Off-shell Jost solution for a Coulomb-like potential

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The theory of ordinary differential equations together with judicious use of boundary conditions and certain properties of higher transcendental functions is exploited to derive a useful analytical expression for the Coulomb-Yamaguchi Jost solution through an r-space approach to the problem. Note that the off-shell Jost solution is expressed in its maximal reduced form involving confluent and Gaussian hypergeometric functions. As an application of the Jost solution the off-shell T matrix is also expressed in terms of Gaussian hypergeometric functions.

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The theory of nonrelativistic scattering is considerably simplified by the use of nonlocal potentials. This simplification is justified by the observation that short-range local potentials can be approximated by finite-rank separable potentials in a mathematically well defined sense and also that the nonlocal potentials can describe a much wider variety of phenomena than that encompassed with short-range local potentials [1]. In view of the importance of experiments which involve charge hadrons, the interest in studying potentials consisting of the sum of a short-range finite-rank separable potential and Coulomb potential is increased [2–4].

In conventional potential scattering theory the physical scattering amplitude can be obtained by taking the on-shell limit of the off-shell T matrix. This is no longer true for Coulomb and Coulomb plus short-range potentials [5] and they exhibit a discontinuity at the energy shell. Nevertheless, it is generally expected that one also can extract, in such a situation, all relevant physical information from the off-shell T matrix. The outgoing-wave off-shell function can be expressed directly in terms of an off-shell Jost solution and Jost function. Also by exploiting the relation that exists between off-shell T matrix and outgoing-wave off-shell function one can obtain an expression for the T matrix. The off-shell Jost function introduced by Fuda and Whiting [6] is also determined from the irregular solution of an inhomogeneous Schödinger equation in the same way as $f_\ell(k)$ is determined from $f_\ell(k,r)$.

Therefore, it is of some importance to have an explicit expression in the literature for the off-shell Jost solution relating to Coulomb-like potentials which are encountered in the physical processes like the proton-proton (p-p) bremsstrahlung and (p-$2p$) reaction.

Separable potentials have been, since the appearance of Yamaguchi's original paper [7], an immensely popular tool in dynamical calculations. In this paper an explicit expression for the off-shell Jost solution for a Coulomb plus Yamaguchi potential will be derived by solving the inhomogeneous Schödinger equation. The result for an off-shell Jost solution to author's knowledge is new. The method proposed will be applicable for a Coulomb plus separable potential of arbitrary rank but a higher partial wave treatment will involve mathematical complication.

Yamaguchi [7] has introduced a one term separable potential

$$V(r,s) = \lambda e^{-\beta(r+s)} \tag{1}$$

to describe the nucleon-nucleon scattering. Here λ and β are the strength and range parameters of the potential. The off-shell Jost solution $f(k,q,r)$ for a Coulomb plus Yamaguchi potential satisfies the differential equation

$$\left[\frac{d^2}{dr^2} + k^2 - \frac{2k\eta}{r}\right] f(k,q,r) = d(k,q)e^{-\beta r} + (k^2 - q^2)e^{iqr} \tag{2}$$

with

$$d(k,q) = \lambda \int_0^\infty ds e^{-\beta s} f(k,q,s). \tag{3}$$

Here η is the well known Sommerfeld parameter.

To solve Eq. (2) the dependent and independent variables are changed as follows:

$$f(k,q,r) = re^{ikr} g(k,q,r), \tag{4a}$$

$$z = -2ikr \tag{4b}$$

to get

$$\left[z\frac{d^2}{dz^2} + (c-z)\frac{d}{dz} - a\right] g(k,q,z) = -\frac{1}{2ik}[d(k,q)e^{\rho z} + (k^2 - q^2)e^{\gamma z}] \tag{5}$$

with

$$a = 1 + i\eta, \quad c = 2, \quad \rho = \frac{(\beta + ik)}{2ik} \text{ and } \gamma = \frac{(k-q)}{2k}. \tag{6}$$

