Levinson theorem for the Dirac equation in one dimension

Zhong-Qi Ma
China Center of Advanced Science and Technology (World Laboratory), P.O. Box 8730, Beijing 100080, China and Institute of High Energy Physics, Beijing 100039, China

Shi-Hai Dong
Departamento de Física, Esc. Sup. de Física y Matemáticas, Instituto Politécnico Nacional, Edificio 9, Unidad Profesional Adolfo López Mateos, México D.F. 07738, México

Lu-Ya Wang
Department of College Computer Education, Hu'nan Normal University, Changsha 410081, China
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The Levinson theorem for the $(1+1)$-dimensional Dirac equation with a symmetric potential is proved with the Sturm-Liouville theorem. The half-bound states at the energies $E=\pm M$, whose wave function is finite but does not decay at infinity fast enough to be square integrable, are discussed. The number $n_{\pm}$ of bound states is equal to the sum of the phase shifts at the energies $E=\pm M$: $\delta_{\pm}(M)+\delta_{\pm}(-M)=(n_{\pm}+a)\pi$, where the subscript $\pm$ denotes the parity and the constant a is equal to $-1/2$ when no half-bound state occurs, to 0 when one half-bound state occurs at $E=M$ or at $E=-M$, and to $1/2$ when two half-bound states occur at both $E=\pm M$.

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I. INTRODUCTION

The Levinson theorem [1] is a fundamental theorem in quantum scattering theory, which shows the relation between the number of bound states and the phase shift at zero momentum. During the past half century more than 100 papers were devoted to generalization of this theorem and to its applications in physics. Since one-dimensional models on scattering theory are often employed to gain deeper insight into physical phenomena in real three-dimensional space, they have attracted the great attention of many authors for a long time. Newton [2–4] studied inverse scattering through a one-dimensional model by the Jost function. The Levinson theorem for the one-dimensional Schrödinger equation was studied by the operator formalism of scattering matrices [5,6], by the orthogonality and completeness relations for the eigenfunctions of the Hamiltonian [7–9], and by the Sturm-Liouville theorem [10]. Furthermore, the Levinson theorem for the $(1+1)$-dimensional Dirac equation was first studied by the Jost function and the relativistic field-theoretic method [11], and then, special attention was shown to the continuity of the S matrix [12]. By analyzing the critical cases with the half-bound states in detail, Lin [13] established the Levinson theorem for the $(1+1)$-dimensional Dirac equation with the Green's function method. The Levinson theorem for the $(1+1)$-dimensional Dirac equation in the presence of a soliton was also discussed [14].

Recently, a stronger version of the Levinson theorem for the $(1+1)$-dimensional Dirac equation has been presented [15], where the scattering phase shifts at $E=M$ and at $E=-M$ were connected to the number of states that have left the positive energy continuum or joined the negative one, respectively. Although it is correct, how does one measure a bound state whether it is transferred from the positive energy continuum or from the negative energy continuum? Such a connection was not very evident in the course of the proof with the Green's function method [13]. However, the connection is only a medium step in the course of the proof with the Sturm-Liouville theorem method. Moreover, it should be pointed out that the latter method is very intuitive in physical meaning and explicit in the demonstration of the connection between the number of the bound states and the scattering phase shifts. This is the reason why we prefer to employ the Sturm-Liouville theorem method to prove the Levinson theorem for the $(1+1)$-dimensional Dirac equation in this work.

The plan of this paper is organized as follows. In Sec. II we sketch the Dirac equation in $1+1$ dimensions. The Sturm-Liouville theorem for the $(1+1)$-dimensional Dirac equation is established in Sec. III. The Levinson theorem is proved with the Sturm-Liouville theorem in Sec. IV. A conclusion is given in Sec. V.

II. DIRAC EQUATION IN 1+1 DIMENSIONS

The Dirac equation in $(1+1)$-dimensional space-time is [13]

$$i\sum_{\mu=0}^{1}\gamma^{\mu}(\partial_{\mu}+ieA_{\mu})\Psi(x,t)=M\Psi(x,t), \tag{1}$$

where the natural units $\hbar=c=1$ are used, $\gamma^0=\gamma_0=\sigma_3$, and $\gamma^1=-\gamma_1=i\sigma_1$. We discuss a special case

