}_1\otimes\mathcal{H}_2$ has Schmidt rank r if its Schmidt decomposition is written as $|\psi\rangle=\Sigma^r_{i=1}\sqrt{\lambda_i}|a_i\rangle|b_i\rangle$. Then, a set of density matrix $S(k)$ is defined such that $\rho\in S(k)$ when there exists a decomposition $\rho=\Sigma_jp_j|\psi_j\rangle\langle\psi_j|$ with $r_j\leq k$ for all j, where r_j is Schmidt rank of $|\psi_j\rangle$. If ρ is not in $S(k-1)$ but in $S(k)$, then we say that ρ has Schmidt number k. Here, $S(1)$ is the set of separable states. For $k\geq2$, $S(k)$ contains $S(k-1)$ and the set of entangled states with Schmidt number k. The set $S(k)$ is a convex compact subset of the entire set of density matrices S, and thus we can simply define entanglement witnesses using the set $S(k)$. Assume that $|\psi\rangle\in\mathcal{H}_1\otimes\mathcal{H}_2$ has Schmidt rank $l>k$. Then, a projector-based Schmidt number witness $\mathcal{W}^{\overline{S(k)}}$ detecting entanglement with Schmidt number $l'>k$ is given by $\mathcal{W}^{\overline{S(k)}}:=c\mathbb{1}-|\psi\rangle\langle\psi|$, where $c:=\max_{\rho\in S(k)}\langle\psi|\rho|\psi\rangle$. This guarantees that $\mathrm{Tr}(\mathcal{W}^{\overline{S(k)}}\rho)\geq0$ for all $\rho\in S(k)$, and $\mathrm{Tr}(\mathcal{W}^{\overline{S(k)}}\rho')<0$ signifies that ρ' has Schmidt number $l'>k$. Here we provide a simple method for calculating c. Since ρ is a convex sum of pure states in $S(k)$, $c=\max_{|\phi\rangle\in S(k)}|\langle\psi|\phi\rangle|^2=\max_{|\phi\rangle\in S(k)}\mathrm{Tr}(|\psi\rangle\langle\psi||\phi\rangle\langle\phi|)$. Let $c':=\max_{\Gamma\in\Omega(k)}\mathrm{Tr}[|\psi\rangle\langle\psi|(\Gamma\otimes\mathbb{1})]$, where $\Omega(k)$ is the set of rank-k projections acting on $\mathcal{H}_1$. It is obvious that $c\leq c'$, but we also have $c\geq c'$ since for every $\Gamma\in\Omega(k)$, we have state $(\Gamma\otimes\mathbb{1})|\psi\rangle/\|(\Gamma\otimes\mathbb{1})|\psi\rangle\|$ in $S(k)$. Hence we obtain

$$c=c'=\max_{\Gamma\in\Omega(k)}\mathrm{Tr}(\rho_{\psi,1}\Gamma),\tag{10}$$

where $\rho_{\psi,1}:=\mathrm{Tr}_2|\psi\rangle\langle\psi|$. Thus, c is the sum of k largest eigenvalues of $\rho_{\psi,1}$ (See also lemma 1 of [20]), or equivalently the sum of squares of k largest Schmidt coefficients of $|\psi\rangle$.

Next we extend the Schmidt number witnesses for multiqubit systems. We can divide n qubits into two groups of qubits with $m=2^{n-1}-1$ kinds of bipartite partitions. Then a set of density matrix $S_t(k_t)$ is defined such that $\rho^t\in S_t(k_t)$ when there exists a decomposition $\rho^t=\Sigma_jp_j|\psi_j\rangle\langle\psi_j|$ and $r^t_j\leq k_t$ for all j, where r^t_j is Schmidt rank of $|\psi_j\rangle$ in bipartite partition t. Then we define a set of density matrices $S(k_1,\ldots,k_m)$ such that $\rho\in S(k_1,\ldots,k_m)$ when ρ is a convex sum of $\rho^t\in S_t(k_t)$: $\rho=\Sigma_tp_t\rho^t$, where $\Sigma_tp_t=1$. For example, $S(1,\ldots,1)$ is the set of biseparable states. A projector-based Schmidt number witness for multiqubit systems is given by $\mathcal{W}^{\overline{S(k_1,\ldots,k_m)}}:=c\mathbb{1}-|\psi\rangle\langle\psi|$, where $c:=\max_{\rho\in S(k_1,\ldots,k_m)}\langle\psi|\rho|\psi\rangle=\max_{\rho^t\in S_t(k_t),t=1,\ldots,m}\langle\psi|\rho^t|\psi\rangle$. This guarantees that $\mathrm{Tr}(\mathcal{W}^{\overline{S(k_1,\ldots,k_m)}}\rho)\geq0$ for all $\rho\in S(k_1,\ldots,k_m)$, and $\mathrm{Tr}(\mathcal{W}^{\overline{S(k_1,\ldots,k_m)}}\rho')<0$ signifies that ρ' has a kind of entanglement that is not shared by the states in $S(k_1,\ldots,k_m)$.