Complementary functions of Eq. (5) are given by confluent hypergeometric functions

$$\Phi(a,c;z) = \frac{\Gamma(c)}{\Gamma(a)} \sum_{n=0}^\infty \frac{\Gamma(a+n)z^n}{\Gamma(c+n)n!} \tag{7}$$

and

$$\bar{\Phi}(a,c;z)=z^{1-c}\Phi(a-c+1,2-c;z). \tag{8}$$

Note that, for $c=2$, Eq. (8) is not an acceptable solution of Eq. (2). However, $\bar{\Phi}$ tends towards the solution [8] of Eq. (2) when c approaches 2. This is no loss of generalization. See, for example, the treatment of the Coulomb field by Newton [9]. Another solution of Eq. (2) defined within the framework of the same limiting procedure is

$$\Psi(a,c;z)=\frac{\Gamma(1-c)}{\Gamma(a-c+1)}\Phi(a,c;z)+\frac{\Gamma(c-1)}{\Gamma(a)}\bar{\Phi}(a,c;z). \tag{9}$$

According to Babister [10] the particular solution of the non-homogeneous confluent hypergeometric equation

$$\left[z\frac{d^2}{dz^2}+(c-z)\frac{d}{dz}-a\right]y=e^{\rho z}z^{\sigma-1}, \tag{10}$$

where a, c, ρ and σ are constants, is written as follows:

$$y_P=\sum_{n=0}^{\infty}\frac{\theta_{\sigma+n}(a,c;z)\rho^n}{n!}, \tag{11a}$$

with

$$\theta_\sigma(a,c;z)=\frac{z^\sigma}{\sigma(\sigma+c-1)}{}_2F_2(1,c+a;\sigma+1,\sigma+c;z). \tag{11b}$$

The series in Eqs. (11a) and (11b) is convergent for all values of ρ and z.

On comparing Eqs. (5) and (10) the particular solution $g_P(z)$ is written as follows:

$$g_P(z)=-\frac{1}{2ik}\sum_{n=0}^{\infty}\left\{\frac{d(k,q)\rho^n}{n!}+\frac{(k^2-q^2)\gamma^n}{n!}\right\}\theta_{n+1}(1+i\eta,2;z). \tag{12}$$

Combining Eqs. (6), (7), (9), and (12) the general solution of Eq. (5) is obtained as follows:

$$g(k,q,z)=A\Phi(1+i\eta,2;z)+B\Psi(1+i\eta,2;z)-\frac{1}{2ik}\sum_{n=0}^{\infty}\left\{\frac{d(k,q)\rho^n}{n!}+\frac{(k^2-q^2)\gamma^n}{n!}\right\}\theta_{n+1}(1+i\eta,2;z), \tag{13}$$

with A and B are two arbitrary constants.

From Eqs. (4a), (4b), and (13) the Jost solution reads

$$f(k,q,r)=re^{ikr}\left[A\Phi(1+i\eta,2;-2ikr)+B\Psi(1+i\eta,2;-2ikr)-\frac{1}{2ik}\sum_{n=0}^{\infty}\left\{\frac{d(k,q)\rho^n}{n!}+\frac{(k^2-q^2)\gamma^n}{n!}\right\}\theta_{n+1}(1+i\eta,2;-2ikr)\right]. \tag{14}$$

The two arbitrary constants A and B will be determined by exploiting the values of $f(k,q,r)$ at two extreme points, i.e., at $r=0$ and ∞.

The on- and off-shell Jost functions $f_\ell(k)$ and $f_\ell(k,q)$ are defined by [9]

$$f_\ell(k)=Lt_{r\to 0}\frac{\ell!(-2ikr)^\ell}{(2\ell)!}f_\ell(k,r), \tag{15}$$

$$f_\ell(k,q)=Lt_{r\to 0}\frac{\ell!(-2ikr)^\ell}{(2\ell)!}f_\ell(k,q,r). \tag{16}$$

The off-shell Jost function for the Coulomb plus Yamaguchi potential has appeared in a number of publications [5,11,12] and is written:

$$f(k,q)=f^C(k,q)+\frac{\lambda f^C(k,q)}{D(k)(1+i\eta)(\beta-ik)(\beta^2+k^2)}\left(\frac{(\beta-ik)}{(\beta+ik)}\right)^{i\eta}\times\left[F\left(1,i\eta;2+i\eta;\frac{(\beta+ik)}{(\beta-ik)}\right)+\frac{e^{i\pi/2}(q-k)}{(\beta-iq)f^C(k,q)}\times F\left(1,i\eta;2+i\eta\frac{(\beta+ik)(q-k)}{(\beta-ik)(q+k)}\right)\right] \tag{17}$$

with $f^C(k,q)$, the off-shell s-wave Coulomb Jost function given by

$$f^C(k,q)=\left(\frac{(q+k)}{(q-k)}\right)^{i\eta} \tag{18}$$

and

$$D(k)=1-\frac{\lambda}{(1+i\eta)(\beta-ik)}\left[\frac{1}{(\beta^2+k^2)}\left(\frac{(\beta-ik)}{(\beta+ik)}\right)^{i\eta}F\left(1,i\eta;2+i\eta;\frac{(\beta+ik)}{(\beta-ik)}\right)-\frac{1}{2\beta(\beta-ik)}F\left(1,i\eta;2+i\eta;\left[\frac{(\beta+ik)}{(\beta-ik)}\right]^2\right)\right]. \tag{19}$$