$$eA_0=\lambda V(x), \quad V(x)=V(-x), \quad A_1=0, \tag{2}$$

where $V(x)$ satisfies the restriction

$$\int_0^{\infty}x|V(x)|dx<\infty, \tag{3}$$

which implies that the potential $V(x)$ is less singular near the origin than x^{-2} and vanishes infinity faster than x^{-2}. λ is a real parameter and eventually is set to 1. Due to the symmetry of $V(x)$, the wave function is classified with its parity, denoted by the subscript "±":

$$\gamma^0 \Psi_\pm(-x,t) = \pm \Psi_\pm(x,t). \tag{4}$$

Letting

$$\Psi_\pm(x,t) = e^{-iEt}\begin{pmatrix} F_\pm(x,E,\lambda) \\ G_\pm(x,E,\lambda) \end{pmatrix}, \tag{5}$$

one has

$$\frac{dG_\pm(x,E,\lambda)}{dx} = [E - \lambda V(x) - M]F_\pm(x,E,\lambda),$$

$$-\frac{dF_\pm(x,E,\lambda)}{dr} = [E - \lambda V(x) + M]G_\pm(r,E,\lambda). \tag{6}$$

Since $\gamma^0 = \sigma_3$, the component functions with a given parity satisfy

$$F_\pm(-x,E,\lambda) = \pm F_\pm(x,E,\lambda), \quad G_\pm(-x,E,\lambda) = \mp G_\pm(x,E,\lambda). \tag{7}$$

Thus, their boundary conditions at the origin are

$$\lim_{x\to 0} F_-(x,E,\lambda) = \lim_{x\to 0} G_+(x,E,\lambda) = 0,$$

$$\lim_{x\to 0} \frac{d}{dx}F_+(x,E,\lambda) = \lim_{x\to 0} \frac{d}{dx}G_-(x,E,\lambda) = 0. \tag{8}$$

Hence, the variant region of x reduces to $[0,\infty)$ equivalently. Since the tail of the potential at large x can be neglected, we will simply discuss the bounded potential

$$V(x) = 0, \quad \text{when } x > x_0. \tag{9}$$

It is easy to see that Eqs. (6)–(8) remain invariant by the replacement $F_\pm(x,E,\lambda) \leftrightarrow G_\mp(x,-E,-\lambda)$. The formulas for even parity can be obtained from those for odd parity by the replacement. In the following we will only study the Levinson theorem for solutions with odd parity and give the main results for even parity.

III. STURM-LIOUVILLE THEOREM

It is easy to show from Eq. (8) that

$$F_\pm(x,E',\lambda')\frac{dG_\pm(x,E,\lambda)}{dx} - G_\pm(x,E,\lambda)\frac{dF_\pm(x,E',\lambda)}{dx} = [E - E' - (\lambda - \lambda')V(x)]F_\pm(x,E',\lambda)G_\pm(r,E,\lambda).$$

The ratio

$$\phi_\pm(x,E,\lambda) = \frac{F_\pm(x,E,\lambda)}{G_\pm(x,E,\lambda)} \tag{10}$$

satisfies the generalized Sturm-Liouville theorem

$$G_\pm(x_0,E,\lambda)^2\frac{\partial}{\partial E}\phi_\pm(x_0-,E,\lambda) = -\int_0^{x_0}\{F_\pm(x,E,\lambda)^2 + G_\pm(x,E,\lambda)^2\}dx < 0, \tag{11}$$

$$G_\pm(x_0,E,\lambda)^2\frac{\partial}{\partial \lambda}\phi_\pm(x_0-,E,\lambda) = \int_0^{x_0} V(x)\{F_\pm(x,E,\lambda)^2 + G_\pm(x,E,\lambda)^2\}dx. \tag{12}$$

Except for some values of λ where $G_\pm(x_0,E,\lambda)=0$, the ratio $\phi_\pm(x_0-,E,\lambda)$ at a given point x_0- decreases monotonically as the energy increases and is monotonic with respect to λ if the potential $V(x)$ does not change its sign in the region $[0,x_0)$. Similarly, if the solution $\Psi(x,t)$ with $|E|<M$ tends to zero as x goes to infinity, one has

$$G_\pm(x_0,E,\lambda)^2\frac{\partial}{\partial E}\phi_\pm(x_0+,E,\lambda) = \int_{x_0}^{\infty}\{F_\pm(x,E,\lambda)^2 + G_\pm(x,E,\lambda)^2\}dx > 0. \tag{13}$$

The ratio $\phi_\pm(x_0+,E,\lambda)$ at a given point x_0+ increases monotonically as the energy increases.