Using the construction of Schmidt number witness for multiqubit systems, we can reconstruct the projector-based entanglement witness (5). The four-qubit system of the parties 1, 2, 3, and 4 can be divided into seven kinds of bipartite systems, 1(234), 2(134), 3(124), 4(123), (12)(34), (13)(24), and (14)(23). For simplicity, we denote them as 1, 2, 3, 4, 12, 13, and 14, respectively. A set of density matrices $S(k_1,k_2,k_3,k_4,k_{12},k_{13},k_{14})$ is defined as described above. Then a projector-based entanglement witness detecting genuine four-qubit entanglement around $|C_4\rangle$ is given by $\mathcal{W}^{\overline{S(1,\ldots,1)}}=\frac{1}{2}\mathbb{1}-|C_4\rangle\langle C_4|$, with 1/2 coming from the maximal eigenvalue of $\rho_A:=\mathrm{Tr}_{\bar{A}}|C_4\rangle\langle C_4|$, where $A=1$, 2, 3, 4, 12, 13, or 14, and $\bar{A}$ is the complement of A.

Here, the maximal eigenvalues of ρ_1, ρ_2, ρ_3, ρ_4, and ρ_{12} are 1/2, but the maximal eigenvalues of ρ_{13} and ρ_{14} are 1/4.

Therefore, the witness of the form $\frac{1}{2}\mathbb{1}-|C_4\rangle\langle C_4|$ may allow one to rule out wider classes of entanglement than biseparable entanglement in partitions 13 and 14. Actually, the sum of two largest eigenvalues of ρ_{13} (ρ_{14}) is 1/2. Thus, it turns out that the operator of the form $\frac{1}{2}\mathbb{1}-|C_4\rangle\langle C_4|$ works as a witness $\mathcal{W}^{\overline{S(1,\ldots,1,2,2)}}$. For example, $S(1,\ldots,1,2,2)$ includes four-qubit GHZ state $|\mathrm{GHZ}\rangle=\frac{1}{\sqrt{2}}(|0000\rangle+|1111\rangle)$ and W state $|W\rangle=\frac{1}{2}(|0001\rangle+|0010\rangle+|0100\rangle+|1000\rangle)$. Such entangled states can be distinguished from $|C_4\rangle$ by $\mathcal{W}^{\overline{S(1,\ldots,1,2,2)}}$.

The states $|C_4\rangle$ has Schmidt number 2 in partitions 1, 2, 3, 4, and 12, and Schmidt number 4 in partitions 13 and 14. Therefore, we can construct another form of Schmidt number witness for detecting entanglement around $|C_4\rangle$. It is $\mathcal{W}^{\overline{S(1,\ldots,1,3,3)}}=\frac{3}{4}\mathbb{1}-|C_4\rangle\langle C_4|$, which detects entanglement of states outside of $S(1,\ldots,1,3,3)$, where 3/4 comes from the sum of three largest eigenvalues of ρ_{13} (ρ_{14}). For example, $|\xi\rangle:=\frac{1}{\sqrt{3}}(|0000\rangle+|0011\rangle+|1100\rangle)$ is in $S(1,\ldots,1,3,3)$ but not in $S(1,\ldots,1,2,2)$. Such entangled states are also distinguished from $|C_4\rangle$ by $\mathcal{W}^{\overline{S(1,\ldots,1,3,3)}}$.

The operator B_4 in Sec. III is also useful for this construction of entanglement witnesses. The witness $\mathcal{W}_4^{\overline{S(1,\ldots,1,2,2)}}:=\frac{1}{2}\mathbb{1}-B_4$ tolerates up to 50% noise for $\rho(p_N)$ as shown in Sec. III. The witness $\mathcal{W}_4^{\overline{S(1,\ldots,1,3,3)}}:=\frac{3}{4}\mathbb{1}-B_4$ tolerates up to 25% noise for $\rho(p_N)$, which is close to the noise tolerance of $\mathcal{W}^{\overline{S(1,\ldots,1,3,3)}}$, 26%.

V. CONCLUSIONS

We have described methods of fidelity estimation and entanglement discrimination for experimentally produced four-qubit cluster states. We have obtained a high lower bound of the fidelity using less than half of local projective measurement settings for obtaining the exact fidelity. We have also constructed witness operators for distinguishing classes of genuine multipartite entanglement using the estimated value of the fidelity. The witness operators are obtained by evaluating the Schmidt number of states in several bipartite partitions. These ideas would also be applied for estimating the fidelity of other multipartite entangled states and verifying the class of entanglement.

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