The function $f(k,q,r)$ satisfies the asymptotic condition

$$Lt_{r\to\infty}f(k,q,r)e^{-iqr}=1. \tag{20}$$

Thus as $r\to 0$ Eq. (14) yields

$$B=-2ik\Gamma(1+i\eta)f(k,q) \tag{21}$$

with the fact that $Lt_{z\to 0}\Psi(a,c;z)\approx\frac{\Gamma(c-1)}{\Gamma(a)}z^{1-c}$ together with Eqs. (16) and (17). Thus, from Eqs. (14), (17), and (21) $f(k,q,r)$ reads

$$f(k,q,r)=re^{ikr}\Bigg[A\Phi(1+i\eta,2;-2ikr)-2ik\Gamma(1+i\eta)f(k,q)\Psi(1+i\eta,2;-2ikr)-\frac{1}{2ik}\sum_{n=0}^{\infty}\left\{\frac{d(k,q)\rho^n}{n!}+\frac{(k^2-q^2)\gamma^n}{n!}\right\}\times\theta_{n+1}(1+i\eta,2;-2ikr)\Bigg]. \tag{22}$$

Evaluation of constant A as $r\to\infty$ is rather tricky and the procedure is as follows.

The function $\theta_\sigma(a,c;z)$ can be expressed in terms of indefinite integrals [10] involving $\Phi(\bullet)$ and $\bar{\Phi}(\bullet)$ as follows:

$$\theta_\sigma(a,c;z)=\frac{1}{(c-1)}\left[\Phi(a,c;z)\int_0^z ds s^{\sigma+c-2}e^{-s}\bar{\Phi}(a,c;s)-\bar{\Phi}(a,c;z)\int_0^z ds s^{\sigma+c-2}e^{-s}\Phi(a,c;s)\right]. \tag{23}$$

The well known Coulomb regular Green's function is written [9]

$$G(r,r')=[\varphi^C(k,r)f^C(k,r')-\varphi^C(k,r')f^C(k,r)]/f^C(k)=2ikrr'e^{ik(r+r')}[\bar{\Phi}(1+i\eta,2;-2ikr)\Phi(1+i\eta,2;-2ikr')-\bar{\Phi}(1+i\eta,2;-2ikr')\Phi(1+i\eta,2;-2ikr)] \tag{24}$$

for $r'<r$ and zero elsewhere. Here $\varphi^C(k,r)$ and $f^C(k,r)$ are the regular and irregular Coulomb solutions, respectively. The last two terms in Eq. (22) in conjunction with Eqs. (23) and (24) can be expressed [13,14]

$$\frac{(k^2-q^2)}{2ik}re^{ikr}\sum_{n=0}^{\infty}\frac{\gamma^n}{n!}\theta_{n+1}(1+i\eta,2;-2ikr)=-(k^2-q^2)\int_0^r G(r,s)e^{iqs}ds \tag{25}$$

and

$$\frac{1}{2ik}re^{ikr}\sum_{n=0}^{\infty}\frac{\rho^n}{n!}\theta_{n+1}(1+i\eta,2;-2ikr)=-\int_0^r G(r,s)e^{-\beta s}ds. \tag{26}$$

In view of Eqs. (25) and (26) and the well known differential equations for $G(r,r')$, $\varphi^C(k,r)$ and $f^C(k,r)$ together with the transposed operator relation $\int\varphi O\psi=\int\psi\tilde{O}\varphi$, where, $\tilde{O}=O$, Eq. (22) is obtained as follows:

$$f(k,q,r)=\left[A-2ik\Gamma(1+i\eta)\left\{d(k,q)\int_0^r ds se^{-(\beta-ik)s}\Psi(1+i\eta,2;-2iks)-(k^2-q^2)\int_0^r ds se^{-i(k+q)s}\Psi(1+i\eta,2;-2iks)\right\}\right]re^{ikr}\Phi(1+i\eta,2;-2ikr)-\left[2ik\Gamma(1+i\eta)f(k,q)-d(k,q)\int_0^r ds se^{-(\beta-ik)s}\Phi(1+i\eta,2;-2iks)-(k^2-q^2)\int_0^r ds se^{i(k+q)s}\Phi(1+i\eta,2;-2iks)\right]re^{ikr}\Psi(1+i\eta,2;-2ikr)+\int_0^r\delta(r-s)e^{iqs}ds. \tag{27}$$