Introduce the momenta k and k_1 for different energies:

$$k = \sqrt{E^2 - M^2}, \quad \text{when } |E| > M,$$

$$k_1 = \sqrt{M^2 - E^2}, \quad \text{when } |E| \leqslant M. \tag{14}$$

For a free particle, $\lambda=0$, the orthonormal regular solution of Eq. (6) with odd parity for $|E|>M$ is

$$F_-(x,E,0) = \frac{E}{|E|}\sqrt{\frac{|E+M|}{2\pi k}}\sin(kx),$$

$$G_-(x,E,0) = -\sqrt{\frac{|E-M|}{2\pi k}}\cos(kx). \tag{15}$$

There is no finite solution with odd parity for $|E|\leqslant M$ and $\lambda=0$ except for a half-bound state at $E=-M$:

$$F_-(x,-M,0) = 0, \quad G_-(x,-M,0) = 1, \quad \phi_-(x,-M,0) = 0. \tag{16}$$

For a given λ, Eq. (6) can be solved in the region (x_0,∞) from Eq. (9). For $|E|>M$ one has

$$F_-(x,E,\lambda) = \frac{E}{|E|}\sqrt{\frac{|E+M|}{2\pi k}}\sin[kx + \delta_-(E,\lambda)],$$

$$G_-(x,E,\lambda) = -\sqrt{\frac{|E-M|}{2\pi k}}\cos[kx + \delta_-(E,\lambda)]. \tag{17}$$

In fact, the normalization factor is not important in the following calculation. A convention for the phase shift $\delta_\pm(E,\lambda)$ that can be accepted is that $\delta_\pm(E,\lambda)$ with $|E|>M$ is a continuous function of λ and vanishes for a free particle:

$$\delta_\pm(E,0) = 0. \tag{18}$$

Through the matching condition at x_0,

$$\phi_\pm(x_0-,E,\lambda) = \phi_\pm(x_0+,E,\lambda), \tag{19}$$

we are able to obtain the phase shift as

$$\tan \delta\_(E,\lambda) = \frac{\phi\_(x_0-,E,\lambda)k\cos(kx_0)+(E/|E|)|E+M|\sin(kx_0)}{\phi\_(x_0-,E,\lambda)k\sin(kx_0)-(E/|E|)|E+M|\cos(kx_0)}, \tag{20}$$

where the phase shift $\delta\_(E,\lambda)$ and the wave functions depend upon λ. For a fixed k,

$$\left.\frac{\partial\delta\_(E,\lambda)}{\partial\phi(x_0-,E,\lambda)}\right|_k = \frac{-k(E/|E|)|E+M|\cos^2[\delta\_(E,\lambda)]}{[\phi\_(x_0-,E,\lambda)k\sin(kx_0)-(E/|E|)|E+M|\cos(kx_0)]^2}. \tag{21}$$

As $\phi(x_0-,E,\lambda)$ decreases, the phase shift $\delta\_(E,\lambda)$ increases when $E>M$ and decreases when $E<-M$. The phase shift $\delta\_(\pm M,\lambda)$ of zero momentum is the limit of $\delta\_(E,\lambda)$ as k goes to zero:

$$\delta\_(\pm M,\lambda) = \lim_{E\to\pm M}\delta\_(E,\lambda), \tag{22}$$

where $\delta\_(\pm M,\lambda)$ changes discontinuously as λ increases.

By expanding Eq. (20) with respect to kx_0, one has

$$\tan\delta\_(E,\lambda) \sim -kx_0\frac{\phi\_(x_0-,M,\lambda)^{-1}+(2Mx_0)^{-1}}{\phi\_(x_0-,M,\lambda)^{-1}+c^2k^2-k^2x_0/(2M)}, \quad \text{when } E>M, \tag{23}$$

and

$$\tan\delta\_(E,\lambda) \sim \frac{1}{kx_0}\frac{\phi\_(x_0-,E,\lambda)-k^2x_0/(2M)}{\phi\_(x_0-,E,\lambda)+1/(2Mx_0)}, \quad \text{when } E<-M. \tag{24}$$

It should be noted that $\phi\_(x_0-,M,\lambda)$ tends to infinity and $\phi\_(x_0-,M,\lambda)^{-1}$ tends to 0 as $G\_(x_0-,M,\lambda)$ goes to 0. When $E>M$ and kx_0 is sufficiently small, as λ increases from 0 to 1, each time $\phi\_(x_0-,M,\lambda)^{-1}$ increases across the value 0, $\delta\_(M,\lambda)$ jumps by π and each time $\phi\_(x_0-,M,\lambda)^{-1}$ decreases across the value 0, $\delta\_(M,\lambda)$ jumps by $-\pi$. For the critical case where $\phi\_(x_0-,M,1)^{-1}=0$, $\delta\_(M,\lambda)$ jumps by an additional $\pi/2$ (or $-\pi/2$) if $\phi(x_0-,M,\lambda)^{-1}$ increases (or decreases) to reach 0 as λ increases to reach 1.