In case of pure Coulomb problem [14] the coefficient of $re^{ikr}\Psi(1+i\eta,2;-2ikr)$ in the expression for Coulomb Jost solution vanishes as $r\to\infty$. Thus, in analogy with the Coulomb case $d(k,q)$ is identified as follows:

$$d(k,q)=\frac{\lambda f^C(k,q)}{D(k)(1+i\eta)(\beta-ik)}\left[F\left(1,i\eta;2+i\eta;\frac{(\beta+ik)}{(\beta-ik)}\right)+\frac{e^{i\pi/2}(q-k)}{(\beta-iq)f^C(k,q)}F\left(1,i\eta;2+i\eta;\frac{(q-k)(\beta+ik)}{(q+k)(\beta-ik)}\right)\right]. \tag{28}$$

As $r\to\infty$, Eq. (27) in conjunction with Eq. (28) and the condition given in Eq. (20) leads to

$$A=\frac{e^{i\pi/2}(q-k)}{(1+i\eta)}F\left(1,i\eta;2+i\eta;\frac{(q-k)}{(q+k)}\right)-\frac{\lambda f^C(k,q)}{D(k)(1+i\eta)^2(\beta-ik)^2}F\left(1,i\eta;2+i\eta;\frac{(\beta+ik)}{(\beta-ik)}\right),$$

$$\left[F\left(1,i\eta;2+i\eta;\frac{(\beta+ik)}{(\beta-ik)}\right)+\frac{e^{i\pi/2}(q-k)}{(\beta-iq)f^C(k,q)}F\left(1,i\eta;2+i\eta;\frac{(q-k)(\beta+ik)}{(q+k)(\beta-ik)}\right)\right]. \tag{29}$$

In deriving the above result the following relations [8,15]:

$$\int_0^\infty e^{-ax}x^{s-1}\Psi(b,d;\mu x)dx=\frac{\Gamma(1+s-d)\Gamma(s)}{a^s\Gamma(1+b+s-d)}F(b,s;1+b+s-d;1-\mu/a), \tag{30}$$

$$\int_0^\infty e^{-\lambda z}z^{\nu}\Phi(a,c;pz)dz=\frac{\Gamma(\nu+1)}{\lambda^{\nu+1}}F\left(a,\nu+1;c;\frac{p}{\lambda}\right) \tag{31}$$

and

$$F(a,b;c;z)=(1-z)^{c-a-b}F(c-a,c-b;c;z) \tag{32}$$

have been used. Therefore, from Eqs. (22), (28), and (29) the desired expression for $f(k,q,r)$ is obtained as follows:

$$\begin{aligned}f(k,q,r)&=f^C(k,q,r)-\frac{\lambda f^C(k,q)}{D(k)(1+i\eta)(\beta-ik)}\left[F\left(1,i\eta;2+i\eta;\frac{(\beta+ik)}{(\beta-ik)}\right)+\frac{e^{i\pi/2}(q-k)}{(\beta-iq)f^C(k,q)}F\left(1,i\eta;2+i\eta;\frac{(q-k)(\beta+ik)}{(q+k)(\beta-ik)}\right)\right]\\&\times\left\{\frac{1}{(1+i\eta)(\beta-ik)}F\left(1,i\eta;2+i\eta;\frac{(\beta+ik)}{(\beta-ik)}\right)\Phi(1+i\eta,2;-2ikr)+\frac{2ik\Gamma(1+i\eta)}{(\beta^2+k^2)}\left(\frac{(\beta-ik)}{(\beta+ik)}\right)^{i\eta}\Psi(1+i\eta,2;-2ikr)\right.\\&\left.+\frac{1}{2ik}\sum_{n=0}^{\infty}\frac{\rho^n}{n!}\theta_{n+1}(1+i\eta,2;-2ikr)\right\}re^{ikr}.\end{aligned} \tag{33}$$