When $E<-M$ and kx_0 is sufficiently small, $|\tan\delta\_(E,\lambda)|$ is very small if $\phi\_(x_0-,-M,\lambda)=0$ and is very large if $\phi\_(x_0-,-M,\lambda)\neq 0$. Note that $\phi\_(x_0-,-M,0)=0$ and $\delta\_(-M,0)=0$. As λ increases, due to Eq. (21), $\delta\_(E,\lambda)$ decreases (or increases) when $\phi\_(x_0-,-M,0)$ decreases (or increases). As λ increases from 0, $\delta\_(-M,\lambda)$ jumps by $\pi/2$ if $\phi\_(x_0-,-M,\lambda)$ increases and jumps by $-\pi/2$ if $\phi\_(x_0-,-M,\lambda)$ decreases. As λ increases again, when $\phi\_(x_0-,-M,\lambda)$ changes across the value $-(2Mx_0)^{-1}$, $\tan\delta\_(E,\lambda)$ changes sign but its absolute value is still very large. Each time $\phi\_(x_0-,-M,\lambda)$ increases across the value 0, $\tan\delta\_(E,\lambda)$ increases from negative to positive across the value 0 and $\delta\_(-M,\lambda)$ jumps by π. Similarly, each time $\phi\_(x_0-,-M,\lambda)$ decreases across the value 0, $\delta\_(-M,\lambda)$ jumps by $-\pi$. For the critical case where $\phi\_(x_0-,-M,1)=0$, $\delta\_(-M,\lambda)$ jumps by an additional $\pi/2$ (or $-\pi/2$) if $\phi\_(x_0-,-M,\lambda)$ increases (or decreases) to reach 0 as λ increases to reach 1.

IV. LEVINSON THEOREM

Now, we turn to discuss the bound states. The solution of Eq. (6) in the region (x_0,∞) with odd parity for $|E|<M$ is

$$F\_(x,E)=\sqrt{M+E}e^{-k_1x}, \quad G\_(x,E)=\sqrt{M-E}e^{-k_1x}, \tag{25}$$

and its ratio is

$$\phi(x_0+,E)=\begin{cases}2M/k_1\sim\infty, & \text{when } E\to M,\\ k_1/(2M)\sim 0, & \text{when } E\to -M.\end{cases} \tag{26}$$

When $E=\pm M$ there are half-bound states in the region (x_0,∞):

$$F\_(x,M)=1,\quad G\_(x,M)=0,\quad \phi\_(x_0+,M,0)=\infty,$$
$$F\_(x,-M)=0,\quad G\_(x,-M)=1,\quad \phi\_(x_0+,-M,0)=0. \tag{27}$$

The solution of Eq. (6) in the region $[0,x_0)$ cannot be solved analytically except for $\lambda=0$. When $\lambda=0$, the solution with odd parity for $|E|<M$ is

$$F\_(x,E,0)=\sqrt{M+E}\sinh(k_1x),$$
$$G\_(x,E,0)=-\sqrt{M-E}\cosh(k_1x), \tag{28}$$

and its ratio is

$$\phi\_(x_0-,E,0)\to\begin{cases}-2Mx_0, & \text{when } E\to M,\\ -k_1^2x_0/(2M)\sim 0, & \text{when } E\to -M.\end{cases} \tag{29}$$

it can be seen from Eqs. (26) and (29) that as E increases from $-M$ to M, $\phi\_(x_0+,E)$ increases monotonically from 0 to positive infinity and $\phi\_(x_0-,E,0)$ decreases monotonically from 0 to $-2Mx_0$. There is a half-bound state at $E=-M$ for a free particle [see Eq. (16)].