Here $f^C(k,q,r)$, the Coulomb off-shell Jost solution [14] reads

$$\begin{aligned}f^C(k,q,r)=&\left\{\frac{e^{i\pi/2}(q-k)}{(1+i\eta)}F\left(1,i\eta;2+i\eta;\frac{(q-k)}{(q+k)}\right)\Phi(1+i\eta,2;\right.\\&-2ikr)-2ik\Gamma(1+i\eta)f^C(k,q)\Psi(1+i\eta,2;-2ikr)\\&\left.-\frac{(k^2-q^2)}{2ik}\sum_{n=0}^{\infty}\frac{\gamma^n}{n!}\theta_{n+1}(1+i\eta,2;-2ikr)\right\}re^{ikr}.\end{aligned} \tag{34}$$

A couple of useful checks is made on the expression for Coulomb plus Yamaguchi off-shell Jost solution with particular emphasis on their limiting behavior and on-shell discontinuity. For example, in absence of Yamaguchi potential i.e. $\lambda=0$, $f(k,q,r)$ goes to pure Coulomb Jost solution [14]. Secondly, in the limit of no Coulomb field, $\eta=0$, Yamaguchi Jost solution is obtained [16]. When both λ and η goes to zero, $f(k,q,r)=e^{iqr}$. Other useful checks on Eq. (33) consists in showing that

$$f(k,q,r)_{r\to 0}\to f(k,q), \tag{35}$$

$$\begin{aligned}f(k,r)&=Lt_{q\to k}\frac{e^{\pi\eta/2}}{\Gamma(1+i\eta)}\left(\frac{(q-k)}{(q+k)}\right)^{i\eta}f(k,q,r)\\&=f^C(k,r)-\lambda\left[\frac{e^{\pi\eta/2}}{D(k)\Gamma(2+i\eta)(\beta-ik)}\right]\\&\times{}_2F_1\left(1,i\eta;2+i\eta;\frac{\beta+ik}{\beta-ik}\right),\\&\left[re^{ikr}\left[\frac{1}{(1+i\eta)(\beta-ik)^2}{}_2F_1\left(1,i\eta;2+i\eta;\frac{\beta+ik}{\beta-ik}\right)\right.\right.\\&\times\Phi(1+i\eta,2;-2ikr)+\frac{2ik\Gamma(1+i\eta)}{(\beta^2+k^2)}\left(\frac{\beta-ik}{\beta+ik}\right)^{i\eta}\\&\left.\times\Psi(1+i\eta,2;-2ikr)-\sum_{n=0}^{\infty}\frac{\rho^n}{n!}\theta_{n+1}(1+i\eta,2;-2ikr)\right]\end{aligned} \tag{36}$$

with

$$f^C(k,r)=-2ik\Gamma(1+i\eta)re^{ikr}\Psi(1+i\eta,2;-2ikr). \tag{37}$$

The Coulomb-Yamaguchi on-shell Jost function is given by

$$\begin{aligned}f(k)&=Lt_{r\to 0}f(k,r)\\&=f^C(k)+\lambda\frac{f^C(k)}{D(k)(1+i\eta)(\beta-ik)(\beta^2+k^2)}\left(\frac{\beta-ik}{\beta+ik}\right)^{i\eta},\end{aligned}$$

$$ {}_2F_1\left(1,i\eta;2+i\eta;\frac{(\beta+ik)}{(\beta-ik)}\right). \quad (38)$$

Equations (35) and (36) can easily be verified from the result in Eq. (33). The result of Eqs. (35), (36), and (38) are in agreement with that of van Haeringen [17] and Talukdar *et al.* [18]. In the following the off-shell T matrix will be calculated to support another useful check on the expression for an off-shell Jost solution for the Coulomb-Yamaguchi potential.

The results for $f(k,r)$ and $f(k,q,r)$ can be used to construct an exact analytical expression for the off-shell outgoing wave solution $\psi^{(+)}(k,q,r)$ by using the relation [18]

$$\psi^{(+)}(k,q,r) = -\frac{1}{2}\pi q T(k,q,k^2) f(k,r) + \frac{1}{2i}[f(k,q,r) - f(k,-q,r)], \quad (39)$$

where the half-shell T matrix is

$$T(k,q,k^2) = \left\{\frac{f(k,q)-f(k,-q)}{i\pi q f(k)}\right\}. \quad (40)$$

Given the expression for $\psi^{(+)}(k,q,r)$ one will be in a position to write a uncomplicated expression for the off-shell T matrix in terms of the formula

$$T(p,q,k^2) = \frac{2}{\pi p q}\left[\int_0^\infty dr\,\mathrm{Sin}(pr)V^C(r)\psi^{(+)}(k,q,r) + \lambda\int_0^\infty dr\,\mathrm{Sin}(pr)e^{-\beta r}\int_0^\infty ds e^{-\beta s}\psi^{(+)}(k,q,s)\right]. \quad (41)$$