As λ increases, $\phi\_(x_0+,E)$ remains invariant, but $\phi\_(x_0-,E,\lambda)$ changes. Based on the Sturm-Liouville theorem (11), one only needs to pay attention to variations of $\phi\_(x_0-,M,\lambda)$ and $\phi\_(x_0-,-M,\lambda)$. As λ increases from 0 to 1, each time $\phi\_(x_0-,M,\lambda)^{-1}$ increases across the value 0, a scattering state in the positive energy continuum transfers to a bound state and the phase shift $\delta\_(M,\lambda)$ jumps by π and vice versa. For the critical case where $\phi\_(x_0-,M,1)^{-1}=0$, if $\phi\_(x_0-,M,\lambda)^{-1}$ increases to reach the value 0 as λ increases to reach 1, a scattering state in the positive energy continuum transfers to a half-bound state and the phase shift $\delta(0,\lambda)$ jumps by $\pi/2$. If $\phi(x_0-,M,\lambda)^{-1}$ decreases to reach 0 as λ increases to reach 1, a bound state becomes a half-bound state and the phase shift $\delta(0,\lambda)$ jumps by $-\pi/2$.

On the other hand, if $\phi\_(x_0-,-M,\lambda)$ increases as λ increases from 0, a half-bound state becomes a bound state and

$\delta_{-}(-M,\lambda)$ jumps from 0 to $\pi/2$. But if $\phi_{-}(x_0-,-M,\lambda)$ decreases, a half-bound state transfers to a scattering state in the negative energy continuum and $\delta_{-}(-M,\lambda)$ jumps from 0 to $-\pi/2$. As λ increases again, each time $\phi_{-}(x_0-,-M,\lambda)$ increases across the value 0, a scattering state in the negative energy continuum transfers to a bound state and $\delta_{-}(-M,\lambda)$ jumps by π and vice versa. For the critical case where $\phi_{-}(x_0-,-M,1)=0$, if $\phi_{-}(x_0-,-M,\lambda)$ increases to reach the value 0 as λ increases to reach 1, a scattering state in the negative energy continuum transfers to a half-bound state and $\delta_{-}(-M,\lambda)$ jumps by $\pi/2$. Conversely, if $\phi_{-}(x_0-,-M,\lambda)$ decreases to reach the value 0, a bound state becomes a half-bound state and $\delta_{-}(-M,\lambda)$ jumps by $-\pi/2$.

Through the replacement $F_{\pm}(x,E,\lambda)\leftrightarrow G_{\mp}(x,-E,-\lambda)$ one can make a similar conclusion for even parity to that for odd parity. Therefore, the Levinson theorem for the Dirac equation in $(1+1)$-dimensional space-time is written as

$$\delta_{\pm}(M)+\delta_{\pm}(-M) = \begin{cases} (n_{\pm}-1/2)\pi, & \text{no half-bound state occurs,} \\ n_{\pm}\pi, & \text{one half-bound state occurs,} \\ (n_{\pm}+1/2)\pi, & \text{two half-bound states occur,} \end{cases} \tag{30}$$

where $\delta_{\pm}(\pm M)=\delta_{\pm}(\pm M,1)$ is the phase shift at $E=\pm M$, $n_{\pm}$ is the number of bound states, and the subscript "±" denotes the parity. A half-bound state at $E=M$ occurs when $\phi_{\pm}(x_0-,M,1)^{-1}=0$, and a half-bound state at $E=-M$ occurs when $\phi_{\pm}(x_0-,-M,1)=0$. When no half-bound state occurs, $\delta_{-}(-M)/\pi$ and $\delta_{+}(M)/\pi$ are half of an odd integer and $\delta_{-}(M)/\pi$ and $\delta_{+}(-M)/\pi$ are an integer. When a half-bound state occurs at $E=M$, $\delta_{-}(M)/\pi$ becomes half of an odd integer and $\delta_{+}(M)/\pi$ becomes an integer. When a half-bound state occurs at $E=-M$, $\delta_{-}(-M)/\pi$ becomes an integer and $\delta_{+}(-M)/\pi$ becomes half of an odd integer. Equation (30) coincides with Eq. (51) in Ref. [13].

V. CONCLUSION

In this paper we reproved the Levinson theorem with the Sturm-Liouville theorem. The phase shift at $E=M$ jumps by π and a scattering state in the positive energy continuum transfers to a bound state as the ratio $\phi_{\pm}(r_0-,M,\lambda)^{-1}$ increases across 0 and vice versa. Similarly, the phase shift at $E=-M$ jumps by $-\pi$ and a bound state transfers to a scattering state in the negative energy continuum as the ratio $\phi_{\pm}(r_0-,-M,\lambda)$ decreases across 0 and vice versa. However, for a given bound state, one cannot distinguish whether it is transferred from the positive energy continuum or from the negative energy continuum, so that the stronger version of the Levinson theorem does not make new sense.

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