Equations (39) and (40) together with Eqs. (17)–(19) and (36)–(38) yield

$$\psi^{(+)}(k,q,r) = \psi^{C(+)}(k,q,r) + K(\beta,q,k^2)X(\beta,k,r), \quad (42)$$

with

$$\psi^{C(+)}(k,q,r) = re^{ikr}\left\{\left[\frac{e^{-i\pi/2}(q-k)}{2(1+i\eta)}\right]F\left(1,i\eta;2+i\eta;\frac{(q-k)}{(q+k)}\right)\Phi(1+i\eta,2;-2ikr) + \frac{(k^2-q^2)}{4k}\Lambda_{\gamma,1}(1+i\eta,2;-2ikr) + \left[\frac{e^{-i\pi/2}(q+k)}{2(1+i\eta)}\right]F\left(1,i\eta;2+i\eta;\frac{(q+k)}{(q-k)}\right),\Phi(1+i\eta,2;-2ikr) - \frac{(k^2-q^2)}{4k}\Lambda_{1-\gamma,1}(1+i\eta,2;-2ikr)\right\}, \quad (43)$$

$$K(\beta,q,k^2) = \lambda\frac{(\beta+ik)(\beta+iq)}{2D^{(+)}(k)(1+i\eta)(\beta^2+k^2)(\beta^2+q^2)}, \left[(k-q)\,{}_2F_1\left(1,i\eta;2+i\eta;\frac{(q-k)(\beta+ik)}{(q+k)(\beta-ik)}\right) - (k+q)\,{}_2F_1\left(1,i\eta;2+i\eta;\frac{(q+k)(\beta+ik)}{(q-k)(\beta-ik)}\right)\right], \quad (44)$$

and

$$X(\beta,k,r) = \left\{\frac{1}{(1+i\eta)(\beta-ik)}\,{}_2F_1\left(1,i\eta;2+i\eta;\frac{(\beta+ik)}{(\beta-ik)}\right)\Phi(1+i\eta,2;-2ikr) + \frac{1}{2ik}\Lambda_{\rho,1}(1+i\eta,2;-2ikr)\right\}re^{ikr}. \quad (45)$$

On expanding $e^{\rho z}$ as a power series in z, the particular integral [10] of Eq. (10) can also be expressed in terms of the function $\Lambda_{\rho,\sigma}(a,c;z)$, where

$$\Lambda_{\rho,\sigma}(a,c;z) = \sum_{n=0}^{\infty}\frac{\rho^n}{n!}\theta_{\sigma+n}(a,c;z) \quad (46)$$

and $\theta_\sigma(a,c;z)$ is related to confluent hypergeometric functions by

$$\theta_\sigma(a,c;z) = \frac{1}{(c-1)}\left[\Phi(a,c;z)\int_0^z e^{-z'}z'^{(\sigma+c-2)}\bar{\Phi}(a,c;z')dz' - \bar{\Phi}(a,c;z')\int_0^z e^{-z'}z'^{(\sigma+c-2)}\Phi(a,c;z')dz'\right]. \quad (47)$$

Utilizing Eqs. (41) and (42) the off-shell T matrix for the Coulomb plus Yamaguchi potential is obtained as follows:

$$T(p,q,k^2) = T^C(p,q,k^2) - \frac{2ik\eta}{\pi p q}K(\beta,q,k^2)[I_1(p,q,k^2) - I_2(p,q,k^2)] - \frac{i\lambda}{\pi p q}I_3(p,\beta)[I_4(\beta,q,k^2) + K(\beta,q,k^2)I_5(\beta,q,k^2)], \quad (48)$$

with the quantities

$$T^C(p,q,k^2) = \frac{4k\eta}{\pi p q}\int_0^\infty \frac{\sin pr}{r}\psi^{C(+)}(k,q,r)dr, \quad (49)$$

$$I_1(p,q,k^2) = \int_0^\infty dr r^{-1}e^{ipr}X(k,\beta,r), \quad (50)$$

$$I_2(p,q,k^2) = [I_1(p,q,k^2)]_{p\to -p}, \quad (51)$$

$$I_3(p,\beta) = \int_0^\infty dr\, \mathrm{Sin}(pr) e^{-\beta r}, \tag{52}$$

$$I_4(\beta,q,k^2) = \int_0^\infty dr e^{-\beta r} \psi^{C(+)}(k,q,r), \tag{53}$$

and

$$I_5(\beta,q,k^2) = \int_0^\infty dr e^{-\beta r} X(k,\beta,r). \tag{54}$$

To calculate the off-shell Coulomb T matrix $T^C(p,q,k^2)$ Eq. (43) is substituted in Eq. (49) to have

$$T^C(p,q,k^2) = T^{C1}(p,q,k^2) + T^{C2}(p,-q,k^2) + T^{C3}(-p,q,k^2) + T^{C4}(-p,-q,k^2), \tag{55}$$

where

$$T^{C1}(p,q,k^2) = \frac{2ik\eta}{\pi pq}\Bigg[\frac{(k-q)}{2(1+i\eta)} F\left(1,i\eta;2+i\eta;\frac{(q-k)}{(q+k)}\right)\int_0^\infty dr e^{i(p+k)r}\Phi(1+i\eta,2;-2ikr) - \frac{(k^2-q^2)}{4k}\int_0^\infty dr e^{i(p+k)r}\Lambda_{\gamma,1}(1+i\eta,2;-2ikr)\Bigg], \tag{56}$$

$$T^{C2}(p,-q,k^2) = T^{C1}(p,q,k^2)|_{q\to -q}, \tag{57}$$

$$T^{C3}(-p,q,k^2) = T^{C1}(p,q,k^2)|_{p\to -p}, \tag{58}$$

and

$$T^{C4}(-p,-q,k^2) = T^{C1}(p,q,k^2)|_{p\to -p;q\to -q}. \tag{59}$$

Use of the following relations [8,10,19,20]:

$$\int_0^\infty dz e^{-\lambda z} z^\nu \Phi(a,c;pz) = \frac{\Gamma(\nu+1)}{\lambda^{\nu+1}} F\left(a,\nu+1;c;\frac{p}{\lambda}\right), \tag{60}$$

$$\int_0^\infty dz z^\nu e^{-bz}\theta_\sigma(a,c;pz) = \frac{\Gamma(\nu+\sigma+1)p^\sigma}{\sigma(\sigma+c-1)b^{\nu+\sigma+1}} {}_3F_2\left(1,\sigma+a,\nu+\sigma+1;\sigma+1\sigma+c;\frac{p}{b}\right),$$

$$\mathrm{Re}\,\sigma > 0, \mathrm{Re}(\sigma+c) > 1, \mathrm{Re}\,\nu > -1, \mathrm{Re}\,b > \mathrm{Re}\,p, \tag{61}$$

$$F(a,b;c;z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a,b;a+b-c+1;1-z) + (1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} F(c-a,c-b;c-a-b+1;1-z) \tag{62}$$

$$c[{}_2F_1(a,b;c;z) - {}_2F_1(a+1,b;c;z)] + bz\,{}_2F_1(a+1,b+1;c+1;z) = 0, \tag{63}$$

$${}_2F_1(a,b;c;z) = (1-z)^{-a}{}_2F_1\left(a,c-b;c;\frac{z}{z-1}\right), \tag{64}$$

$${}_2F_1(a,b;c;z) = (1-z)^{c-a-b}{}_2F_1(c-a,c-b;c;z) \tag{65}$$

and the integral representation

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 dt t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} \tag{66}$$

and certain algebraic manipulations in Eq. (56) lead to

$$T^{C1}(p,q,k^2) = \frac{e^{i\pi/2}k}{\pi pq} F\left(1,i\eta;1+i\eta;\frac{(q-k)(p-k)}{(q+k)(p+k)}\right) - \frac{e^{i\pi/2}}{2\pi pq}\left[(k+q) + \frac{(k-q)}{(1+i\eta)} \times F\left(1,i\eta;2+i\eta;\frac{(q-k)}{(q+k)}\right)\right]. \tag{67}$$

In view of Eqs. (57)–(59) and (67) equation (55) is obtained as follows:

$$T^C(p,q,k^2) = \frac{e^{i\pi/2}k}{\pi pq}\Bigg[F\left(1,i\eta;1+i\eta;\frac{(q-k)(p-k)}{(q+k)(p+k)}\right) - F\left(1,i\eta;1+i\eta;\frac{(q+k)(p-k)}{(q-k)(p+k)}\right) - F\left(1,i\eta;1+i\eta;\frac{(q-k)(p+k)}{(q+k)(p-k)}\right) + F\left(1,i\eta;1+i\eta;\frac{(q+k)(p+k)}{(q-k)(p-k)}\right)\Bigg]. \tag{68}$$

The result in the above equation is in exact agreement with that of van Haeringen and van Wageningen [21] derived in the momentum space approach. Utilizing the relations (60)–(66) the other integrals involved in Eqs. (50)–(54) can easily be calculated to obtain

$$I_1(p,\beta,k^2) = \frac{1}{2k\eta}\left[\frac{1}{(\beta+ik)} - \frac{1}{(1+i\eta)(\beta-ik)} \times {}_2F_1\left(1,i\eta;2+i\eta;\frac{(\beta+ik)}{(\beta-ik)}\right) + \frac{2ik}{(\beta^2+k^2)} \times {}_2F_1\left(1,i\eta;1+i\eta;\frac{(p-k)(\beta+ik)}{(p+k)(\beta-ik)}\right)\right], \tag{69}$$

$$I_3(p,\beta) = \frac{2ip}{(\beta^2+p^2)}, \tag{70}$$

$$I_4(q,\beta,k^2) = \frac{q}{(\beta^2+q^2)} - \frac{k}{(\beta^2+k^2)} \times \left\{ {}_2F_1\left(1,i\eta;1+i\eta;\frac{(q+k)(\beta+ik)}{(q-k)(\beta-ik)}\right) - {}_2F_1\left(1,i\eta;1+i\eta;\frac{(q-k)(\beta+ik)}{(q+k)(\beta-ik)}\right)\right\} \tag{71}$$

and

$$I_5(q,\beta,k^2) = \frac{1}{(\beta^2+k^2)^2}\left\{\frac{(\beta-ik)^2}{2\beta} + 2ik\,{}_2F_1\left(1,i\eta;1+i\eta;\frac{(\beta+ik)^2}{(\beta-ik)^2}\right)\right\}. \tag{72}$$

The integral $I_2(p,q,k^2)$ is obtained by replacing p by $-p$ in Eq. (69). Therefore, Eq. (48) together with Eqs. (44) and (68)–(72) gives the desired expression for the off-shell T matrix for Coulomb-Yamaguchi potential expressed as follows:

$$T(p,q,k^2) = T^C(p,q,k^2) + \frac{2}{\pi pq}K(\beta,q,k^2)\left[-\frac{p}{(\beta^2+p^2)} + \frac{k}{(\beta^2+k^2)}\left\{ {}_2F_1\left(1,i\eta;1+i\eta;\frac{(p-k)(\beta+ik)}{(p+k)(\beta-ik)}\right) - {}_2F_1\left(1,i\eta;1+i\eta;\frac{(p+k)(\beta+ik)}{(p-k)(\beta-ik)}\right)\right\}\right]. \tag{73}$$

In writing $T(p,q,k^2)$ in its maximal reduced form the following relations are used:

$$I_4(\beta,q,k^2) = -\frac{1}{\lambda}K(\beta,q,k^2)D^{(+)}(k), \tag{74}$$

and

$$I_5(\beta,q,k^2) = \frac{1}{\lambda}[D^{(+)}(k) - 1] \tag{75}$$

together with

$${}_2F_1(1,i\eta;2+i\eta;z) = (1+i\eta)\left\{\frac{1}{z} + \left(1-\frac{1}{z}\right){}_2F_1(1,i\eta;1+i\eta;z)\right\}. \tag{76}$$

The expression in Eq. (73) agrees well with that of Ref. [21] and produces correct limiting behaviors. However, $T(p,q,k^2)$ has been derived via the momentum space approach to the problem in [21] whereas the r-space approach is adopted here.

The present approach can easily be extended to deal with potentials of higher rank and the restriction to symmetric form factors is not compelling. By exploiting the relation that exists between an off-shell physical solution and off-shell Jost solutions and functions, one will be in a position to write an expression for $\psi^{(+)}(k,q,r)$ and thereby an off-shell T matrix for Coulomb and Coulomb-like potentials. This conjecture represents a straightforward approach to deal with off-shell scattering on the same class of potentials. Thus, the exact analytical expression for an off-shell Jost solution for scattering by a Coulomb plus a Yamaguchi potential is believed to be useful for the description of the charged particle scattering/reaction processes.

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